Solution 10

1. Prove that a geodesically convex subset $U \subset (M, g)$ is contractible.

Solution: Pick a point $p \in U$. Then for any other point q in U, there is a unique geodesic γ from p to q in U. This implies that $\Omega := \exp_p^{-1}(U)$ is star-shaped with respect to the origin and $\exp_p : \Omega \to U$ is bijective. Therefore, we can define $f_t : U \to U$ for $t \in [0, 1]$ by $f_t(q) := \exp_p(t \exp^{-1}(q))$ for $q \in U$. Then $f_1 = \mathrm{id}_U$ and f_0 is a constant map. Thus U is contractible.

2. Let $\alpha \in \Omega^1(M)$. Equivalent are

a) α is exact.

b) For any two smooth curves $\gamma_0, \gamma_1 : [0, 1] \to M$ which agree at the end, we have

$$\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha.$$

c) For any smooth loop $\gamma : S^1 \to M$, we have

$$\int_{\gamma} \alpha = 0.$$

Solution: We may assume M is connected without loss of generality. By reparametrisation at the ends, we can make the two curves in $b)$ into a smooth loop as in $c)$, and so we immediately see that $c)$ implies $b)$.

For $a)$ implies $c)$, we simply calculate for $\alpha = df$ that for a loop $\gamma : S^1 \to M$

$$\int_{\gamma} \alpha = \int_0^{2\pi} df_{\gamma(\theta)}(\dot{\gamma}(\theta))\, d\theta = f(\gamma(2\pi)) - f(\gamma(0)) = 0.$$

We are left with proving $b)$ implies $a)$. For this, we define for $q \in M$ a function $f_q : M \to \mathbb{R}$ defined by

$$f_q(p) := \int_{\gamma_{q,p}} \alpha$$

where $\gamma_{q,p}$ is a path from q to p. By assumptions in $b)$, $f_q(p)$ does not depend on the choice of path $\gamma_{p,q}$ and so is well-defined.

Now pick a chart $\varphi : U \subset M \to \mathbb{R}^m$ with $\varphi(q) = 0$. Then, let $(\varphi^{-1})^*\alpha =: \sum_{i=1}^m \beta_i dx^i$ and let $\gamma_{q,p} : [0, 1] \to M : t \mapsto \varphi^{-1}(tx)$ be our choice of path connecting q to $p := \varphi^{-1}(x)$. Then, we have

$$f_q(p) = \int_{\gamma_{q,p}} \alpha = \int_{\varphi \circ \gamma_{p,q}} (\varphi^{-1})^*\alpha = \int_0^1 (\sum_{i=1}^m \beta_i(tx)x^i)\, dt = \sum_{i=1}^m (\int_0^1 \beta_i(tx)\, dt)x^i.$$

This means that f_q is smooth on U and we also have for $d\varphi(q)\hat{q} = \hat{x}$

$$df_q(q)\hat{q} = d(f \circ \varphi^{-1})(0)\hat{x} = \sum_{i=1}^m \beta_i(0)(\hat{x})^i = d((\varphi^{-1})^*\alpha)(0)\hat{x} = d\alpha(q)\hat{q}.$$

So $df_q(q) = \alpha(q)$.

Next fix $q_0 \in M$. For $q \in M$, we can take a fixed path $\gamma_{q_0,q}$ which is constant at the ends. Then for any third point $p \in M$, we take a path $\gamma_{q,p}$ which is constant at the ends. Then the concatinated path $\gamma_{q_0,q}\#\gamma_{q,p}$ is a smooth path from p_0 to p. Thus, we get for any $p \in M$ the relationship

$$f_{q_0}(p) = \int_{\gamma_{q_0,q}\#\gamma_{q,p}} \alpha = \int_{\gamma_{q_0,q}} \alpha + \int_{\gamma_{q,p}} \alpha = f_{q_0}(q) + f_q(p).$$

Therefore, from f_q smooth in a neighborhood of q also follows smoothness of f_{q_0} in a neighborhood of q. From the relation $df_q(q) = \alpha(q)$, we also have $df_{q_0}(q) = \alpha(q)$. Thus α is exact and $\alpha = df_{q_0}$.

3. Let $\alpha \in \Omega^1(M)$. Equivalent are

a) α is closed.

b) For any two smooth curves $\gamma_0, \gamma_1 : [0,1] \to M$ homotopic with fixed end points, we have

$$\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha.$$

c) For any contractible, smooth loop $\gamma : S^1 \to M$, we have

$$\int_\gamma \alpha = 0.$$

Solution: Without loss of generality, we may assume that M is connected.

We prove that $a)$ implies $c)$. Let $\gamma : S^1 \to M$ be a smooth contractible loop. Then there is a map $f : D^1 \to M$ such that $f_{\partial D^1} = \gamma$. An application of Stokes' theorem then gives

$$\int_\gamma \alpha = \int_{S^1} \gamma^*\alpha = \int_{D^1} d(f^*\alpha) = 0.$$

We prove that $c)$ implies $b)$. Given two homotopic two smooth curves $\gamma_0, \gamma_1 : [0,1] \to M$ homotopic with fixed end points, we can reparametrize such that they are constant near the ends. This reparametrisation does not change the integrals of α over γ_i. Also take a homotopy h_t which is constant near the ends. Now we can define a associated loops, by defining $\psi_t : S^1 \to M$ for $t \in [0,1]$ by

$$\psi_t(e^{i\theta}) = \begin{cases} h_{\frac{t}{2}}(\frac{\theta}{\pi}), & \text{, for } \theta \in [0,\pi], \\ h_{1-\frac{t}{2}}(\frac{2\pi-\theta}{\pi}), & \text{, for } \theta \in [\pi, 2\pi]. \end{cases}$$

Then ψ_0 is the loop obtained by concatenating γ_0 to the γ_1 in reverse direction. ψ_1 is a contractible loop, since it is $h_{1/2}$ concatenated with itself in reverse direction. Thus ψ_0 is also contractible. Thus

$$\int_{\gamma_0} \alpha - \int_{\gamma_1} \alpha = \int_\gamma \alpha \overset{c)}{=} 0.$$

We are left with proving that $b)$ implies $a)$. This is a local statement, so we may pass to a chart $M = \mathbb{R}^m$. Put $\omega = d\alpha$. We want to prove that $\omega = 0$. Assume to the contrary, that there is $p \in M$ such that $\omega(p) \neq 0$. Then there is $v_1, v_2 \in \mathbb{R}^m$ such that $\omega(p)(v_1, v_2) > 0$. This remains true for $q \in B_{2r}(p)$ where r is sufficiently small. Define the curves $\gamma_0(t) = p + r(\cos(t)v_1 + \sin(t)v_2)$ and $\gamma_1(t) = p + r(\cos(t)v_1 - \sin(t)v_2)$. Then by Stokes' Theorem, we have for $U_r := \{w \in \mathbb{R}^m \,:\, w \in \mathrm{span}(v_1, v_2) \cap B_r(p)\}$, that

$$0 < \int_{U_r} \omega = \int_{\gamma_0} \alpha - \int_{\gamma_1} \alpha = 0.$$

This is a contradiction. So α has to be closed.

4. Show that $\alpha = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\})$ is closed 1-form, but it is not exact. What does this tells us about the de-Rham complex of $\mathbb{R}^2 \setminus \{(0,0)\}$?

Solution: For closeness, we calculate

$$d\alpha = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} dy \wedge dx + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} dx \wedge dy = 0.$$

For exactness, we use the criterion from Exercise 2. Namely, take the loop $\gamma : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\} : e^{i\theta} \mapsto (\cos(\theta), \sin(\theta))$, and calculate

$$\int_\gamma \alpha = \int_0^{2\pi} \alpha_\gamma(\theta)(\dot\gamma(\theta))\, d\theta = \int_0^{2\pi} \sin^2(\theta) + \cos^2(\theta)\, d\theta = 2\pi \neq 0.$$

So α is closed, but not exact. This proves that $H^1_{dR}(\mathbb{R}^2 \setminus \{(0,0)\}) \neq 0$. Furthermore, $\mathbb{R}^2 \setminus \{(0,0)\}$ is homotopy equivalent to S^1, so we have $H^1_{dR}(\mathbb{R}^2 \setminus \{(0,0)\}) = \mathbb{R}$ and α is a generator of H^1.

5. Let M be a compact manifold with non-empty boundary.

a) Prove that there is $X \in \mathrm{Vect}(M)$ such that X points in on the boundary.

b) Prove that X as in a) has a smooth semi-flow

$$\varphi : [0,\infty) \times M \to M : (t,p) \mapsto \varphi_t(p).$$

c) Every map $f : M \to M$ is homotopic to a map without fixed points on the boundary.

d) Let $f_0, f_1 : M \to M$ be homotopic maps without fixed points on the boundary. Prove there is a homotopy $(f_t)_{t\in[0,1]}$ such that $f_t(q) \neq q$ for all $t \in [0,1]$ and all $q \in \partial M$.

e) Define the Lefschetz number $L(f)$ for $f : M \to M$ without fixed points on the boundary.

f) Prove that there is an open neighborhood $U \subset M$ of the boundary ∂M such that $[0,1) \times \partial M \cong U$.

Solution:

a) Let $p \in \partial M$ and choose a chart $\phi : U_p \to \Omega_p \subset \mathbb{H}^n$ which maps p to the origin. Let $\rho : \mathbb{H}^n \to [0,1]$ be a smooth function with $\rho(0) = 1$ and $\mathrm{supp}(\rho) \subset \Omega_p$. Then define $\tilde X \in \mathrm{Vect}(\Omega_p)$ by

$$\tilde X(x_1, \dots, x_m) = \rho(x_1, \dots, x_m) e_m$$

where $e_m = (0, \dots, 0, 1)$ is the m-th basis vector. Pulling $\tilde X$ back to U_p and extending by zero yields the vector field $X \in \mathrm{Vect}(M)$ defined by

$$X^{(p)}(q) = \begin{cases} \phi^* \tilde X(q) & \text{for } q \in U_p \\ 0 & \text{for } q \in M \setminus U_p \end{cases}$$

By continuity, this vector field points inwards on an open neighborhood $p \in V_p \subset \partial M$ and it vanish or points inwards on every point of ∂M.

Since ∂M is compact, there exists a finite collection of points $p_1, \ldots, p_N \in \partial M$ and vector field $X^{(p_i)} \in \mathrm{Vect}(M)$ as above, such that the open sets V_{p_i} cover ∂M. Then

$$X := X^{(p_1)} + X^{(p_2)} + \cdots + X^{(p_N)} \in \mathrm{Vect}(M)$$

is a vector field which points inwards at every boundary point.

b) In local coordinates, the vector field is a smooth function

$$X_{\mathrm{loc}} : \mathbb{H}^m \to \mathbb{R}^m.$$

Now recall that by definition of smoothness, there exists a vector field $\tilde{X} : W \to \mathbb{R}^m$ defined on a slightly larger open set $\mathbb{H}^m \subset W$ which restricts to X_{loc} on $\mathbb{H}^m$. The flow lines starting at any point in $\mathbb{H}^m$ under $\tilde{X}$ remain in $\mathbb{H}^m$ for $t > 0$ and therefore agree with the flow under X_{loc}.

We have thus show that for every $p \in M$ there exists an $\epsilon > 0$ such that the equation

$$\gamma(0) = p, \qquad \dot{\gamma}(t) = X(\gamma(t))$$

has a unique solution $\gamma : [0, \epsilon) \to M$. By smooth dependence of solutions to ODEs on the initial values, we see that the flow is smooth for small times. Smoothness of the whole flow follows as in the case of manifolds without boundary (see lecture notes DG1, Theorem 2.4.9)

The same argument which shows that every vector field on a compact manifold without boundary is complete applies to our present situation and shows that all integral curves exist for all times $t > 0$. We gave a careful argument of this in Exercise 2 on Exercise Sheet 3 in the last semester. The main idea is to use smoothness of the flow to prove that the there exists a uniform lower bound $\epsilon > 0$ such that all solutions starting at any point $p \in M$ exists for $t \in [0, \epsilon)$. Then concatenating the integral curves, it is not hard to see that the maximal existence interval is $[0, \infty)$.

c) Let $X \in \mathrm{Vect}(M)$ be a vector field which points inward at the boundary and denote its semi-flow by φ_t. For $t > 0$ this is not a diffeomorphism of M onto itself, but rather a smooth map $\varphi_t : M \to M \backslash \partial M$ from the manifold into its interior. Consider

$$f_t := f \circ \varphi_t : M \to M.$$

Then $f = f_0$ and $f_t(M) \subset M \backslash \partial M$ for all $t > 0$. This uses the fact that a diffeomorphism f maps boundary points of M to boundary points and interior points of M to interior points. In particular f_t has no boundary fixed points.

d) Let $X \in \mathrm{Vect}(M)$ be a vector field which points inward at the boundary and denote its semi-flow by φ_t. Let $h_t : M \to M$ be any smooth homotopy with $h_0 = f_0$ and $h_1 = f_1$. By compactness of ∂M there exists $\epsilon > 0$ such that h_t has no boundary fixed point for $t \in [0, \epsilon]$ and $t \in [1 - \epsilon, 1]$. Let $\eta : [0, 1] \to [0, 1]$ be a smooth function with $\eta(0) = 0 = \eta(1)$ and $\eta(t) > 0$ for $t \in [\epsilon, 1 - \epsilon]$. Then

$$f_t := h_t \circ \varphi_{\eta(t)} : M \to M$$

is a smooth homotopy between f_0 and f_1. The same argument as in (c) shows that f_t has no boundary fixed points for all $t \in [0, 1]$.

e) In order to define the Lefschetz number $L(f)$, we associate a fixed point index $\iota(p, f)$ to every fix point $p \in \mathrm{Fix}(f)$ and define

$$L(f) := \sum_{p \in \mathrm{Fix}(f)} \iota(p, f) \in \mathbb{Z}.$$

Since the fixed point index of f at p is locally defined, the same definition works for manifolds with boundary provided that f has no fixed points on the boundary. It follows from part (d) that the Lefschetz number $L(f)$ is invariant under homotopies.

f) Let $X \in \mathrm{Vect}(M)$ be a vector field which points inward at the boundary and denote its semi-flow by φ_t. Consider the map

$$F : [0, \infty) \times \partial M \to M, \qquad F(t, p) := \varphi_t(p).$$

As restriction of the semi-flow $[0, \infty) \times M \to M$, this is a smooth map. Moreover for every $p \in \partial M$ the map

$$dF(0, p) : \mathbb{R} \oplus T_p(\partial M) \to T_pM, \qquad dF(0, p)(\hat{t}, \hat{p}) := \hat{t}X(p) + \hat{p}$$

is an isomorphism. Since ∂M is compact and the set of pairs (t, p) for which $dF(t, p)$ is bijective is open in $[0, \infty) \times \partial M$, there exists $\epsilon > 0$ such that $dF(t, p)$ is bijective for all $t \in [0, \epsilon]$ and $p \in \partial M$. (Compare Exercise 1 on Exercise Sheet 4 from last semester.) Now consider

$$F_\epsilon : [0, 1) \times \partial M \to M, \qquad F_\epsilon(t, p) := \varphi_{\epsilon t}(p).$$

Then $dF_\epsilon(t, p)$ is an isomorphism for all $(t, p) \in [0, 1) \times \partial M$. Hence, by the implicit function theorem, F_ϵ is a local diffeomorphism around every point in its domain. It is not hard to see that F_ϵ is injective: Following the flow of $q \in \mathrm{Image}(F_\epsilon)$ under X backwards, its maximal existence time in negative direction is $t\epsilon$ and its point of extinction is p. It follows that F_ϵ is a diffeomorphism onto its image and this proves the claim with $U := \mathrm{Image}(F_\epsilon)$.