Solution 12

1. Let M be a compact manifold M without boundary and consider the map

$$H^1(M) \to \mathrm{Hom}(\pi_1(M), \mathbb{R}) : \ [\alpha] \mapsto \left([\gamma] \mapsto \int_\gamma \alpha \right)$$

a) Show that the map is well-defined.

b) Show that the map is injective.

c) ** Show that the map is surjective.

Hint: Use Exercise 3 on Sheet 10 to prove injectivity of the map. Surjectivity is the hard part and we did not succeed in finding any rigorous solution, which works without assuming some non-trivial facts from algebraic geometry and the de Rham theorem. There is a convincing heuristic which uses the universal cover $\tilde{M} \to M$ and $H^1(\tilde{M}) = 0$ to construct a preimage.

Solution:

a) Suppose $\gamma_0, \gamma_1 : S^1 \to M$ are two smoothly homotopic loops. Then it follows from Theorem 6.3.1 in the lecture notes that there exists a collection of linear maps $h : H^k(M) \to H^{k-1}(S^1)$ such that

$$\gamma_1^* - \gamma_0^* : d \circ h + h \circ d : \Omega^k(M) \to \Omega^k(S^1).$$

In particular, when $\alpha \in \Omega^1(M)$ is closed, it follows

$$\int_{\gamma_1} \alpha - \int_{\gamma_1} \alpha = \int_{S^1} \gamma_1^*\alpha - \gamma_0^*\alpha = \int_{S^1} d(h\alpha) = 0$$

by Stokes theorem. It also follows $\int_\gamma \alpha = 0$ for every exact 1-form by Stokes theorem and hence the map above is well-defined.

b) Suppose $\alpha \in \Omega^1(M)$ is closed and $\int_\gamma \alpha = 0$ for all loops $\gamma : S^1 \to M$. Then it follows from Exercise 3 on Sheet 10 that α is exact. Hence $[\alpha] = 0 \in H^1(M)$ and this establishes injectivity.

c) Before we proceed to show surjectivity, we recall some facts for the universal cover. Fix a base-point $p_0 \in M$. The universal cover can then be defined as

$$\tilde{M} := \{\beta : [0,1] \to M \mid \beta(0) = p_0\} / \sim$$

where two paths are $\beta_0 \sim \beta_1$ are equivalent when they are homotopic with fixed end-points. The fundamental group $\pi_1(M, p_0)$ acts on $\tilde{M}$ via concatenation

$$\pi_1(M, p_0) \times \tilde{M} \to \tilde{M}, \qquad ([\gamma], [\beta]) \mapsto [\gamma * \beta]$$

where $\gamma * \beta$ is the path obtained by first following γ and then β. To obtain a smooth curve, let $\eta : [0,1] \to [0,1]$ be a smooth monotone increasing function with $\eta(t) = 0$ for $t \in [0, \epsilon]$ and $\eta(t) = 1$ for $t \in [1-\epsilon, 1]$ and define

$$\gamma * \beta : [0,1] \to M, \qquad (\gamma * \beta)(t) = \begin{cases} \gamma(\eta(2t)) & \text{for } t \in [0, \frac{1}{2}] \\ \beta(\eta(2t-1)) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

Evaluation on the endpoint of a homotopy class of paths $[\beta]$ yields a projection

$$\pi : \tilde{M} \to M, \qquad \pi([\beta]) = \beta(1).$$

The action of $\pi_1(M, p_0)$ on $\tilde{M}$ is a properly discontinuous and free. Hence π is a covering map and yields a diffeomorphism $\tilde{M}/\pi_1(M, p_0) \cong M$.

After this preliminary discussion of the universal cover, we can prove surjectivity. Let $\phi : \pi_1(M, p_0) \to \mathbb{R}$ be a homomorphism and we have a function $f : \tilde{M} \to \mathbb{R}$ such that

$$f([\gamma * x]) = f([x]) + \phi([\gamma]) \qquad \text{for all } [\gamma] \in \pi_1(M, p_0) \text{ and } [x] \in \tilde{M}.$$

Now the problem with this solution is, that it is not entirely obvious why such a function f exists. If it exists then we can construct a preimage as follows. Note that $df \in \Omega^1(\tilde{M})$ is invariant under the action of $\pi_1(M, p_0)$ and hence there exists $\alpha \in \Omega^1(M)$ such that $df = \pi^*\alpha$. Since π is a local diffeomorphism and $d(\pi^*\alpha) = ddf = 0$, we also have $d\alpha = 0$. Finally, any loop $[\gamma] \in \pi_1(M, p_0)$ based at p_0 lifts to a path $\tilde{\gamma} : [0, 1] \to \tilde{M}$ with $[\tilde{\gamma}(1)] = [\gamma * \tilde{\gamma}(0)]$ and therefore

$$\int_\gamma \alpha = \int_{\tilde{\gamma}} \pi^*\alpha = f([\tilde{\gamma}(1)]) - f([\tilde{\gamma}(0)]) = f([\gamma * \tilde{\gamma}(0)]) - f([\tilde{\gamma}(0)]) = \phi([\gamma]).$$

Hence $[\alpha] \in H^1(M)$ is a preimage of ϕ and this completes the proof of surjectivity.

2. Let M_0 and M_1 be smooth, closed and oriented m-manifolds. Let

$$\iota_0 : D^m \to M_0, \qquad \iota_1 : D^m \to M_1$$

be two embeddings of the unit ball $D^m \subset \mathbb{R}^m$, where ι_0 is orientation preserving and ι_1 is orientation reversing. The connected sum $M_0\#M_1$ is defined as

$$M_0\#M_1 := ((M_0 \setminus \{\iota_0(0)\}) \,\dot\cup\, (M_1 \setminus \{\iota_1(0)\})) \,/\sim$$

where $\dot\cup$ denotes the disjoint union and $\iota_0(tx) \sim \iota_1((1-t)x)$ for $x \in S^{m-1}$ and $0 < t < 1$.

a) Let M be a smooth closed and oriented m-manifold and let $p \in M$. Compute the de Rham cohomology of $M\backslash\{p\}$ using the Mayer–Vietoris sequence

b) Compute the de Rham cohomology $H^k(M_0\#M_1)$ using the Mayer-Vietoris sequence.

c) Calculate the de Rham cohomology of the connected sum $\Sigma_\ell := T^2\#\ldots\#T^2$ of $\ell \geq 2$ copies of the two torus.

Solution:

a) We show that

$$H^k(M\backslash\{p\}) \cong \begin{cases} H^k(M) & k \neq m \\ 0 & k = m \end{cases}$$

Let $p \in U \subset M$ be a contractible open neighborhood and let $V = M\backslash\{p\}$. We take a look at the start of the Mayer–Vietoris sequence:

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(M) \to \cdots$$

Since $M, U, V, U\cap V$ are all connected, their 0-th cohomoloy groups are all isomorphic to $\mathbb{R}$. It then follows from counting dimensions that the map $H^0(U \cap V) \to H^1(M)$ is zero. The sequences then continues as

$$0 \to H^1(M) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V).$$

and

$$H^{k-1}(U \cap V) \to H^k(M) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V).$$

Since U is contractible, we have $H^k(U) = 0$ for all $k \geq 0$. Moreover, $U \cap V$ is homotopy equivalent to S^{m-1} and we thus have $H^k(U \cap V) = 0$ for $2 \leq k \leq m-2$. It thus follows from the sequence above that $H^k(M) \cong H^k(V) = H^k(M \backslash \{p\})$ for $0 \leq k \leq m-2$.

For $k = m$, we have by Poincare duality $H^m(M\backslash\{p\}) \cong H_c^0(M\backslash\{p\}) = 0$, where the last equality follows from the fact that $M\backslash\{p\}$ is not compact.

For $k = m-1$ we then have

$$0 \to H^{m-1}(M) \to H^{m-1}(U) \oplus H^{m-1}(V) \to H^{m-1}(U \cap V) \to H^m(M) \to 0.$$

Since $H^{m-1}(U \cap V) \cong \mathbb{R}$, $H^m(M) \cong \mathbb{R}$, and $H^{m-1}(V) = 0$, it follows $H^{m-1}(M) \cong H^{m-1}(V)$ and this completes the proof.

b) We show that

$$H^k(M_0 \# M_1) \cong \begin{cases} H^k(M_0) \oplus H^k(M_1) & 1 \leq k \leq m-1 \\ \mathbb{R} & k = 0 \text{ or } k = m \end{cases}$$

Since $M_0 \# M_1$ is again an orientable closed m-manifold, we known that $H^m(M_0 \# M_1) \cong \mathbb{R}$. For the other groups we apply the Mayer–Vietoris sequence for $U := M_0 \setminus \{\iota_0(0)\}$ and $V = M_1 \setminus \{\iota_1(0)\}$, which we identify with open subsets in the connected sum. By part (a) follows

$$H^k(U) \cong \begin{cases} H^k(M_0) & k \neq m \\ 0 & k = m \end{cases} \quad \text{and} \quad H^k(V) \cong \begin{cases} H^k(M_1) & k \neq m \\ 0 & k = m \end{cases}$$

The intersection $U \cap V$ is homotopy equivalent to S^{m-1} and thus $H^k(U \cap V) = 0$ for $1 \leq k \leq m-2$ and $H^k(U \cap V) \cong \mathbb{R}$ for $k = 0$ or $k = m-1$.

As in part (a), we see that in the sequence

$$0 \to H^0(M_0 \# M_1) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(M_0 \# M_1) \to \cdots$$

the final map $H^0(U \cap V) \to H^1(M_0 \# M_1)$ is zero, since all 0-th cohomology groups in this sequence are isomorphic to $\mathbb{R}$.

The sequences then continues as

$$0 \to H^1(M_0 \# M_1) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V).$$

and

$$H^{k-1}(U \cap V) \to H^k(M_0 \# M_1) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V).$$

This yields

$$H^k(M_0 \# M_1) \cong H^k(U) \oplus H^k(V) \cong H^k(M_0) \oplus H^k(M_1) \qquad \text{for } k = 1, \ldots, k-2.$$

Finally, the end of the sequences yields

$$0 \to H^{m-1}(M_0 \# M_1) \to H^{m-1}(U) \oplus H^{m-1}(V) \to H^{m-1}(U \cap V) \to H^m(M_0 \# M_1) \to 0$$

since $H^m(U) =$ and $H^m(V) = 0$. Since $H^{m-1}(U \cap V) \cong \mathbb{R}$ and $H^m(M_0 \# M_1) \cong \mathbb{R}$, we thus conclude

$$H^{m-1}(M_0 \# M_1) \cong H^{m-1}(U) \oplus H^{m-1}(V) \cong H^{m-1}(M_0) \oplus H^{m-1}(M_1).$$

c) The de Rham cohomology of the torus is

$$H^k(T^2) \cong \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R}^2 & k = 1 \\ \mathbb{R} & k = 2 \end{cases}$$

It then follows from part (b) for the ℓ-fold direct sum

$$H^k(\Sigma_\ell) = H^k(T^2 \# \cdots \# T^2) \cong \begin{cases} \mathbb{R} & \ell = 0 \\ \mathbb{R}^{2\ell} & k = 1 \\ \mathbb{R} & k = 2 \end{cases}$$

3. **a)** Specify a basis of $H^1(T^2)$ and compute the Poicaré pairing on this basis.

b) Prove that both $\dim H^1(\Sigma)$ and the Euler characteristic $\chi(\Sigma)$ is even for every oriented, compact 2-manifold Σ without boundary, a so-called Riemann surface.

c) We call $g := \frac{1}{2} \dim H^1(\Sigma)$ the genus of a Riemann surface Σ. Give an example of a Riemann surface Σ_g for every $g \in \mathbb{N}$.

Hint: For b), use Poincaré duality and the expression of the Euler characteristic in terms of Betti numbers.

Note: The classification of Riemann surfaces proves that all Riemann surfaces of the same genus are diffeomorphic.

Solution:

a) Let $\theta \in \Omega^1(S^1)$ be a volume form be the standard angular volume form on S^1 with $\int_{S^1} \theta = 2\pi$. Denote by $\pi_1, \pi_2 : T^2 = S^1 \times S^1 \to S^1$ the projection onto the first and onto the second coordinate. Then $\theta_1 := \pi_1^* \theta$ and $\theta_2 := \pi_2^* \theta$ are closed 1-forms on T^2 and $\theta_1 \wedge \theta_2 \in \Omega^2(T^2)$ is a volume form. Since $H^1(T^2) \cong \mathbb{R}^2$, it then follows that $[\theta_1], [\theta_2] \in H^1(T^2)$ is a basis. This satisfies

$$([\theta_i], [\theta_j]) = \int_{T^2} \theta_i \wedge \theta_j = \begin{cases} 0 & i = j \\ 4\pi & (i,j) = (1,2) \\ -4\pi & (i,j) = (2,1) \end{cases}.$$

b) Since Σ is closed, connected and oriented, we have $H^0(\Sigma) \cong \mathbb{R}$ and $H^2(\Sigma) \cong \mathbb{R}$, and hence

$$\chi(\Sigma) = 2 - \dim\left(H^1(\Sigma, \mathbb{R})\right) \leq 2.$$

Hence it remains to show that $\dim(H^1(\Sigma, \mathbb{R}))$ is even. By Poincare duality, it follows that the pairing

$$H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R}, \qquad ([\alpha], [\beta]) := \int_\Sigma \alpha \wedge \beta$$

is a nondegenerated, skew-symmetric bilinear form on $H^1(\Sigma)$. By the classification of skew-symmetric bilinear forms, there exists a basis of $H^1(\Sigma)$ such that the pairing above is represented by a block-diagonal matrix with blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and 0. The form is non-degnerated, when the normal form consists only of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ blocks. In particular, $H^1(\Sigma)$ must be even dimensional.

c) For $g = 0$, one can take the sphere S^2 with $H^1(S^2) = 0$. For $g = 1$, one can take the torus T^2 with $H^1(T^2) \cong \mathbb{R}^2$. For $g \geq 2$, one can take $\Sigma_g = T^2 \# \cdots \# T^2$ to be the g-fold connected sum of T^2. It follows from exercise 2 that $H^1(\Sigma_g) \cong \mathbb{R}^{2g}$.

4. **a)** Give an example of a smooth map $f : M \to N$ between manifolds without boundary and a compactly supported form $\omega \in \Omega^k_c(N)$ such that $f^*\omega$ is not compactly supported.

b) Let M be the open Möbius strip. Compute the de-Rham cohomology groups – the compactly supported and also the full one.

c) Give a counter-example to Poincaré duality in the non-oriented case.

Solution:

a) Take $M = N = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R} : t \to \tanh(t)$. Take a non-zero compactly supported function g with support $[-1, 1]$. Then $g \in \Omega^0(N)$ is compactly supported, but f^*g has support equal to $M = \mathbb{R}$ which is not compact.

b) We recall that M can be defined as

$$M := \{(e^{it}, re^{it/2}) \in \mathbb{C}^2 \, : \, t \in \mathbb{R}, \, r \in (-1, 1)\}.$$

and that $\pi : M \to S^1 : (z_1, z_2) \mapsto z_1$ is a submersion. We claim that π is also a homotopy equivalence. Let $\sigma : S^1 \to M$ be the zero section. Then $\pi \circ \sigma = \mathrm{id}_{S^1}$ and $\sigma \circ \pi(z_1, z_2) = (z_1, 0)$. This last map is homotopic to the identity. Indeed, take the homotopy $H : M \times [0, 1] \to M : ((z_1, z_2), t) \mapsto (z_1, tz_2)$. So M is homotopy equivalent to S^1, and therefore

$$H^i(M) = H^i(S^1) = \begin{cases} \mathbb{R} & , \text{ for } i = 0, 1, \\ 0 & , \text{ else.} \end{cases}$$

This trick does not work for compactly supported cohomology since we loose the functoriality. We start by seeing that since M is connected and open, there are no locally constant, compactly supported functions and so $H^0_c(M) = 0$. To calculate $H^1_c(M)$, we use Mayer-Vietoris with $D^2 \cup M = \mathbb{R}P^2$. This can be seen by taking the universal cover $f : S^2 \to \mathbb{R}P^2$ and looking at

$$U = f(\{p = (x_0, x_1, x_2) \in S^2 \, : x_2 > 0\}) \cong D^2 \text{ and } V = f(\{p \in S^2 : -1 < x_2 < 1\}) \cong M.$$

Also $U \cap V \cong S^1 \times \mathbb{R}$. Thus the Mayer-Vietoris sequence for compactly supported cohomology gives an exact sequence

$$\cdots \to H^0_c(M) \oplus H^0_c(D^2) \to H^0_c(\mathbb{R}P^2) \to H^1_c(S^1 \times \mathbb{R}) \to H^1_c(M) \oplus H^1_c(D^2) \to H^1_c(\mathbb{R}P^2) \to \cdots$$

This reads

$$\cdots \to 0 \oplus 0 \to \mathbb{R} \to \mathbb{R} \to H^1_c(M) \oplus 0 \to 0 \to \cdots$$

where we used $H_c^1(\mathbb{R}P^2) = 0$, $H_c^0(\mathbb{R}P^2) = \mathbb{R}$, $H_c^1(S^1 \times \mathbb{R}) = H_c^0(S^1) = \mathbb{R}$ by Künneth, $H_c^0(D^2) = 0$, and $H_c^1(D^2) = 0$ by Poincaré duality. So $H_c^1(M) = 0$. Since every top form on a non-orientable manifold is exact, we have $H_c^2(M) = 0$.

c) We take the Möbius strip M and see that since $H_c^i(M) = 0$ for all $i = 0, 1, 2$, the pairing cannot be non-degenerate since $H^1(M) = \mathbb{R}$.

5. **[Poincaré lemma]** Find an explicit formula for an operator

$$h : \Omega^k(\mathbb{R}^m) \to \Omega^{k-1}(\mathbb{R}^m)$$

that satisfies $d \circ h + h \circ d = \mathrm{Id}$.

Hint: Look carefully at the proof of Theorem 1 in Chapter XI. Use the homotopy $f_t(x) = tx$ and Cartan's formula.

Solution: We take $f_t(x) = tx$ on $M = \mathbb{R}^m$. Then we have for $\omega \in \Omega^k(M)$ that

$$\omega = f_1^*\omega = f_1^*\omega - f_0^*\omega = \int_0^1 \frac{d}{dt}(f_t^*\omega)\,dt = \int_0^1 f_t^*(\mathcal{L}_{X_t}\omega)\,dt = \int_0^1 f_t^*\Big(\iota(X_t)d\omega + d(\iota(X_t)\omega)\Big), \tag{1}$$

where we used Exercise 2 b) of Sheet 9 and Cartan's formula with

$$X_t(x) = \frac{d}{dt}f_t(x) = x.$$

So take

$$h : \Omega^k(M) \to \Omega^{k-1}(M) : \omega \to \int_0^1 f_t^*(\iota(X_t)\omega)\,dt.$$

Then equation (1) reads for $\omega \in \Omega^k(M)$ that

$$\omega = dh(\omega) + h(d\omega).$$

In explicit terms, we can thus calculate

$$h(\omega)_x(v_1, \ldots, v_{k-1}) = \int_0^1 \Big(f_t^*(\iota(X_t)\omega)\Big)_x(v_1, \ldots, v_{k-1})\,dt = \int_0^1 \omega_{tx}(x, tv_1, \ldots, tv_{k-1})\,dt.$$

6. Let M be a compact manifold without boundary and $U \subset M$ be an open set. Assume that $f : M \to M$ has $f(M) \subset U$. Prove that for every $k \in \mathbb{N}$,

$$\mathrm{tr}\left(f^* : H^k(M) \to H^k(M)\right) = \mathrm{tr}\left((f|_U)^* : H^k(U) \to H^k(U)\right).$$

Hint: We use the Mayer-Vietoris sequence with U and $V = M \setminus f(M)$. Complete the following diagram such that the resulting diagram commutes.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^{k-1}(U\cap V) & \xrightarrow{d^*} & H^k(M) & \xrightarrow{i^*} & H^k(U)\oplus H^k(V) & \xrightarrow{j^*} & H^k(U\cap V) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow f^* & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^{k-1}(U\cap V) & \xrightarrow{d^*} & H^k(M) & \xrightarrow{i^*} & H^k(U)\oplus H^k(V) & \xrightarrow{j^*} & H^k(U\cap V) & \longrightarrow & \cdots
\end{array}$$

Solution: The completed diagram is the following.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^{k-1}(U\cap V) & \xrightarrow{d^*} & H^k(M) & \xrightarrow{i^*} & H^k(U)\oplus H^k(V) & \xrightarrow{j^*} & H^k(U\cap V) & \longrightarrow & \cdots \\
 & & \downarrow 0 & & \downarrow f^* & & \downarrow \begin{pmatrix} (f|_U)^* & 0 \\ (f|_V)^* & 0 \end{pmatrix} & & \downarrow 0 & & \\
\cdots & \longrightarrow & H^{k-1}(U\cap V) & \xrightarrow{d^*} & H^k(M) & \xrightarrow{i^*} & H^k(U)\oplus H^k(V) & \xrightarrow{j^*} & H^k(U\cap V) & \longrightarrow & \cdots
\end{array}$$

We need to prove that this diagram commutes.

For the first square, let ρ_U, ρ_V be a partition of unity. We need to prove that $f^*(d^*\omega)$ is exact for all closed $\omega \in \Omega^k(U \cap V)$. We note that $f(M) \cap V = \emptyset$ by definition and that $d^*\omega$ is supported in $U \cap V$, so $f^*d^*\omega = f^*(d\rho_U \wedge \omega) = 0$.

For the second square, we note that from $f(M) \subset U$, we get that $(f^*\omega)|_U = (f|_U)^*\omega|_U$ and $(f^*\omega)|_V = (f|_V)^*\omega|_U$.

For the third square, we note that for all $\omega \in \Omega^k(U)$,

$$((f|_V)^*\omega)|_{U\cap V} = (f^*\omega)|_{U\cap V} = ((f|_U)^*\omega)|_{U\cap V}.$$

For the statement about the traces decompose

$$H^k(M) = B_0 \oplus B_1, \qquad H^k(U) \oplus H^k(V) = C_0 \oplus C_1$$

where $B_0 = \operatorname{im}(d^*) = \ker(i^*)$ and $C_0 = \operatorname{im}(i^*) = \ker(j^*)$. By comutativity of the diagram, we have $f^*([\omega]) = 0$ for $[\omega] \in B_0$ and hence

$$\operatorname{tr}\Big(f^* : H^k(M) \to H^k(M)\Big) = \operatorname{tr}\Big(f^* : H^k(M)/B_0 \to H^k(M)/B_0\Big).$$

Similarly, we have by commutativiy of the diagram that $j^*((f|_U)^*[\omega_U], (f|_V)^*[\omega_U]) = 0$. Hence $((f|_U)^*[\omega_U], (f|_V)^*[\omega_U]) \in C_0$ and therefore

$$\operatorname{tr}\Big(\alpha : H^k(U) \oplus H^k(V) \to H^k(U) \oplus H^k(V)\Big) = \operatorname{tr}\Big(\alpha|_{C_0} : C_0 \to C_0\Big),$$

where

$$\alpha : H^k(U) \oplus H^k(V) \to H^k(U) \oplus H^k(V) : (\omega, \tau) \mapsto \begin{pmatrix} (f|_U)^* & 0 \\ (f|_V)^* & 0 \end{pmatrix} \begin{pmatrix} \omega \\ \tau \end{pmatrix}$$

is the map in the diagram. Since $i^* : H^k(M)/B_0 \to C_0$ induces is an isomorphism, we then conclude

$$\begin{aligned} \operatorname{tr}\Big(f^* : H^k(M) \to H^k(M)\Big) &= \operatorname{tr}\Big(f^* : H^k(M)/B_0 \to H^k(M)/B_0\Big) \\ &= \operatorname{tr}\Big(\alpha|_{C_0} : C_0 \to C_0\Big) \\ &= \operatorname{tr}\Big(\alpha : H^k(U) \oplus H^k(V) \to H^k(U) \oplus H^k(V)\Big) \\ &= \operatorname{tr}\Big((f|_U)^* : H^k(U) \to H^k(U)\Big). \end{aligned}$$