Solution 11

1. Let M be a smooth m-manifold (with or without boundary). Let $U, V \subset M$ be open sets such that $M = U \cup V$.

a) Let ρ_U, ρ_V be a partition of unity subordinate to the U, V. Let $\omega \in \Omega^k(U \cap V)$ and define $d^* : \Omega^k(U \cap V) \to \Omega^{k+1}(M)$ by

$$d^*\omega = \begin{cases} d\rho_U \wedge \omega & \text{on } U \cap V \\ 0 & \text{on } M \setminus (U \cap V). \end{cases}$$

Find an example of M, U, V and $\omega \in \Omega^k(U \cap V)$ such that $d^*\omega$ is not exact.

b) Prove that the cohomology class $[d^*\omega] \in H^{k+1}(M)$ is independent of the choice of partition of unity.

Hint: For a) take for example $M = S^1$ and $k = 0$.

Solution:

a) We take $M = S^1$, the open cover $M = U \cap V$ with

$$U := \left\{ e^{\mathbf{i}t} \,\middle|\, \frac{\pi}{4} < t < \frac{7\pi}{4} \right\}, \qquad V := \left\{ e^{\mathbf{i}t} \,\middle|\, -\frac{3\pi}{4} < t < \frac{3\pi}{4} \right\}.$$

Then $U \cap V$ is the disjoint union of the following two sets

$$W_1 = \left\{ e^{\mathbf{i}t} \,\middle|\, \frac{\pi}{4} < t < \frac{3\pi}{4} \right\}, \qquad W_2 = \left\{ e^{\mathbf{i}t} \,\middle|\, \frac{5\pi}{4} < t < \frac{7\pi}{4} \right\}.$$

Now let $\omega \in \Omega^0(U \cap V)$ be the smooth function which is constant 1 on W_1 and vanishes on W_2. Then

$$d^*\omega = \begin{cases} d\rho_U, & \text{on } W_1, \\ 0, & \text{on } M \setminus W_1. \end{cases}$$

and therefore by Stokes theorem

$$\int_{S^1} d^*\omega = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\rho_U(e^{\mathbf{i}t})dt = \rho_U(e^{\frac{3}{4}\pi\mathbf{i}}) - \rho_U(e^{\frac{1}{4}\pi\mathbf{i}}) = 1 - 0 = 1.$$

Again by Stokes theorem, it follows that $d^*\omega \in \Omega^1(S^1)$ cannot be exact, as its integral does not vanish.

b) Let $\{\rho_U, \rho_V\}$ and $\{\rho'_U, \rho'_V\}$ be two partitions of unity subordinate to the cover $M = U \cup V$ and define $\alpha \in \Omega^k(M)$ by

$$\alpha = \left\{ \begin{array}{ll} (\rho_U - \rho'_U) \wedge \omega & \text{on } U \cap V \\ 0 & \text{on } M \setminus (U \cap V) \end{array} \right\}.$$

This is well-defined and smooth, since $\text{supp}(\rho_U - \rho'_U) \subset U \cap V$, and it holds:

$$d\alpha = d\rho_U \wedge \omega - d\rho'_U \wedge \omega$$

Therefore $[d\rho_U \wedge \omega] = [d\rho'_U \wedge \omega] \in H^{k+1}(M)$.

2. **a)** Calculate the dimensions of the de Rham cohomology groups of $M \times S^1$.

b) Calculate the dimensions of the de Rham cohomology groups of the torus T^m.

Hint: For a): Use the Künneth formula or Mayer–Vietoris: For the later approach cover S^1 by two open intervals to obtain an open cover $M \times S^1 = U \cup V$. Have a careful look at the maps $H^k(U) \oplus H^k(V) \to H^k(U \cap V)$. What can you say about their kernel and image?

Solution:

a) This exercise follows directly from the Künneth formula. We give an alternative argument in the following which uses the Mayer–Vietoris long exact sequence.

Consider the cover $M \times S^1 = U \cup V$ with

$$U := \left\{ \left(x, e^{\mathbf{i}t}\right) \in M \times S^1 \,\middle|\, x \in M,\, \frac{\pi}{4} < t < \frac{7\pi}{4} \right\},$$
$$V := \left\{ \left(x, e^{\mathbf{i}t}\right) \,\middle|\, x \in M, -\frac{3\pi}{4} < t < \frac{3\pi}{4} \right\}.$$

Then $U \cap V$ is the disjoint union of

$$W_1 := \left\{ \left(x, e^{\mathbf{i}t}\right) \in M \times S^1 \,\middle|\, x \in M,\, \frac{\pi}{4} < t < \frac{3\pi}{4} \right\},$$
$$W_2 := \left\{ \left(x, e^{\mathbf{i}t}\right) \in M \times S^1 \,\middle|\, x \in M,\, \frac{5\pi}{4} < t < \frac{7\pi}{4} \right\}.$$

Since W_1, W_2, U , V, are all diffeomorphic to $M \times I$ for some open intervall $I \subset \mathbb{R}$, they are all homotopy equivalent to M and it follows

$$H^k(U \cap V) \cong H^k(M) \oplus H^k(M), \qquad H^k(U) \cong H^k(V) \cong H^k(M).$$

The Mayer–Vietoris long exact sequence is given by

$$\cdots \to H^{k-1}(U \cap V) \to H^k(M \times S^1) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to \cdots$$

and we claim that in this sequence the maps

$$f_k : H^k(U) \oplus H^k(V) \to H^k(U \cap V)$$

have $\dim(H^k(M))$ dimensional kernels and images. From this claim and the exactness of the long exact sequence, it then follows that

$$\dim(H^k(M \times S^1)) = \dim(H^k(M)) + \dim(H^{k-1}(M)).$$

We prove the claim next. Denote by $\pi_U : U \to M$ the projection onto the first coordinate. Then every cohomology class in $H^k(U)$ can be represented by a form $\pi_U^*\omega$ for some $\omega \in \Omega^k(M)$. (That is what's behind our earlier claim $H^k(U) \cong H^k(M)$.) The same holds for the other sets V, W_1, W_2 and we have

$$f_k([\pi_U^*\omega_1], [\pi_V^*\omega_2]) = [\pi_{W_1}^*(\omega_1 - \omega_2) + \pi_{W_2}^*(\omega_1 - \omega_2)].$$

for $\omega_i \in \Omega^k(M)$. From this follows

$$\ker(f_k) = \left\{ ([\pi_U^*\omega], [\pi_V^*\omega]) \in H^k(U) \oplus H^k(V) \,|\, \omega \in \Omega^k(M) \right\}$$

and

$$\operatorname{image}(f_k) = \left\{ ([\pi_{W_1}^*\omega + \pi_{W_2}^*\omega]) \in H^k(U \cap V) \,|\, \omega \in \Omega^k(M) \right\}.$$

From this description it is obvious that both have dimension $\dim(H^k(M))$.

b) We show by induction on n that

$$\dim\left(H^k(T^n)\right) \cong \binom{n}{k}$$

Since S^1 is a compact connection oriented 1-manifold without boundary, it follows

$$H^0(S^1) \cong \mathbb{R}, \qquad H^1(S^1) \cong \mathbb{R}.$$

This proves the claim for $n = 1$. Now assume $n > 1$ and that the claim holds for $n - 1$, then follows by part (a):

$$\begin{aligned}\dim\left(H^k(T^n)\right) &= \dim\left(H^k(T^{n-1} \times S^1)\right) \\ &= \dim\left(H^k(T^{n-1})\right) + \dim\left(H^{k-1}(T^{n-1})\right) \\ &= \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}\end{aligned}$$

3. Calculate the dimensions of the de Rham cohomology groups for the following manifolds.

a) $\mathbb{R}P^2$ by using the double covering $S^2 \to \mathbb{R}P^2$.

b) $\mathbb{R}P^n$ for $n \geq 2$.

c) $\mathbb{C}P^n$ for $n \geq 1$ by induction.

Solution:

a) Since $\mathbb{R}P^2$ is connected, we have $H^0(\mathbb{R}P^2) \cong \mathbb{R}$. Moreover, since $\mathbb{R}P^2$ is non-orientable, it follows from Exercise 1 on Sheet 9 that every differential form $\omega \in \Omega^2(M)$ is exact, and hence $H^2(\mathbb{R}P^2) = 0$.

We show next that $H^1(\mathbb{R}P^2) = 0$. Let $\pi : S^2 \to \mathbb{R}P^2$ be the double covering, which is obtained by identifying antipodal points on S^2. Since S^2 is simply connected, it follows from Exercise 5 on Sheet 9 that $H^1(S^2) = 0$. Now let $\alpha \in \Omega^1(\mathbb{R}P^2)$ be closed. Then $\pi^*\alpha \in \Omega^1(S^2)$ is closed and hence exact. In particular, there exists a function $f : S^2 \to \mathbb{R}$ such that $\pi^*\alpha = df$. Then define

$$\bar{f} : \mathbb{R}P^2 \to \mathbb{R}, \qquad \bar{f}([x]) := \frac{f(x) + f(-x)}{2}.$$

This function satisfies $\pi^* d\bar{f} = d\pi^*\bar{f} = df = \pi^*\alpha$. Since π is a local diffeomorphism, it follows $d\bar{f} = \alpha$. We have thus shown that every closed 1-form on $\mathbb{R}P^2$ is exact and therefore $H^1(\mathbb{R}P^2) = 0$.

b) We show that

$$H^k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & 1 \leq k \leq n-1 \\ \mathbb{R} & k = n \text{ and } n \text{ odd} \\ 0 & k = n \text{ and } n \text{ even} \end{cases}$$

One way to prove this is as in a) by considering the double cover $\pi : S^n \to \mathbb{R}P^n$. The cohomology of $H^n(\mathbb{R}P^n)$ reflects the fact that $\mathbb{R}P^n$ is orientable when n is odd and non-orientable when n is even.

We present an alternative proof by induction which uses the Mayer–Vietoris long exact sequence. Define

$$U := \mathbb{R}P^n \backslash \{[0 : \cdots : 0 : 1]\}, \qquad V := \{[x_0 : \cdots : x_n] \in \mathbb{R}P^n \,|\, x_n \neq 0.\}$$

Then $M = U \cup V$, and U homotopy equivalent to $\mathbb{R}P^{n-1}$, V contractible, and $U \cap V$ is homotopy equivalent to S^{n-1}.

Since $\mathbb{R}P^n$ is connected we clearly have $H^0(\mathbb{R}P^n) \cong \mathbb{R}$. Next, we look at the start of the Mayer–Vietoris sequence

$$0 \to H^0(\mathbb{R}P^n) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(\mathbb{R}P^n) \to H^1(U) \oplus H^1(V) \to \cdots$$

Since $H^0(\mathbb{R}P^n) \cong H^0(U) \cong H^0(V) \cong H^0(U \cap V) \cong \mathbb{R}$, it follows by counting dimensions that the map $H^0(U \cap V) \to H^1(\mathbb{R}P^n)$ vanishes. Since $H^1(U) = H^1(V) = 0$, it then follows $H^1(\mathbb{R}P^n) = 0$.

For $1 < k < n-1$ we have $H^{k-1}(U \cap V) \cong H^{k-1}(S^{n-1}) = 0$, $H^k(U) \cong H^k(\mathbb{R}P^{n-1}) = 0$, and $H^k(V) = 0$. Hence it follows from the exact sequence

$$0 = H^{k-1}(U \cap V) \to H^k(\mathbb{R}P^n) \to H^k(U) \oplus H^k(V) = 0$$

that $H^k(\mathbb{R}P^n) = 0$ for $1 < k < n - 1$.

Next look at the right end of the Mayer–Vietoris sequence

$$\begin{aligned} 0 = H^{n-2}(U \cap V) \to H^{n-1}(\mathbb{R}P^n) \to H^{n-1}(U) \oplus H^{n-1}(V) \to H^{n-1}(U \cap V) \\ \to H^n(\mathbb{R}P^n) \to 0. \end{aligned}$$

Here, we have $H^{n-2}(U \cap V) \cong H^{n-2}(S^{n-1}) = 0$, $H^{n-1}(U \cap V) \cong H^{n-1}(S^{n-1}) \cong \mathbb{R}$ and, $H^{n-1}(V) = 0$.

Suppose first that n is odd. Then $H^{n-1}(\mathbb{R}P^{n-1}) = 0$ and it follows from the sequence above that $H^{n-1}(\mathbb{R}P^n) = 0$ and $H^n(\mathbb{R}P^n) \cong \mathbb{R}$.

Suppose next that n is even. Then $H^{n-1}(\mathbb{R}P^{n-1}) \cong \mathbb{R}$, but this is not quite enough to deduce the cohomology groups from the sequence above. We show that the map $H^{n-1}(U) \to H^{n-1}(U \cap V)$ is non-trivial. With this additional information it then follows $H^{n-1}(\mathbb{R}P^n) = 0$ and $H^n(\mathbb{R}P^n) = 0$. Consider the explicit homotopy equivalences

$$f : U \to \mathbb{R}P^{n-1}, \qquad f([x_0 : \cdots : x_n]) = [x_0 : \cdots : x_{n-1}]$$

$$g : S^{n-1} \to U \cap V, \qquad g(x_1, \ldots, x_n) = \left[\frac{x_1}{\sqrt{2}} : \cdots : \frac{x_n}{\sqrt{2}} : \frac{1}{\sqrt{2}}\right].$$

The composition

$$h : S^{n-1} \xrightarrow{g} U \cap V \hookrightarrow U \xrightarrow{f} \mathbb{R}P^{n-1}$$

is then the natural projection of S^{n-1} onto $\mathbb{R}P^{n-1}$. Since n is even, $\mathbb{R}P^{n-1}$ is orientable, and h is an orientation preserving double cover. Hence it has degree $\deg(h) = 2$ and induces a nontrivial map

$$h^* : H^{n-1}(\mathbb{R}P^{n-1}) \to H^{n-1}(S^{n-1})$$

in cohomology. Since f^* and g^* are isomorphism in cohomology, it follows that the inclusion map $U \cap V \to U$ induces an isomorphism $H^{n-1}(U) \to H^{n-1}(U \cap V)$. This proves the claim, since $H^{n-1}(V) = 0$.

c) We show by induction on n that

$$H^k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{R} & k \text{ even and } 0 \le k \le 2n \\ 0 & \text{otherwise} \end{cases}$$

As base case one can either take $\mathbb{C}P^0$ (which is a point) or $\mathbb{C}P^1 \cong S^2$. In both cases we have calculated the cohomology before.

Suppose $n \ge 2$ and define

$$U := \mathbb{C}P^n \backslash \{[0 : \cdots : 0 : 1]\}, \qquad V := \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_n \neq 0.\}$$

Then U is homotopy equivalent to $\mathbb{C}P^{n-1}$, V is contractible, and $U \cap V$ is homotopy equivalent to S^{2n-1}.

Since $\mathbb{C}P^n$ is connected, we have $H^0(\mathbb{C}P^n) \cong \mathbb{R}$.

Next, consider the start of the Mayer–Vietoris sequence:

$$0 \to H^0(\mathbb{C}P^n) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(\mathbb{C}P^n) \to H^1(U) \oplus H^1(V) \to \cdots$$

Since $H^0(\mathbb{C}P^n) \cong H^0(U) \cong H^0(V) \cong H^0(U \cap V) \cong \mathbb{R}$, it follows by counting dimensions that the map $H^0(U \cap V) \to H^1(\mathbb{C}P^n)$ vanishes. Since $H^1(U) \cong H^1(\mathbb{C}P^{n-1}) = 0$ and $H^1(V) = 0$, it then follows $H^1(\mathbb{C}P^n) = 0$.

For $1 < k < 2n-1$, we have $H^{k-1}(U \cap V) \cong H^{k-1}(S^{2n-1}) = 0$, $H^k(U \cap V) \cong H^k(S^{2n-1}) = 0$, and $H^k(V) = 0$. Hence it follows from

$$0 = H^{k-1}(U \cap V) \to H^k(\mathbb{C}P^n) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) = 0$$

that $H^k(\mathbb{C}P^n) \cong H^k(U) \cong H^k(\mathbb{C}P^{n-1})$ and this is isomorphic to $\mathbb{R}$ for even k and vanishes for odd k.

For $k = 2n-1$, we look at the sequence

$$0 = H^{2n-2}(U \cap V) \to H^{2n-1}(\mathbb{C}P^n) \to H^{2n-1}(U) \oplus H^{2n-1}(V) = 0$$

to conclude $H^{2n-1}(\mathbb{C}P^n) = 0$.

Finally, for $k = 2n$, we look at the last piece of the sequence

$$0 = H^{2n-1}(U) \oplus H^{2n-1}(V) \to H^{2n-1}(U \cap V) \to H^{2n}(\mathbb{C}P^n) \to H^{2n}(U) \oplus H^{2n}(V) = 0$$

to conclude $H^{2n}(\mathbb{C}P^n) \cong H^{2n-1}(U \cap V) \cong H^{2n-1}(S^{2n-1}) \cong \mathbb{R}$.

4. Prove the five-lemma stated below.

Let $A_i, B_i,\ i = 1, 2, 3, 4, 5$ be abelian groups. Let

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

be a commutative diagram in which the horizontal sequences are exact. If h_1, h_2, h_4 and h_5 are isomorphisms prove that h_3 is also an isomorphism.

Hint: Chasing diagrams is fun.

Solution: See Hatcher - Algebraic Topology on p:129. It is good to have done this exercise once in your life ;)

5. The goal is to prove the following.

Every non-empty geodesically convex open subset of a Riemannian m-manifold M without boundary is diffeomorphic to $\mathbb{R}^m$.

We argue in several steps.

a) Prove or assume that it is diffeomorphic to a bounded **star shaped** open set $U \subset \mathbb{R}^m$ centred at the origin, so that if $x \in U$, then $tx \in U$ for $0 \le t \le 1$.

b) Prove that there exists a smooth function $g : \mathbb{R}^m \to \mathbb{R}$ such that $g(x) > 0$ for every $x \in U$, $g(x) = 1$ for $|x|$ sufficiently small, and $g(x) = 0$ for $x \in \mathbb{R}^m \setminus U$.

c) Given $g : \mathbb{R}^m \to \mathbb{R}$ as in b). Define $h : U \to [0, \infty)$ by

$$h(x) := \int_0^1 \frac{dt}{g(tx)}.$$

Prove that the map $\varphi : U \to \mathbb{R}^m, \varphi(x) := h(x)x$, is a diffeomorphism.

Hint: For b), use the fact that U is second-countable and take a countable cover by balls of small radius $r_i \le 1$. Fix one cut off function ρ which is > 0 on $B_1(0)$ and 0 outside of $B_1(0)$. The constants C_i bounding all partial derivatives up to order i can come in handy. You can define g (dropping the condition of constant near 0 for now) through an infinite sum where summands involve ρ, C_i, r_i and some weighting. Use Weierstrass M-test a lot.

Note: There are contractible manifolds without boundary which are not diffeomorphic to $\mathbb{R}^m$, e.g. an exotic $\mathbb{R}^4$.

Solution:

a) Denote by V the geodesically convex open subset and take $p \in V$. As explained last week, $U = \exp_p^{-1}(V)$ is star shaped and $\exp_p : U \to V$ is a homeomorphism. That it is a diffeomorphism requires the subject of conjugate points and cut locus which was not covered in the first semester. Thus we admit the fact that if $\exp_p$ is injective on an open set, then it is also a local diffeomorphism on this open set. So $\exp_p : U \to V$ is a diffeomorphism. (Think about why it is called the injectivity radius and not the diffeomorphism radius. You can look it up in Milnor's book on 'Morse Theory'.) If $U \subset T_pM \cong \mathbb{R}^m$ is not bounded, then we can replace U by its image under a diffeomorphism between $\mathbb{R}^m$ and $B_1(0)$ which is obtained by using c) for example.

b) This is a brute force construction. Namely, for every point $p \in U$, there is an open ball of radius $r \le 1$ with centre p still contained in U. Since U is second countable, we can cover U by countably many $(B_{r_i}(p_i))_{i \in \mathbb{N}}$ of these balls. Now choose one particular cut-off function $\rho : \mathbb{R}^m \to \mathbb{R}$ with $\rho > 0$ on $B_1(0)$ and $\rho = 0$ on $\mathbb{R}^m \setminus B_1(0)$. Let $C_i > 0$ be constants which bound ρ and all its partial derivatives up to order i. Then we define $g : \mathbb{R}^m \to \mathbb{R}$ by

$$g(x) := \sum_{i=0}^{\infty} \frac{(r_i)^i}{2^i C_i} \rho\left(\frac{x - p_i}{r_i}\right).$$

Every term is bounded by $\frac{1}{2^i}$, so by Weierstrass M-test, g is a finite-valued and continuous. By construction, $g > 0$ on U and $g = 0$ on $\mathbb{R}^m \setminus U$. The only thing remaining is to prove that g is smooth. For this we note that any formal multi-indexed partial derivative of the infinite series is given by

$$\sum_{i=0}^{\infty} \frac{(r_i)^{i-|\alpha|}}{2^i C_i} (\partial^\alpha \rho) \left(\frac{x - p_i}{r_i} \right)$$

for all multi-indices $\alpha \in \mathbb{N}^m$, $|\alpha| = \sum_{i=1}^m \alpha_i$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m}$. Then by definition of C_i, we have that for all $i \geq |\alpha|$,

$$\frac{(r_i)^{i-|\alpha|}}{2^i C_i} (\partial^\alpha \rho) \left(\frac{x - p_i}{r_i} \right) \leq \frac{1}{2^i}.$$

So by M-test, all these multi-indexed partial derivatives of the infinite series converge to continuous functions. So the result on the on derivatives of function defined by infinite series, we see that g is smooth.

To get a function which is constant equal to 1 near 0, we simply take a ball $B_\epsilon(0) \subset U$ and a non-negative cutoff function $\tilde{\rho}$ which is equal one on $B_{\epsilon/2}(0)$ and has support in $B_\epsilon(0)$. Then we take $\tilde{g} = \tilde{\rho} + (1 - \tilde{\rho})g$.

c) We need to prove three things about φ. Namely, that φ is injective, surjective and that its differential is an isomorphism at any point of U. The statement will then follow by inverse function theorem.

- **Injective:** Since $g > 0$ on U, $h > 0$ on U. So 0 is the only point mapped into 0 under φ. Also any ray $\{tx : t > 0\}$ maps into itself under φ. Thus assume $x = (1 + s)y$ for $y \in U \setminus \{0\}$ and $-1 \leq s < 0$. Then

  $$h(x) = \frac{1}{1+s} \int_0^{1+s} \frac{dt}{g(ty)}.$$

  If $\varphi(x) = \varphi(y)$, then we must have $\frac{h(x)}{h(y)}(1 + s) = 1$. But

  $$\frac{h(x)}{h(y)}(1+s) = \frac{\int_0^{1+s} \frac{dt}{g(ty)}}{\int_0^1 \frac{dt}{g(ty)}} < 1.$$

  So φ is injective.

- **Surjective:** Denote by $\lambda_\infty(x) = \sup\{t > 0 : tx \in U\}$ for all $x \in U \setminus \{0\}$. Since U is bounded and $0 \in U$, $0 < \lambda_\infty(x) < \infty$. We need to prove that $h(\lambda x) = \infty$ as λ tends to $\lambda_\infty(x)$ from below.

  Fix $x \in U \setminus \{0\}$. By its properties, we have $g(\lambda_\infty(x)x) = 0$. Since g is smooth, we have

  $$\begin{aligned} |g(tx)| = |g(tx) - g(\lambda_\infty(x)x)| &= \Big| \int_0^1 \frac{d}{dt} g(\lambda_\infty(x)x + s(t - \lambda_\infty(x))x) \, ds \Big| \\ &\leq \Big| \int_0^1 dg(\lambda_\infty(x)x + s(t - \lambda_\infty(x))x) \, ds (t - \lambda_\infty(x))x \Big| \\ &\leq |x| \left(\sup_{s \in [0, \lambda_\infty(x)]} \|dg(sx)\| \right) |t - \lambda_\infty(x)| \leq M|t - \lambda_\infty(x)| \end{aligned}$$

This means that by monotone convergence theorem

$$\lim_{\lambda\to\lambda_\infty(x)} h(\lambda x) = \lim_{\lambda\to\lambda_\infty(x)} \frac{1}{\lambda}\int_0^\lambda \frac{dt}{g(tx)} = \frac{1}{\lambda_\infty(x)}\int_0^{\lambda_\infty(x)} \frac{dt}{g(tx)}$$
$$\geq \frac{1}{\lambda_\infty(x)}\int_0^{\lambda_\infty(x)} \frac{dt}{M|t-\lambda_\infty(x)|} = \infty.$$

This proves that φ is surjective, since x was arbitrary.

- **Bijective Differential:** We have $d\varphi(0)$ is the identity. So fix $x \in U \setminus \{0\}$. Since an open subset of $\mathbb{R}^m$ is an m-manifold, we only need to prove that $d\varphi(x)$ is injective. We calculate that

$$dh(x)x = \frac{d}{dt}\bigg|_{t=0} \frac{1}{1+t}\int_0^{1+t} \frac{ds}{g(sx)} = -h(x) + \frac{1}{g(x)}, \quad d\varphi(x)\hat{x} = (dh(x)\hat{x})x + h(x)\hat{x}.$$

First assume $d\varphi(x)x = 0$, then we have the equation

$$0 = (dh(x)x)x + h(x)x = 0 + \frac{x}{g(x)}.$$

which is a contradiction to $x \neq 0$ and $g(x) > 0$. So $x \notin \ker(d\varphi(x))$.

Take any $\hat{x} \perp x$ and look at the equation $d\varphi(x)\hat{x} = 0$, hence

$$0 = \langle (dh(x)\hat{x})x + h(x)\hat{x}, \hat{x}\rangle = h(x)\|\hat{x}\|^2$$

which can only hold if $\hat{x} = 0$, since $h(x) > 0$. Thus $d\varphi(x)$ is injective.

6. Prove that for M compact, $M \setminus \partial M$ has a finite good cover.

Hint: Use small geodesically convex balls and Exercise 5. Small can mean of radius less than one fourth the injectivity radius of M for $\partial M = \emptyset$.

Solution: We know from the first semester that $U_p := \exp_p(B_{\rho_p}(0))$ is geodesically convex for $\rho_p > 0$ sufficiently small and $p \in M \setminus \partial M$. (Note that we automatically have the diffeomorphism in a) of Exercise 5, so we do not use any material not covered in this lecture course :))

Let us first assume $\partial M = \emptyset$. To avoid disconnecting the intersection of two such balls, take $\rho_p < \frac{\mathrm{inj}(M)}{4}$. Then by compactness finitely many of these ball U_{p_i} cover M. Take a finite intersection $U_{i_1} \cap \ldots \cap U_{i_N}$ which is non-empty. Then this set is connected and geodesically convex. Indeed, geodesic convexity follows by uniqueness of geodesics.

Connectivity follows from the fact that the intersection of two geodesically convex sets V_1, V_2 with diameter less than half the injectivity radius is connected. Indeed, for assume there would be no path inside $V_1 \cap V_2$ between $p, q \in V_1 \cap V_2$. Since V_1, V_2 are geodesically convex, there are vectors $v_1 \neq v_2 \in T_pM$ of length $\leq \frac{\mathrm{inj}(M)}{2} \leq \frac{\mathrm{inj}(M,p)}{2}$ with $\exp_p(v_1) = q = \exp_p(v_2)$, which contradicts the definition of injectivity radius. Here $\gamma_i(t) = \exp_p(tv_i)$ is the unique geodesic in V_i connecting p and q.

So by exercise 5, it is diffeomorphic to $\mathbb{R}^m$ and thus M has a finite good cover.

Now we need an extra argument for the case $\partial M \neq \emptyset$. Let φ_t be the semi-flow of X as last week and $U_\epsilon \cong \partial M \times [0, 1)$ be a collar neighbourhood as discussed last week. Then the manifold $M \setminus U_\epsilon$ is diffeomorphic to $M \setminus \partial M$. So one has a good cover if the other has. Now take $\delta = \mathrm{dist}(\partial M, \varphi_\epsilon(\partial M))$ and choose all $\rho_p < \delta$. We also require

$$\rho_p < \frac{\mathrm{inj}(M, p)}{2}, \text{ for } p \in M \setminus U_\epsilon.$$

Then the geodesically convex ball U_p for $p \in M \setminus U_\epsilon$ together with U_ϵ is a cover of the compact space M. Thus there are finitely many U_{p_i} which cover a bit more that $M \setminus U_\epsilon$. The collection of the U_{p_i} are a good cover by the argument above. We thus get a good cover of $M \setminus U_\epsilon$ by taking $\varphi_\epsilon(U_{p_i})$.