Solution 9

1. **a)** If $f : M \to N$ is an orientation preserving diffeomorphism between oriented m-manifolds then $\int_M f^*\omega = \int_N \omega$ for every $\omega \in \Omega_c^m(N)$. If $f : M \to N$ is an orientation reversing diffeomorphism between oriented m-manifolds then $\int_M f^*\omega = -\int_N \omega$ for every $\omega \in \Omega_c^m(N)$.

b) The orientation double cover of a manifold M is

$$\tilde{M} := \{(p, o) \,:\, p \in M,\, o \text{ is an orientation of } T_pM\}.$$

Show that $\tilde{M}$ has a natural smooth manifold structure and is orientable. Moreover, interchaning the orientation yields an orientation reversing a diffeomorphism $\varphi : \tilde{M} \to \tilde{M}$ with $\varphi^2 = \mathrm{id}_{\tilde{M}}$.

c) Let $\pi : \tilde{M} \to M$ denote the canonical projection. Then $\tilde{\eta} \in \Omega^k(\tilde{M})$ satsifies $\tilde{\eta} = \phi^*\tilde{\eta}$ if and only if there exists $\eta \in \Omega^k(M)$ with $\tilde{\eta} = \pi^*\eta$.

d) Let M be a compact connected non-orientable m-manifold without boundary. Prove that every m-form on M is exact.

Solution:

a) Let $\psi : V \to \mathbb{R}^m$ be a chart of N and suppose first that ω is supported in V. Then $\phi := \psi \circ f : f^{-1}(V) \to \mathbb{R}^m$ yields a chart for M which is compatible with the given orientation if and only if f is orientation preserving. It now follows from the definition of the integral that

$$\int_N \omega = \int_{\mathbb{R}^m} (\psi^{-1})^*\omega$$

and

$$\int_M f^*\omega = \pm \int_{\mathbb{R}^m} (\phi^{-1})^*(f^*\omega) = \pm \int_{\mathbb{R}^m} (f \circ \phi^{-1})^*\omega = \pm \int_{\mathbb{R}^m} (\psi^{-1})^*\omega$$

where the sign is $+$ when f is orientation preserving and the sign is $-$ when f orientation reversing. This proves the result when ω is contained in a single chart of N.

For the general case, we may cover N by finitely many chart V_i and choose a partition of unity $\{\rho_i\}$ subordinate to this cover. Then follows from the special case above

$$\int_N \omega := \sum_{i=1}^r \int_N \rho_i\omega = \sum_{i=1}^r \pm \int_M f^*(\rho_i\omega) = \pm \sum_{i=1}^r \int_M (\rho_i \circ f) f^*\omega = \pm \int_M f^*\omega$$

where the sign is $+$ when f is orientation preserving and the sign is $-$ when f orientation reversing. This proves the general case.

b) We construct an atlas for $\tilde{M}$. Let $p \in M$ and let o be an orientation of T_pM. Let $\phi : U \to \mathbb{R}^m$ be any chart with $p \in U \subset M$. Suppose $d\phi(p) : T_pM \to \mathbb{R}^m$ is orientation preserving with respect to the orientation o on T_pM and the standard orientation of $\mathbb{R}^m$. Then define

$$\tilde{U} := \left\{ (q, o) \in \tilde{M} \,:\, \begin{array}{c} q \in U \text{ and } o \text{ is the (unique) orientation on } T_qM \\ \text{such that } d\phi(q) : T_qM \to \mathbb{R}^m \text{ is orientation preserving} \end{array} \right\}$$

Then $\tilde{\phi} : \tilde{U} \to \mathbb{R}^m$ defined by $\tilde{\phi}(q, o) := \phi(q)$ yields a chart for $\tilde{M}$ defined on a neighbourhood of (p, o).

Now suppose ϕ is a chart of M around p and $d\phi(p) : T_pM \to \mathbb{R}^m$ does not intertwine the orientation o and the standard orientation on $\mathbb{R}^m$. Then choose any orientation reversing diffeomorphism $\tau : \mathbb{R}^m \to \mathbb{R}^m$, e.g. $\tau(x_1, \dots, x_m) = (-x_1, x_2, \dots, x_m)$. Then $\tau \circ \phi$ is again a chart for M which now intertwines the orientation o on T_pM with the orientation of $\mathbb{R}^m$. We then apply the procedure above to obtain a chart around (p, o).

The transition maps of the atlas obtained from all these charts are either transition maps of the atlas of M or obtained as the composition of such transition maps with an orientation reversing diffeomorphism τ of $\mathbb{R}^n$. Moreover, all transition maps of this atlas for $\tilde{M}$ are orientation preserving diffeomorphism of $\mathbb{R}^m$. We thus have constructed an oriented atlas for $\tilde{M}$ which defines a natural orientation. This is the tautological orientation which yields on $T_{(p,o)}\tilde{M} \cong T_pM$ the orientation o.

Finally define $\varphi : \tilde{M} \to \tilde{M}$ by $\varphi(p, o) = (p, \bar{o})$, where $\bar{o}$ is the opposite orientation of o for T_pM. This is clearly a diffeomorphism which satisfies $\varphi^2(p, o) = (p, o)$ for all $(p, o) \in \tilde{M}$. Moreover, let $\tilde{\phi} : \tilde{U} \to \mathbb{R}^m$ be an orientation preserving chart around (p, o). Then $\tilde{\phi}' := \tau \circ \tilde{\phi} : \varphi(\tilde{U}) \to \mathbb{R}^m$ is an orientation preserving chart around $(p, \bar{o})$, where $\tau : \mathbb{R}^n \to \mathbb{R}^n$ denotes any orientation reversing diffeomorphism. It follows that $\tilde{\phi}' \circ \varphi \circ \tilde{\phi}^{-1} = \tau$ is orientation reversing and so is φ.

c) Since $\pi \circ \varphi = \pi$, we have

$$\varphi^*(\pi^*\eta) = (\pi \circ \varphi)^*\eta = \pi^*\eta$$

for all $\eta \in \Omega^k(M)$.

Assume conversely, that $\tilde{\eta} \in \Omega^k(\tilde{M})$ satisfies $\varphi^*\tilde{\eta} = \tilde{\eta}$. Then define $\eta \in \Omega^k(M)$ by

$$\eta_p(v_1, \dots, v_k) = \tilde{\eta}_{(p,o)}(\tilde{v}_1, \dots, \tilde{v}_n)$$

where o is any orientation of T_pM and $\tilde{v}_i \in T_{(p,o)}\tilde{M}$ are the unique vectors with $d\pi(p, o)\tilde{v}_i = v_i$. We need to show that this does not depend on the chosen orientation o for T_pM. Denote by $\bar{o}$ the oppostive orientation of T_pM and by $\bar{v}_i \in T_{(p,\bar{o})}\tilde{M}$ the unique vectors with $d\pi(p, \bar{o})\bar{v}_i = v_i$. Then follows

$$\begin{aligned}
\tilde{\eta}_{(p,\bar{o})}(\bar{v}_1, \dots, \bar{v}_n) &= \tilde{\eta}_{\varphi(p,o)}(d\varphi(p, o)\tilde{v}_1, \dots, d\varphi(p, o)\tilde{v}_n) \\
&= (\varphi^*\tilde{\eta})_{(p,o)}(\tilde{v}_1, \dots, \tilde{v}_n) \\
&= \tilde{\eta}_{(p,o)}(\tilde{v}_1, \dots, \tilde{v}_n)
\end{aligned}$$

and hence both choices for the orientation yield the same value for $\eta_p(v_1, \dots, v_k)$.

d) Let $\omega \in \Omega^m(M)$ and define $\tilde{\omega} := \pi^*\omega$. Since $\varphi : \tilde{M} \to \tilde{M}$ is orientation reversing, it follows from part (a) and (c):

$$\int_{\tilde{M}} \tilde{\omega} = \int_{\tilde{M}} \varphi^*\tilde{\omega} = -\int_{\tilde{M}} \tilde{\omega}.$$

This shows $\int_{\tilde{M}} \tilde{\omega} = 0$.

Now note that $\tilde{M}$ is connected if and only if M is non-orientable. There are various ways to see this and we argue by using the theory of covering spaces. It follows from the path lifting property for covering spaces that the double cover $\tilde{M}$ is either connected or the disjoint union of two copies of M. Since $\tilde{M}$ is orientable, it would follow in the later case that M is also orientable and this gives a contradiction.

Since $\tilde{M}$ is connected and $\int_{\tilde{M}} \tilde{\omega} = 0$, it follows from a theorem of the lecture that $\tilde{\omega}$ is exact and there exits $\tilde{\tau} \in \Omega^{m-1}(\tilde{M})$ with $d\tilde{\tau} = \tilde{\omega}$. Define

$$\tilde{\tau}' := \frac{1}{2}\left(\tilde{\tau} + \varphi^*\tilde{\tau}\right) \in \tilde{\Omega}^{n-1}(\tilde{M}).$$

Then $\varphi^*\tilde{\tau}' = \tilde{\tau}'$ and it follows from part (c) that there exists $\tau \in \Omega^{n-1}(M)$ with $\pi^*\tau = \tilde{\tau}'$. Moreover,

$$\pi^* d\tau = d\tau' = \frac{1}{2}\left(d\tilde{\tau} + \varphi^* d\tilde{\tau}\right) = \tilde{\omega} = \pi^*\omega.$$

Since π is a surjective submersion, this implies $d\tau = \omega$ and therefore ω is exact.

2. Let M, N be smooth manifolds.

a) Prove that given $\varphi : M \to N$ a diffeomorphism, $Y \in \mathrm{Vect}(N)$, and ω a form on M, we have

$$\mathcal{L}_{\varphi^* Y}(\varphi^*\omega) = \varphi^*(\mathcal{L}_Y \omega).$$

b) Let $Y_t \in \mathrm{Vect}(N)$ be a smooth family of vector fields and let ψ_t be the isotopy generated by Y_t via

$$\partial_t \psi_t = Y_t \circ \psi_t,\ \psi_0 = \mathrm{id}.$$

Prove that

$$\frac{d}{dt}\psi_t^*\omega = \psi_t^*\mathcal{L}_{Y_t}\omega.$$

c) Let $\omega \in \Omega^k(N)$, $Y \in \mathrm{Vect}(N)$ and $\varphi : [0,1] \times M \to N : (t,p) \mapsto \varphi_t(p)$ a smooth map. Deduce from

$$\mathcal{L}_Y \omega = d(\iota(Y)\omega) + \iota(Y)d\omega, \tag{1}$$

the formula

$$\frac{d}{dt}\varphi_t^*\omega = dh_t\omega + h_t d\omega \tag{2}$$

where we recall $h_t : \Omega^k(N) \to \Omega^{k-1}(M)$ is given by

$$(h_t\omega)_p(v_1, \ldots, v_{k-1}) := \omega_{\varphi_t(p)}(\partial_t\varphi_t(p), d\varphi_t(p)v_1, \ldots, d\varphi_t(p)v_{k-1})$$

for $p \in M$ and $v_1, \ldots, v_{k-1} \in T_pM$.

Hint: Assume first that the map $\varphi_t : M \to N$ is an embedding for every t. Then there is a smooth family of vector field such that

$$Y_t \in \mathrm{Vect}(N),\ Y_t \circ \varphi_t = \partial_t \varphi_t.$$

Let ψ_t be isotopy of N generated by Y_t as above. Then $\varphi_t = \psi_t \circ \varphi_0$. Now deduce (2) from (1) for $\mathcal{L}_{Y_t}\omega$. To prove (2) in general replace the map $\varphi_t : M \to N$ by the embedding $\psi_t : M \to \tilde{N} := M \times N, \tilde{\psi}_t(p) := (p, \varphi_t(p))$ and argue as above.

Solution:

a) This is simply recalling from the first semester that $\varphi^{-1} \circ \psi_t \circ \varphi$ is the isotopy generated by $\varphi^* Y$, if ψ_t is the one of Y. So by definition

$$\mathcal{L}_{\varphi^* Y}(\varphi^*\omega) = \left.\frac{d}{dt}\right|_{t=0} (\varphi^{-1} \circ \psi_t \circ \varphi)^*\varphi^*\omega = \left.\frac{d}{dt}\right|_{t=0} (\psi_t \circ \varphi)^*\omega = \varphi^* \left.\frac{d}{dt}\right|_{t=0} \psi_t^*\omega = \varphi^*\mathcal{L}_Y\omega.$$

b) By part a) and $(\varphi^{-1}\psi_t)^*\varphi^*\omega = \psi_t^*\omega$, we can reduce to the case $t = 0$ by taking $\varphi = \psi_s$. Furthermore by a) and since it is a local statement, it is enough to prove it in charts, if we take φ a chart. Thus assume $N = \mathbb{R}^m$ and $\omega \in \Omega^k(\mathbb{R}^m)$. For $k = 0$, we have ω is equal to a smooth function f on N. Then we calculate for $p \in N$

$$\frac{d}{dt}\Big|_{t=0}(\psi_t^* f)(p) = \frac{d}{dt}\Big|_{t=0}(f \circ \psi_t)(p) = df(p)(\partial_t|_{t=0}\psi_t(p)) = df(p)Y_0(p) =: \mathcal{L}_{Y_0}f(p).$$

So it is true for $k = 0$.

For $k = 1$, we take $\omega = gdf$ for f, g a zero form, and get

$$\frac{d}{dt}\Big|_{t=0}(\psi_t^* gdf) = \frac{d}{dt}\Big|_{t=0}(g \circ \psi_t)d(\psi_t^* f) = (\mathcal{L}_{Y_0}g)df + d(\mathcal{L}_{Y_0}f).$$

To check our formula in this case, let us calculate the other side of our equation.

$$\begin{aligned}
\mathcal{L}_{Y_0}(gdf) &= \sum_{j=1}^m \frac{d}{dt}\Big|_{t=0}\psi_t^*(g\partial_j f dx^j) = \sum_{i,j=1}^m \frac{d}{dt}\Big|_{t=0}(g \circ \psi_t)\,(\partial_j f \circ \psi_t)\,\partial_i \psi_t^j\; dx^i) \\
&= (\mathcal{L}_{Y_0}g)df + \sum_{i,j=1}^m \Big(\partial_i\partial_j f (Y_0)^j + \partial_j f \partial_i (Y_0)^j\Big)dx^i \\
&= (\mathcal{L}_{Y_0}g)df + \sum_{i,j=1}^m \partial_i(\partial_j f\; (Y_0)^j)dx^i = (\mathcal{L}_{Y_0}g)df + d(\mathcal{L}_{Y_0}f).
\end{aligned}$$

This proves that by linearity, the result is true for $k = 1$. The general case now follows from the facts

$$\psi_t^*(\alpha \wedge \beta) = \psi_t^*\alpha \wedge \psi_t^*\beta, \text{ and } \mathcal{L}_{Y_0}(\alpha \wedge \beta) = \mathcal{L}_{Y_0}\alpha \wedge \beta + \alpha \wedge \mathcal{L}_{Y_0}\beta.$$

c) We start by assuming that φ_t is an embedding. Then we get a map ψ_t with $\varphi_t = \psi_t \circ \varphi_0$ as in the hint. Here Y_t can be constructed by a partition of unity of N and adapted charts to the submanifold $\varphi_0(M)$ in N. Thus, we get

$$\frac{d}{dt}\varphi_t^*\omega = (\varphi_0)^*\frac{d}{dt}\psi_t^*\omega \overset{b)}{=} (\varphi_t)^*\mathcal{L}_{Y_t}\omega \overset{(1)}{=} (\varphi_t)^*(d(\iota(Y_t)\omega) + \iota(Y_t)d\omega) = dh_t\omega + h_t d\omega,$$

where we used that for $p \in M$ and $v_1, \ldots, v_{k-1} \in T_pM$

$$\begin{aligned}
\varphi_t^*(\iota(Y_t)\omega)_p(v_1, \ldots, v_{k-1}) &= (\iota(\partial_t\varphi_t(p))\omega)_{\varphi_t(p)}(d\varphi_t(p)v_1, \ldots, d\varphi_t(p)v_{k-1}) \\
&= (h_t\omega)_p(v_1, \ldots, v_{k-1}).
\end{aligned}$$

Now for the general case, we take $\omega \in \Omega^k(N)$ and define $\pi : \tilde{N} := N \times M \to N$ the projection on N. Then we have that $\pi\tilde{\varphi}_t = \varphi_t$ and so by the embedded case above

$$\frac{d}{dt}\varphi_t^*\omega = \frac{d}{dt}\tilde{\varphi}_t^*(\pi^*\omega) = d((\tilde{h}_t\pi^*)\omega)) + (\tilde{h}_t\pi^*)d\omega.$$

where $\tilde{h}_t$ is the map associated to $\tilde{\varphi}_t$. Thus to finish the proof, we need to prove that $\tilde{h}_t\pi^* = h_t$. Thus take $p \in M$ and $v_1, \ldots, v_{k-1} \in T_pM$, to get

$$\begin{aligned}
&(\tilde{h}_t)_p(\pi^*\omega)(v_1, \ldots, v_{k-1}) = (\pi^*\omega)_{\tilde{\varphi}_t(p)}(\partial_t\tilde{\varphi}_t(p), d\tilde{\varphi}_t(p)v_1, \ldots, d\tilde{\varphi}_t(p)v_{k-1}) \\
&= \omega_{\varphi_t(p)}(d\pi(\tilde{\varphi}_t(p))\partial_t\tilde{\varphi}_t(p), d\pi(\tilde{\varphi}_t(p))d\tilde{\varphi}_t(p)v_1, \ldots, d\pi(\tilde{\varphi}_t(p))d\tilde{\varphi}_t(p)v_{k-1}) \\
&= (h_t)_p(v_1, \ldots, v_{k-1}).
\end{aligned}$$

3. Let M be a manifold, $X \in \mathrm{Vect}(M)$ and τ, ω be forms on M. Prove the following:

a) $\iota(X)(\omega \wedge \tau) = (\iota(X)\omega) \wedge \tau + (-1)^{\deg(\omega)} \omega \wedge (\iota(X)\tau)$.

b) $\mathcal{L}_X(\omega \wedge \tau) = \mathcal{L}_X\omega \wedge \tau + \omega \wedge \mathcal{L}_X\tau$.

Solution:

a) We start by proving the result for $\omega \in \Omega^1(M)$ and $\tau \in \Omega^k(M)$.

Take $v_1, \ldots, v_k \in T_pM$ and plug in

$$\begin{aligned}
&(\iota(X)(\omega \wedge \tau))_p(v_1, \ldots, v_k) = (\omega \wedge \tau)_p(X(p), v_1, \ldots, v_k) \\
&= \omega(X(p))\tau(v_1, \ldots, v_k) + \sum_{i=1}^k (-1)^i \omega(v_i)\tau(X(p), v_1, \ldots, \hat{v}_i, \ldots, v_k) \\
&= (\iota(X)\omega \wedge \tau)_p(v_1, \ldots, v_k) + \sum_{i=1}^k (-1)^i \omega(v_i)\tau(X(p), v_1, \ldots, \hat{v}_i, \ldots, v_k).
\end{aligned}$$

where $(-1)^i = \mathrm{sgn}(i, \mathbf{0}, 1, \ldots, i-1, i+1, \ldots, k)$. Now we calculate the other side of the equation

$$\begin{aligned}
(\omega \wedge \iota(X)\tau)_p(v_1, \ldots, v_k) &= \sum_{i=1}^k (-1)^{i-1} \omega(v_i)\tau(X(p), v_1 \ldots, \hat{v}_i, \ldots, v_k) \\
&= (-1)^{\deg(\omega)} \sum_{i=1}^k (-1)^i \omega(v_i)\tau(X(p), v_1, \ldots, \hat{v}_i, \ldots, v_k)
\end{aligned}$$

where $(-1)^{i-1} = \mathrm{sgn}(i, 1, \ldots, i-1, i+1, \ldots, k)$. Now we proceed by induction. By linearity, we may assume that $\omega = \alpha \wedge \beta$ for $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^{\ell-1}$. So assume the formula holds for β, then by the one dimensional case, we have

$$\begin{aligned}
&\iota(X)(\omega \wedge \tau) = (\iota(X)\alpha) \wedge \beta \wedge \tau - \alpha \wedge (\iota(X)(\beta \wedge \tau)) \\
&= (\iota(X)\alpha) \wedge \beta \wedge \tau - (\iota(X)\beta) \wedge \tau + (-1)^\ell \omega \wedge (\iota(X)\tau) \\
&= \Big((\iota(X)\alpha) \wedge \beta - \alpha \wedge (\iota(X)\beta)\Big) \wedge \tau + (-1)^\ell \omega \wedge (\iota(X)\tau) \\
&= (\iota(X)\omega) \wedge \tau + (-1)^\ell \omega \wedge (\iota(X)\tau).
\end{aligned}$$

b) We get by Leibnitz for φ_t the flow of X that

$$\begin{aligned}
\mathcal{L}_X(\omega \wedge \tau) &= \left.\frac{d}{dt}\right|_{t=0} \varphi_t^*(\omega \wedge \tau) \\
&= \left.\frac{d}{dt}\right|_{t=0} (\varphi_t^*\omega \wedge \varphi_t^*\tau) \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} \varphi_t^*\omega\right) \wedge \tau + \omega \wedge \left(\left.\frac{d}{dt}\right|_{t=0} \varphi_t^*\tau\right) \\
&= \mathcal{L}_X\omega \wedge \tau + \omega \wedge \mathcal{L}_X\tau.
\end{aligned}$$

4. Prove that for $\beta \in \Omega^1(M)$, $\omega \in \Omega^2(M)$, and $X, Y, Z \in \mathrm{Vect}(M)$, we have

a) $d\beta(X, Y) = \mathcal{L}_X(\beta(Y)) - \mathcal{L}_Y(\beta(X)) + \beta([X, Y])$,

b) $d\omega(X,Y,Z) = \mathcal{L}_X(\omega(Y,Z)) + \mathcal{L}_Y(\omega(Z,X)) + \mathcal{L}_Z(\omega(X,Y))$
$+ \omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y)$.

Solution:

a) Both sides are skew-symmetric. So let us take a function f on M and calculate both sides for fX and Y.

$$
\begin{aligned}
&d\beta(fX,Y) = fd\beta(X,Y),\\
&\mathcal{L}_{fX}(\beta(Y)) - \mathcal{L}_Y(\beta(fX)) + \beta([fX,Y])\\
&= f\mathcal{L}_X(\beta(Y)) - d(f\beta(X))Y + \beta((\mathcal{L}_Y f)X) + f\beta([X,Y])\\
&= f\mathcal{L}_X(\beta(Y)) - \mathcal{L}_Y f\beta(X) - f\mathcal{L}_Y\beta(X))Y + \beta((\mathcal{L}_Y f)X) + f\beta([X,Y])\\
&= f\Big(\mathcal{L}_X(\beta(Y)) - \mathcal{L}_Y(\beta(X)) + \beta([X,Y])\Big)
\end{aligned}
$$

Since the statement is local, we can work over a coordinate chart $\varphi : U \subset M \to \Omega \subset \mathbb{R}^m$. Denote by $e_i(p) := d\varphi^{-1}(p)\partial_i$ for $p \in U$, $i = 1,\dots,m$ and where ∂_i is the i-th vector of the standard basis of $\mathbb{R}^m$. Then we can write $X = \sum_{i=1}^m X^i e_i$ and $Y = \sum_{j=1}^m Y^i e_j$. Thus by what we just showed, we reduce to showing the identity on the vector fields e_i. Thus we calculate in the chart $\tilde{\beta} = (\varphi^{-1})^*\beta = \sum_{k=1}^m \beta_k dx^k$ and get

$$
\begin{aligned}
&d\beta(e_i,e_j) = d\tilde{\beta}(\partial_i,\partial_j) = \Big(\sum_{k,\ell=1}^m \partial_\ell\beta_k dx^\ell \wedge dx^k\Big)(\partial_i,\partial_j) = \partial_i\beta_j - \partial_j\beta_i,\\
&\mathcal{L}_{e_i}(\beta(e_j)) - \mathcal{L}_{e_j}(\beta(e_i)) + \beta(0) = \mathcal{L}_{\partial_i}(\tilde{\beta}(\partial_j)) - \mathcal{L}_{\partial_j}(\tilde{\beta}(\partial_i))\\
&= \mathcal{L}_{\partial_i}\beta_j - \mathcal{L}_{\partial_j}\beta_i = \partial_i\beta_j - \partial_j\beta_i.
\end{aligned}
$$

So the formula holds.

b) Just as in a), we can show that both sides are linear over functions. So we reduce to local coordinates just as before. Also by linearity, it is enough to prove the result for $\omega = f dx^a \wedge dx^b$ for $1 \le a < b \le m$. So take $X = \partial_i, Y = \partial_j, Z = \partial_k$ and calculate

$$
\begin{aligned}
&d\omega(\partial_i,\partial_j,\partial_k) = \Big(\sum_{p=1}^m \partial_p f dx^p \wedge dx^a \wedge dx^b\Big)(\partial_i,\partial_j,\partial_k)\\
&= \partial_i f(\delta_j^a\delta_k^b - \delta_j^b\delta_k^a) + \partial_j f(\delta_i^b\delta_k^a - \delta_i^a\delta_k^b) + \partial_k f(\delta_i^a\delta_j^b - \delta_i^b\delta_j^a),\\
&\mathcal{L}_{\partial_i}\omega(\partial_j,\partial_k) + \mathcal{L}_{\partial_j}\omega(\partial_k,\partial_i) + \mathcal{L}_{\partial_k}\omega(\partial_i,\partial_j)\\
&= \mathcal{L}_{\partial_i}(f(\delta_j^a\delta_k^b - \delta_j^b\delta_k^a)) + \mathcal{L}_{\partial_j}(f(\delta_i^b\delta_k^a - \delta_i^a\delta_k^b)) + \mathcal{L}_{\partial_k}(f(\delta_i^a\delta_j^b - \delta_i^b\delta_j^a))\\
&= \partial_i f(\delta_j^a\delta_k^b - \delta_j^b\delta_k^a) + \partial_j f(\delta_i^b\delta_k^a - \delta_i^a\delta_k^b) + \partial_k f(\delta_i^a\delta_j^b - \delta_i^b\delta_j^a).
\end{aligned}
$$

So the formula holds.

5. Let M be a simply connected compact m-manifold.

a) Prove that any closed 1-form is exact.

b) Prove that any map $M \to T^m$ has degree zero.

Hint: For b), choose a product of 1-forms as volume form on T^m.

Solution:

a) Let $\alpha \in \Omega^1(M)$ be a closed 1-form. Fix a point $p_0 \in M$ and define $f : M \to \mathbb{R}$ by

$$f(p) := \int_\gamma \alpha := \int_{[0,1]} \gamma^*\alpha = \int_0^1 \alpha_{\gamma(t)}(\dot\gamma(t))dt$$

where $\gamma : [0,1] \to M$ is any path satisfying $\gamma(0) = p_0$ and $\gamma(1) = p$. We claim that f is well-defined, i.e. it does not depend on the chosen path, and that f is smooth with $df = \alpha$.

Suppose that $\gamma_0, \gamma_1 : [0,1] \to M$ are two paths with $\gamma_i(0) = p_0$ and $\gamma_i(1) = p$. Since M is simply connected there exists a smooth homotopy

$$\gamma : [0,1] \times [0,1] \to M, \qquad (s,t) \mapsto \gamma_s(t)$$

with $\gamma_s(0) = p_0$ and $\gamma_s(1) = p$. It then follows from Cartan's formula that

$$\partial_s \gamma_s^*\alpha = dh_s\alpha + h_s d\alpha$$

Since $d\alpha = 0$, it follows by Stoke's theorem

$$\partial_s \int_{[0,1]} \gamma_s^*\alpha = \int_{[0,1]} d(h_s\alpha) = (h_s\alpha)(1) - (h_s\alpha)(0)$$

where

$$(h_s\alpha)(t) = \alpha_{\gamma_s(t)}(\partial_s\gamma_s(t)).$$

In particular, $(h_s\alpha)(0) = 0$ and $(h_s\alpha)(1) = 0$, and therefore

$$f(p) = \int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha$$

shows that $f(p)$ is independent of the chosen path.

Next let $\phi : U \to \mathbb{R}^m$ be a chart with $p_0 \in U \subset M$ and $\phi(p_0) = 0$. Suppose that in this chart

$$(\phi^{-1})^*(\alpha|_U) = \alpha_1 dx^1 + \cdots + \alpha_m dx^m$$

We use for $p = \phi^{-1}(x) \in U$ the path $\gamma(t) = \phi^{-1}(tx)$ to calculate $f(p)$. This yields the formula

$$f(\phi(x)) = \int_0^1 \alpha_{\phi^{-1}(tx)} d\phi^{-1}(tx)x dt = \sum_{i=1}^m \int_0^1 \alpha_i(tx)x_i dt$$

and it follows that f is a smooth function on U. Moreover, differentiating $(f \circ \phi^{-1})$ at the origin yields $\partial_j d(f \circ \phi^{-1})(0) = \alpha_j(0)$ which implies $df(p_0) = \alpha(p_0)$.

Next let $q \in M$ be any point and define

$$f_q(p) := \int_{\gamma_{q,p}} \alpha := \int_{[0,1]} \gamma_{q,p}^*\alpha := \int_0^1 \alpha_{\gamma_{q,p}(t)}(\dot\gamma_{q,p}(t))\, dt$$

where $\gamma_{q,p} : [0,1] \to M$ is any smooth path with $\gamma_{q,p}(0) = q$ and $\gamma_{q,p}(1) = p$. By concatenating paths and by invariance of these path integrals under endpoint fixing homotopies, we see that

$$f_q(p) - f(p) = \int_{\gamma_{q,p_0}} \alpha = \int_{[0,1]} \gamma_{q,p_0}^*\alpha.$$

It follows that f_q and f differ only by a constant. By the argument above, we see that f_q is smooth in a neighbourhood of q and satisfies $df_q(q) = \alpha(q)$. Then then also f is smooth in a neighborhood of q and satisfies $df(q) = \alpha(q)$.

b) Let $T = S^1 \times \cdots \times S^1$ be the m-torus and denote by $\pi_i : T \to S^1$ the projection onto the i-th factor. Denote by $\theta \in \Omega^1(S^1)$ the standard volume form given by $\theta_z(\hat{z}) = -\mathbf{i}\hat{z}/z$. Then define

$$\theta_i := \pi_i^*\theta \in \Omega^1(T), \qquad \omega := \theta_1 \wedge \cdots \wedge \theta_m \in \Omega^m(T).$$

Now let $\varphi : M \to T$ be a smooth map. As the $\theta_i \in \Omega^1(T)$ are closed, $\varphi^*\theta_i \in \Omega^1(M)$ are also closed and hence exact by part (a). In particular there exists a function $f_1 : M \to \mathbb{R}$ with $df_1 = \varphi^*\theta_1$ and hence

$$\varphi^*\omega = d\left(f_1\varphi^*\theta_2 \wedge \cdots \wedge \varphi^*\theta_n\right)$$

It follows from Stokes theorem and the degree Theorem that

$$0 = \int_M \varphi^*\omega = \deg(\varphi)\int_T \omega.$$

Since ω is a volume form on T, this shows $\deg(\varphi) = 0$.

6. Let M be a smooth compact connected oriented m-manifold without boundary and denote by

$$\mathcal{V}(M) := \left\{\omega \in \Omega^m(M) \,:\, \omega \text{ is a volume form with } \int_M \omega = 1.\right\}$$

the space of volume forms with volume 1. Fix throughout the exercise $\omega_0 \in \mathcal{V}(M)$ and define

$$\mathrm{Diff}(M,\omega_0) := \{\varphi \in \mathrm{Diff}(M) \,:\, \varphi^*\omega_0 = \omega_0\}$$

The goal of this exercise is to show that the inclusion of $\mathrm{Diff}(M,\omega_0)$ into the space of orientation-preserving diffeomorphisms $\mathrm{Diff}^+(M)$ is a homotopy equivalence.

a) Prove that for every $\omega \in \mathcal{V}(M)$ the linear map

$$I_\omega : \mathrm{Vect}(M) \to \Omega^{m-1}(M), \text{ given by } I_{\omega_0}(X) := \iota(X)\omega$$

is an isomorphism.

b) * Show that there exists a continuous map

$$\mathcal{V}(M) \to \mathrm{Diff}^+(M), \qquad \omega \mapsto \psi_\omega$$

such that $\psi_{\omega_0} = \mathrm{id}_M$ and $\psi_\omega^*\omega = \omega_0$ for every $\omega \in \mathcal{V}(M)$.

c) * Show that the map

$$\mathrm{Diff}^+(M) \to \mathcal{V}(M) \times \mathrm{Diff}(M,\omega_0), \qquad \psi \mapsto (\psi^*\omega_0, \psi \circ \psi_{\psi^*\omega_0})$$

is a well-defined homeomorphism with inverse $(\omega,\phi) \mapsto \phi \circ \psi_\omega^{-1}$.

d) Conclude that the inclusion of $\mathrm{Diff}(M,\omega_0)$ into $\mathrm{Diff}^+(M)$ is a homotopy equivalence.

Solution:

a) For every point $p \in M$, the map

$$T_pM \to \Lambda^{m-1}(T_pM^*), \qquad v \mapsto \iota(v)\omega_0$$

is linear and injective. Since both spaces have dimension m, this map is also bijective. From this follows directly that the map $\mathrm{Vect}(M) \to \Omega^{m-1}(M)$ is injective. Conversely, given $\eta \in \Omega^{m-1}(M)$ there exists for every $p \in M$ a unique vector $X(p) \in T_pM$ with $\iota(X(p))\omega_0(p) = \eta(p)$. It remains to show that $X(p)$ is a smooth vector field. This can be done most easily in local coordinates: In these we have

$$\eta = \eta_1 dx^2 \wedge \cdots \wedge dx^m + \cdots + \eta_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m + \cdots + \eta_m dx^1 \wedge \cdots \wedge dx^{m-1}$$

for some real valued functions η_i and

$$\omega_0 = \lambda dx^1 \wedge \ldots \wedge dx^m$$

for a smooth positive real valued functions λ. Then X is given by

$$X = \frac{1}{\lambda}\Big(\eta_1, -\eta_2, \ldots, (-1)^{m-1}\eta_m\Big)$$

and this is indeed smooth.

b) It follows from Moser isotopy that for every $\omega \in \mathcal{V}(M)$ there exists a diffeomorphism $\phi_\omega \in \mathrm{Diff}(M)$ with $\phi^*\omega = \omega$ and that ϕ is isotopic to the identity on M. The proof of this exercise, consists of carefully checking the various steps in the proof in order to convince ourselves that the construction can be done in a continuous fashion when ω varies.

First, we investigate the proof of the theorem which shows that every m-form on M with vanishing integral is exact. After choosing several auxiliary data (which does not depend on the corresponding m-form with vanishing integral), the proof actually yields a linear map from the forms with vanishing integral into the space of $(m-1)$-forms. In particular, there exists an affine map

$$\mathcal{V}(M) \to \Omega^{m-1}(M), \qquad \omega \mapsto \tau_\omega$$

such that $d\tau_\omega = \omega - \omega_0$.

Next use part (a) to define $X^t_\omega \in \mathrm{Vect}(M)$ by

$$I_{(1-t)\omega_0 + t\omega}(X^t_\omega) = -\tau_\omega.$$

We see from the formula in part (a) for the vector field X in local coordinates, that it also depends smoothly on the volume form. Hence X^t_ω is a smooth family of vector fields on M. We talk in the exercise only about continuity, since it somewhat involved to define smooth functions on $\mathcal{V}(M)$, which is an open subset of an infinite dimensional Frechet space.

Finally define $\psi^t_\omega \in \mathrm{Diff}(M)$ for $0 \le t \le 1$ by

$$\psi^0_\omega = \mathrm{id}_M, \qquad \partial_t \psi^t_\omega = X^t_\omega(\psi^t_\omega).$$

By smooth dependency of solutions to ODEs on initial conditions, it follows that this is a smooth family of diffeomorphism is both parameters t and $\omega \in \mathcal{V}(M)$. It then follows from the proof of Moser isotopy that

$$(\psi^t_\omega)^*[(1-t)\omega_0 + t\omega] = \omega_0.$$

Indeed, this is satisfied for $t = 0$ and by Cartan's formula we have

$$\begin{aligned}\partial_t(\psi_\omega^t)^*[(1-t)\omega_0 + t\omega] &= (\psi_\omega^t)^* \Big(d\iota(X_\omega^t)[(1-t)\omega_0 + t\omega] + \omega - \omega_0\Big) \\ &= (\psi_\omega^t)^* \left(d\tau_\omega + \omega - \omega_0\right) \\ &= 0\end{aligned}$$

Since ψ_ω^1 is isotopic to the identiy, the map is orientation preserving. It follows that $\psi_\omega := \psi_\omega^1$ does the job and this proves part (b).

c) The map is well-defined since

$$(\psi \circ \psi_{\psi^*\omega_0})^* \omega_0 = \psi^*_{\psi^*\omega_0}(\psi^*\omega_0) = \omega_0$$

and due to ψ being orientation-preserving

$$\int_M \psi^*\omega_0 = \int_M \omega_0 = 1.$$

We show that $(\omega, \phi) \mapsto \phi \circ \psi_\omega^{-1}$ defines an inverse map. This follows from a direct verification. Since

$$(\phi \circ \psi_\omega^{-1})^*\omega_0 = (\psi_\omega^{-1})^*\phi^*\omega_0 = (\psi_\omega^{-1})^*\omega_0 = \omega$$

for all $(\omega, \phi) \in \mathcal{V}(M) \times \mathrm{Diff}(M, \omega_0)$, it follows

$$\Big((\phi \circ \psi_\omega^{-1})^*\omega_0, (\phi \circ \psi_\omega^{-1}) \circ \psi_{(\phi\circ\psi_\omega^{-1})^*\omega_0}\Big) = (\omega, \phi \circ \psi_\omega^{-1} \circ \psi_\omega) = (\omega, \phi).$$

for all $(\omega, \phi) \in \mathcal{V}(M) \times \mathrm{Diff}(M, \omega_0)$. Composing both map the other way around yields

$$\psi \circ \psi_{\psi^*\omega_0} \circ \psi^{-1}_{\psi^*\omega_0} = \psi$$

for all $\psi \in \mathrm{Diff}(M)$ and therefore both maps are inverse to each other.

Since both maps are clearly continuous, they are therefore homeomorphisms.

d) It follows from part (c) that $\mathrm{Diff}^+(M)$ is homeomorphic to $\mathcal{V}(M) \times \mathrm{Diff}(M, \omega_0)$. The natural inclusion of $\mathrm{Diff}(M, \omega_0)$ into $\mathrm{Diff}^+(M)$ corresponds under this homeomorphism to the inclusion

$$f : \mathrm{Diff}(M, \omega_0) \hookrightarrow \mathcal{V}(M) \times \mathrm{Diff}(M, \omega_0), \qquad f(\phi) = (\omega_0, \phi).$$

This map is a homotopy equivalence with homotopy inverse

$$g : \mathcal{V}(M) \times \mathrm{Diff}(M, \omega_0) \hookrightarrow \mathrm{Diff}(M, \omega_0), \qquad g(\omega, \phi) = \phi.$$

Indeed, $g \circ f$ is the identity map on $\mathrm{Diff}(M, \omega_0)$ and $f \circ g$ is homotopic to the identity map on $\mathcal{V}(M) \times \mathrm{Diff}(M, \omega_0)$ through the homotopy

$$h_t(\omega, \phi) = (t\omega + (1-t)\omega_0,\, \phi)$$

which satisfies $h_0 = f \circ g$ and $h_1(\omega, \phi) = (\omega, \phi)$.