Solution 13

1.

a) Let $E \to M$ be a vector bundle. Show that for every $p \in M$ and for every $e \in E_p$ there exists a smooth section s such that $s(p) = e$.

b) For E a vector bundle. We introduce two notions of orientatability.

(i) E has transition maps in $\mathrm{GL}_+(\mathbb{R}^n)$.

(ii) There exists a collection of orientations of the fibres of E, such that for every $p_0 \in M$ there exists an open neighbourhood of $U \subset M$ of p_0 and there exist smooth sections $s_1, s_2, \cdots, s_n : U \to E|_U$ such that $s_1(p), s_2(p), \cdots, s_n(p)$ form a positive basis of E_p for all $p \in U$.

Prove that these two definitions are equivalent.

c) Assume that M is an oriented manifold and $E \to M$ is a vector bundle. Show that E is oriented as a vector bundle if and only if E is oriented as a manifold.

Solution:

a) Let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be a local trivialisation of E with $p \in U$. Then let $(p, v) := \Phi(e)$ and take a cut-off function $\rho : M \to \mathbb{R}$ with support in U and $\rho(p) = 1$. Then define a map $s : M \to E$ by

$$s(q) = \begin{cases} \Phi^{-1}(q, \rho(q)v), & \text{if } q \in U \\ (q, 0) & \text{if } q \notin U \end{cases}.$$

By construction, s is a section and $s(p) = \Phi^{-1}(p, v) = e$.

b) We start by proving that (i) implies (ii). Given a collection of local trivialisations $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ with transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Gl}_+(\mathbb{R}^n)$, we define an orientation on E_p by saying that $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^n$ is orientation preserving, where $\mathbb{R}^n$ has the standard positive basis $e_1, \ldots, e_n$. With this orientation in place, the sections $s_i^\alpha : U_\alpha \to E|_{U_\alpha}$ given by $s_i^\alpha(p) := \Phi_\alpha^{-1}(p, e_i)$ for $p \in U_\alpha$ form a positive basis of sections over any point in U_α. Thus, we have (ii).

We prove that (ii) implies (i). For every point $p \in M$, pick an open set U_p such that there is a positive basis of sections $(s_i^p : U_p \to E|_{U_p})_{i=1}^n$. This means that we can get local trivialisations

$$\Phi_p : \pi^{-1}(U_p) \to U_p \times \mathbb{R}^n \text{ which is defined by } \Phi_p^{-1}(q, v) = \sum_{i=1}^n v^i s_i^p(q). \tag{1}$$

This defines local trivialisations over every point in M. We need to check that the transition maps g_{qp} take values in $\mathrm{Gl}_+(\mathbb{R}^n)$. This is basically by construction. Namely, by definition

$$\Phi_q \circ \Phi_p^{-1}(r, v) = (r, g_{qp}(r)v) \text{ for all } (r, v) \in (U_p \cap U_q) \times \mathbb{R}^n$$

and both $\Phi_p|_{E_r} : E_r \to \{r\} \times \mathbb{R}^n$ and $\Phi_q|_{E_r} : E_r \to \{r\} \times \mathbb{R}^n$ are orientation preserving. Thus $g_{qp}(r) \in \mathrm{Gl}_+(\mathbb{R}^n)$ for all $r \in U_p \cap U_q$.

c) We start by taking an oriented atlas $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$. Taking a finer covering, we may assume that there are also local trivialisations $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$. Then we get charts of E by looking at

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^m \times \mathbb{R}^n \text{given by} \psi_\alpha(e) := ((\varphi_\alpha \times \mathrm{id}) \circ \Psi_\alpha)(e).$$

Now the change of variable of these charts is given by

$$\psi_\beta \circ \psi_\alpha^{-1}(x,v) = (\varphi_\beta \circ \varphi_\alpha^{-1}(x), g_{\beta\alpha}(\varphi_\alpha^{-1}(x))v),$$

for all $x \in \varphi_\alpha(U_\alpha \cap U_\beta)$ and $v \in \mathbb{R}^n$. The differential is given in block form by

$$d(\psi_\beta \circ \psi_\alpha^{-1})(x,v) = \begin{pmatrix} d(\varphi_\beta \circ \varphi_\alpha^{-1})(x) & 0 \\ * & g_{\beta\alpha}(\varphi_\alpha^{-1}(x)) \end{pmatrix}.$$

Thus,

$$\det(d(\psi_\beta \circ \psi_\alpha^{-1})(x,v)) = \det(d(\varphi_\beta \circ \varphi_\alpha^{-1})(x)) \det(g_{\beta\alpha}(\varphi_\alpha^{-1}(x))).$$

Since we started with an oriented atlas of M, the sign of

$$\operatorname{sign}\Big(\det(d\psi_\beta \circ \psi_\alpha^{-1}(x,v))\Big) = \operatorname{sign}\Big(\det(g_{\beta\alpha}(\varphi_\alpha^{-1}(x)))\Big).$$

Therefore the atlas ψ_α is oriented if and only if the transition maps map into $\mathrm{Gl}_+(\mathbb{R}^n)$. Hence, E is oriented as a manifold if and only if E is oriented as a vector bundle.

2. Let M be a compact manifold and let $\pi^E : E \to M$ and $\pi^F : F \to M$ be vector bundles. A vector bundle homomorphism is a smooth map $\Phi : E \to F$ such that $\pi^F \circ \Phi = \pi^E$ and $\Phi|_{E_p} : E_p \to F_p$ is a linear map for every p.

a) If Φ is injective prove that Φ is an embedding.

b) If Φ is bijective prove that Φ is a diffeomorphism.

c) Prove that for every vector bundle E over M, there exists an injective vector bundle homomorphism $\Phi : E \to M \times \mathbb{R}^N$ for some N.

Hint: In a), for properness use the following two ingredients.

- On a manifolds, $f : M \to N$ is proper, iff any sequence (x_k) such that $f(x_k)$ is bounded, is itself bounded. Here boundedness is with respect to any metric inducing the manifold topology on M and N.
- For any smooth family $A(x) \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^\ell)$ of injective linear maps, there is a constant $C > 0$ such that $|v| \le C|A(x)v|$ for $v \in \mathbb{R}^m$ and any x on a small neighbourhood. For this construct smooth families of left inverses to $A(x)$.

Solution:

a) For $p \in M$, there is an open set U_p such that $p \in U_p$ and

- There is a chart $\varphi_p : U_p \to \mathbb{R}^m$.
- There is a local trivialisation $\Phi_p^E : (\pi^E)^{-1}(U_p) \to U_p \times \mathbb{R}^n$.
- There is a local trivialisation $\Phi_p^F : (\pi^F)^{-1}(U_p) \to U_p \times \mathbb{R}^\ell$.

Since U_p cover M and M is compact, finitely many of these suffice to cover M, say $(U_{p_i})_{i=1}^N$. Denote by $\psi_i^E := (\varphi_{p_i} \times \mathrm{id}) \circ \Phi_{p_i}^E$ and $\psi_i^F := (\varphi_{p_i} \times \mathrm{id}) \circ \Phi_{p_i}^F$ which are chart for E resp. F. In these charts, we have by Φ being a homomorphism, that for $(x,v) \in \mathbb{R}^m \times \mathbb{R}^n$, we get

$$(\psi_i^F) \circ \Phi \circ (\psi_i^E)^{-1}(x,v) = (x, A_i(x)v),$$

where $A_i : \mathbb{R}^m \to \mathbb{R}^{\ell \times n}$ is a map of matrices. Φ injective, means that the map $\Phi|_{E_p}$ is injective, which in turn implies that $A_i(x)$ is injective for all $x \in \mathbb{R}^m$. (So $n \le \ell$.) We need to prove that Φ is an immersion. In block form the differential is given by

$$d((\psi_i^F) \circ \Phi \circ (\psi_i^E)^{-1})(x,v) = \begin{pmatrix} \mathbb{1} & 0 \\ * & A_i(x) \end{pmatrix}. \tag{2}$$

So the differential is injective exactly if $A_i(x)$ is injective. Hence, Φ is an immersion.

It remains to prove that Φ is proper. Choose any distances on E and F inducing the manifold topologies. Now we claim that Φ is proper if

Any sequence $(x_k)_k \subset E$ with $(\Phi(x_k))_k \subset F$ is bounded, is bounded.

We prove that such a map is proper. Take $K \subset F$ a compact set and a sequence $x_k \in \Phi^{-1}(K)$. Since $\Phi(x_k)$ is contained in a compact set, there is a subsequence $f(x_k)$ converging to some $y \in K$. Any compact set is also bounded, so by assumption on Φ, (x_k) is bounded. Hence a subsequence converges to a point $x \in E$, since E has the Heine-Borel property. Since Φ is continuous, we also have that $\Phi(x_k)$ converges to $\Phi(x)$. Since F is Hausdorff, $\Phi(x) = y \in K$ and so $x \in \Phi^{-1}(K)$. Thus $\Phi^{-1}(K)$ is compact.

Thus let $(x_k) \subset E$ be a sequence, such that $\Phi(x_k)$ is bounded. The sequence $y_k = \pi^E(x_k)$ has a subsequence converging to some $y \in M$. Then there is some U_i such that $y \in U_i$. Thus up to taking a subsequence, we assume that $y_k \in U_i$. Define by $(z_k, v_k) := \psi_i^E(x_k)$. We claim that there is a constant $C > 0$ such that

$$|v| \leq C|A_i(x)v|$$

for all $(x, v) \in V \times \mathbb{R}^n$ for V a small neighbourhood of $z := \varphi_{p_i}(y)$. Given this claim, we can conclude as follows.

Since $\Phi(x_k)$ is bounded, $A_i(z_k)v_k$ is bounded. Since z_k converges to z, we may assume that $z_k \in V$. Thus

$$|v_k| \leq C|A_i(z_k)v_k| \leq CM.$$

So v_k is bounded, which also implies that x_k is bounded. We conclude that Φ is proper.

We are left with proving our claim. This is a statement about injective linear maps. Namely, take $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$, then $w_1(x) := A_i(x)e_1, \ldots, w_n(x) := A_i(x)e_n$ is a basis of $\mathrm{im}(A_i(x))$, since $A_i(x)$ is injective. Complete $A_i(z)e_1, \ldots, A_i(z)e_n$ by $w_{n+1}, \ldots, w_\ell$ into a basis of $\mathbb{R}^\ell$. Then by continuity of det, $w_1(x), \ldots, w_n(x), w_{n+1}, \ldots, w_\ell$ is a basis of $\mathbb{R}^\ell$ for all $x \in V$, a small neighborhood of z. Define

$$F_i(x) : \mathbb{R}^\ell \to \mathbb{R}^n \text{ given by } F_i(x)\left(\sum_{i=1}^n \lambda^i w_i(x) + \sum_{i=n+1}^\ell \lambda^i w_i\right) = (\lambda^1, \ldots, \lambda^n).$$

This means that $F_i(x) \circ A_i(x) = \mathrm{id}$ and for V maybe smaller, we have $\|F_i(x)\| \leq C$ for some constant $C > 0$. This implies that

$$|v| = |F_i(A_i(x))| \leq C|A_i(x)v|.$$

b) By a), we already know that Φ is bijective exactly if $A_i(x)$ is invertible for all $x \in \mathbb{R}^m$. Furthermore, this implies by (2) that the differential of Φ is an isomorphism at every point. So by the inverse function theorem, Φ is a local diffeomorphism and is also bijective. So Φ is a diffeomorphism.

c) Take a partition of unity ρ_i of M subordinate to U_{p_i} from a). Then we define for $(i,j) \in I := \{(i,j) \ : \ i = 1, \dots, N, j = 1, \dots, N\}$, $s_{ij}^* : M \to \mathrm{Hom}(E, \mathbb{R})$ by

$$s_{ij}(q)v = \begin{cases} \rho(q)\langle e_j, v_{p_i}\rangle & \text{if } q \in U_{p_i} \\ 0 & \text{if } q \notin U_{p_i} \end{cases} \text{ where } v \in E_q \text{ and } \Phi_{p_i}(v) = (p, v_{p_i}).$$

Thus define the map $\Phi : E \to M \times \mathbb{R}^{|I|}$ where $|I| = N \cdot n$ given by

$$\Phi(v) = (\pi^E(v), (s_{ij}^*(\pi^E(v))v)_{(i,j)\in I}).$$

By construction, Φ is an injective homomorphism, so by a), it is an embedding.

Note: Close inspection of our solution, shows that we did not need M compact for a) and b). Our proof of c) though relies heavily on it, although the statement remains true for non-compact manifolds. Only the proof become way more technical. Think of the difference between Whitney embedding of compact manifolds and that of non-compact manifolds.

3. **a)** Let $Q \subset (M, g)$ be a submanifold. Prove that the normal bundle $TQ^{\perp_g}$ is a vector bundle over Q.

b) Let $E, F \to M$ be two vector bundles. Prove that the Whitney sum

$$E \oplus F := \{(p, e, f) : p \in M, e \in E, f \in F, \pi^E(e) = \pi^F(f) = p\}$$

is a vector bundle over M.

c) Let $E, F \to M$ be two vector bundles. Prove that the homomorphism bundle

$$\mathrm{Hom}(E, F) := \{(p, \Phi) : p \in M, \ \Phi : E_p \to F_p \text{ is linear}\}$$

is a vector bundle over M.

Hint: For a), use the Gram-Schmidt method on a suitably chosen basis of vector fields around any point and see that smoothness is preserved.

Solution:

a) Let $q \in Q$. We want to construct a local trivialisation $\Phi_q : \pi^{-1}(U_q) \to U_q \times \mathbb{R}^{m-k}$ where U_q is a neighbourhood of q, $\dim(M) = m$ and $\dim(Q) = k$. TQ is a vector bundle over Q. Hence there is a basis of sections $X_i : U_q \to TQ|_{U_q}$ (also called vector fields) over some small neighbourhood of q. Now complete $X_1(q), \dots, X_k(q)$ to a basis of T_qM, by adding $w_{k+1}, \dots, w_m$. Now choose sections $X_{k+1}, \dots, X_m$ over U_q of TM such that $X_i(q) = w_i$ for $i = k+1, \dots, m$ which exist by Exercise 1. By continuity of the determinant and up to shrinking U_q, $X_1, \dots, X_m$ form a basis of TM over U_q. Now we apply the Gram-Schmidt algorithm with respect to g to this basis point-wise, which results in a collection of smooth vector fields $Y_1, \dots, Y_m$, such that these form an orthonormal basis of TM over U_q and $X_1, \dots, X_k$ are sections of TQ. Hence $Y_{k+1}, \dots, Y_m$ form a basis of sections of $TQ^{\perp_g}$ over U_q. This gives the wanted local trivialisation of the normal bundle of Q.

To see that the Gram-Schmidt algorithm indeed transforms vector fields into vector fields, we start by writing down the first two steps.

$$Y_1 := \frac{X_1}{\sqrt{g(X_1, X_1)}}, \quad Z_2 = X_2 - g(X_2, Y_1)Y_1, \quad Y_2 := \frac{Z_2}{\sqrt{g(Z_2, Z_2)}}.$$

Since g is smooth and X_1, X_2 are linearly independent, Y_1, Z_2, Y_2 are all smooth. This also shows that $Y_i \in \operatorname{span}(X_1, \ldots, X_i)$, and so since $\operatorname{span}(X_1, \ldots, X_i) \subset TQ$ for $i \leq k$, the resulting $Y_1, \ldots, Y_k$ take values in TQ.

b) Take a collection of open sets $(U_\alpha)_{\alpha \in A}$ small such that the U_α cover and such that both $E|_{U_\alpha}$ and $F|_{U_\alpha}$ are trivialisable for all $\alpha \in A$. Let $g^E_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Gl}(V)$, $g^F_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Gl}(W)$ be the transition map data for E, F. Then

$$g^E_{\beta\alpha} \oplus g^F_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Gl}(V \oplus W)$$

is the transition map data for $E \oplus F$.

c) Let U_α be as in b). Take local trivialisations $\Phi^E_\alpha : (\pi^E)^{-1}(U_\alpha) \to U_\alpha \times V$ and $\Phi^F_\alpha : (\pi^F)^{-1}(U_\alpha) \to U_\alpha \times W$. Then for any $\Phi \in (\pi^{\mathrm{Hom}(E,F)})^{-1}(U_\alpha)$, we define $A^\Phi_\alpha \in \mathrm{Hom}(V, W)$

$$\Phi^F_\alpha \circ \Phi \circ (\Phi^E_\alpha)^{-1}(p, v) = (p, A^\Phi_\alpha v)$$

Thus, we can define the local trivialisation

$$\Phi^{\mathrm{Hom}(E,F)}_\alpha : (\pi^{\mathrm{Hom}(E,F)})^{-1}(U_\alpha) \to \mathrm{Hom}(V, W) : (p, \Phi) \to (p, A^\Phi_\alpha).$$

That this fits together in smooth transition maps can be seen easiest by identifying $\mathrm{Hom}(V, W) \cong V^* \otimes W$. Thus the transition data is given by

$$(g^E_{\beta\alpha})^* \otimes g^F_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Gl}(V \otimes W^*).$$

(This also shows the bundle isomorphism $E^* \otimes F \cong \mathrm{Hom}(E, F)$.)

4. Let E be a real rank-n vector bundle over a smooth m-manifold M and let $s : M \to E$ be a smooth section of E. Assume s is transverse to the zero section. Then the zero set

$$s^{-1}(0) := \{p \in M : s(p) = 0_p\}$$

of s is a smooth submanifold of M of dimension $m - n$ and

$$T_p s^{-1}(0) = \ker Ds(p)$$

for every $p \in M$ with $s(p) = 0_p$.

Solution: We recall some preliminaries first. The result follows then rather directly from transversality theory. Denote by $Z := \{0_p \in E \,|\, p \in M\} \subset E$ the zero section. The tangent space of E at a zero splits canonically as

$$T_{0_p} E \cong T_p M \oplus E_p.$$

where $T_p M \to T_{0_p} E$ is obtained as derivative of the canonical map $M \to Z \subset E$ which maps p to 0_p. Suppose $s : M \to E$ is a section with $s(p) = 0$. The splitting above yields

$$ds(p) : T_p M \to T_{0_p} E \cong T_p M \oplus E_p, \qquad ds(p)v = (v, Ds(p)v)$$

where $Ds(p) : T_p M \to E_p$ is by definition the vertical derivative of s.

Suppose $s : M \to E$ is a section transverse to Z. It follows from Lemma 4.1.3. on transverse intersections in the lecture notes that $s^{-1}(0) := s^{-1}(Z)$ is a smooth $(m - n)$-dimensional submanifold with tangent spaces

$$T_p s^{-1}(0) = \{v \in T_p M \,|\, ds(p)v \in T_{0_p} E\} = \{v \in T_p M \mid Ds(p)v = 0_p\}$$

for every $p \in M$ with $s(p) = 0_p$.

5. In this exercise, we introduce some important line bundles over $\mathbb{C}P^n$. Define for $d \in \mathbb{Z}$ the quotient

$$H^d := \left((\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}\right) / \sim$$

where $(z_0, \dots, z_n; \zeta) \sim (\lambda z_0, \dots, \lambda z_n; \lambda^d \zeta)$.

a) Show that H^d is a real 2-dimensional vector bundle over $\mathbb{C}P^n$. Find explicit trivializations of this bundle and compute the corresponding transition maps.

b) Find an isomorphism between H^{-1} and the tautological line bundle

$$E := \{(\ell, w) \in \mathbb{C}P^n \times \mathbb{C}^n \ : \ w \in \ell\}$$

c) Find an isomorphism between H^1 and the canonical line bundle

$$H := \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C}).$$

d) Show that $T\mathbb{C}P^n \oplus \mathbb{C} \cong H \oplus \ldots \oplus H$ with $(n+1)$ copies of H.

Hint: For d): It holds $T_\ell \mathbb{C}P^n \cong \mathrm{Hom}_{\mathbb{C}}(\ell, \ell^\perp)$, see Exercise 6 on Sheet 1.

Solution:

a) H^d is a vector bundle of $\mathbb{C}P^n$ where the projection map is given by

$$\pi : H^d \to \mathbb{C}P^n, \qquad \pi(z_0, \dots, z_n, \zeta) = [z_0 : \dots : z_n].$$

Define $U_i = \{[z_0 : \dots : z_n] \in \mathbb{C}P^n \mid z_i \neq 0\}$ and consider the local trivializations

$$\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}, \qquad \psi_i(z_0, \dots, z_n, \zeta) = \left([z_0 : \dots : z_n], z_i^{-d}\zeta\right).$$

The transition maps are given by

$$\psi_i \circ \psi_j^{-1}\left([z_0 : \dots : z_n], v\right) \left([z_0 : \dots : z_n], \left(\frac{z_j}{z_i}\right)^{-d} v\right)$$

and give rise to the functions

$$g_{ij} : U_i \cap U_j \to \mathbb{C}^*, \qquad g_{ij}([z_0 : \dots : z_n]) := \left(\frac{z_j}{z_i}\right)^{-d}.$$

The structure group for our set of trivializations is hence $\mathbb{C}^* = \mathrm{GL}(1, \mathbb{C}) \subset \mathrm{GL}(2, \mathbb{R})$.

b) We give in the following explicit formulas for the various isomorphism. In each case they define clearly smooth maps and are linear isomorphism on the fibres. By Exercise 2, it then follows that they are also diffeomorphism. The isomorphism $H^{-1} \overset{\cong}{\to} E$ is given by

$$\Psi_{-1} : H^{-1} \to E, \qquad \Psi_{-1}(z_0, \dots, z_n, \zeta) = \left([z_0 : \dots : z_n], (z_0\zeta, \dots, z_n\zeta)\right).$$

c) Define

$$\Psi_1 : H^1 \to H := \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C}), \qquad \Psi_1(z_0, \dots, z_n, \zeta) \in \mathrm{Hom}_{\mathbb{C}}(E_{[z_0:\dots:z_n]}, \mathbb{C})$$

where $E_{[z_0:\dots:z_n]}$ is the complex line spanned by $(z_0, \dots, z_n)$ in $\mathbb{C}^{n+1}$ and

$$\Psi_1(z_0, \dots, z_n, \zeta) : E_{[z_0:\dots:z_n]} \to \mathbb{C}, \qquad \alpha(z_0, \dots, z_n) \mapsto \alpha\zeta.$$

d) Finally, as in Exercise Sheet 1 Exercise 6, there exists a natural identification

$$T_\ell \mathbb{C}P^n = \mathrm{Hom}_{\mathbb{C}}(\ell, \ell^\perp).$$

Moreover there is a canonical isomorphism $\mathrm{Hom}_{\mathbb{C}}(\ell,\ell) \cong \mathbb{C}$ given by scaling the identity, it follows that

$$\begin{aligned} T_\ell \mathbb{C}P^n \oplus \mathbb{C} &= \mathrm{Hom}_{\mathbb{C}}(\ell, \ell^\perp) \oplus \mathrm{Hom}_{\mathbb{C}}(\ell,\ell) = \mathrm{Hom}_{\mathbb{C}}(\ell, \mathbb{C}^{n+1}) \\ &= \mathrm{Hom}_{\mathbb{C}}(\ell,\mathbb{C}) \oplus \cdots \oplus \mathrm{Hom}_{\mathbb{C}}(\ell,\mathbb{C}) \\ &= H_\ell \oplus \cdots \oplus H_\ell. \end{aligned}$$

6. Let $E \to M$ be an oriented smooth vector bundle over and oriented compact m-manifold M without boundary. For $\omega \in \Omega^\ell(M)$ and $\tau \in \Omega_c^{n+k}(E)$ show that $\pi_*(\pi^*\omega \wedge \tau) = \omega \wedge \pi_*\tau$.

Solution: It suffices to verify the formula locally: Choose a local trivialisation $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$. After shrinking U_α if necessary, we may suppose that $U_\alpha \subset M$ is a chart domain and there exists a chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$. These combine to a chart on $\pi^{-1}(U_\alpha)$ defined by

$$\tilde{\varphi} : \pi^{-1}(U_\alpha) \to \mathbb{R}^m \times \mathbb{R}^n, \qquad \tilde{\varphi}(\psi_\alpha^{-1}(p,v)) = (\varphi_\alpha(p), v).$$

It then suffices to show the following local formula

$$\pi_* \left((\tilde{\varphi}_*\omega) \wedge (\tilde{\varphi}_*\tau) \right) = (\tilde{\varphi}_*\omega) \wedge \pi_*(\tilde{\varphi}_*\tau).$$

(since $\pi_*\tilde{\phi}_* = \tilde{\phi}_*\pi_*$.)

Denote the coordinates on $\mathbb{R}^m \times \mathbb{R}^n$ by $(x_1, \ldots, x_m; y_1, \ldots, y_n)$ and write in these coordinates

$$\omega = \sum_{|I|=\ell} \omega^I(x) dx^I, \qquad \tau = \sum_{|J|+|K|=n+k} \tau^{J,K}(x,y) dx^J \wedge dy^K.$$

Moreover, $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is the projection onto the first factor and

$$\pi^*\omega \wedge \tau = \sum_{|I|=\ell, |J|+|K|=n+k} \omega^I(x)\tau^{J,K}(x,y) dx^I \wedge dx^J \wedge dy^K.$$

The push-forward operator π_* integrates terms with $dy^K = dy^1 \wedge \cdots dy^n$ in y-direction and sends all other terms to zero. In other words, only those terms with $K = K_0 := \{1, \ldots, n\}$ survive and we get

$$\begin{aligned} \pi_*\left(\pi^*\omega \wedge \tau\right) &= \sum_{|I|=\ell, |J|=k} \left(\int_{\mathbb{R}^n} \omega^I(x) \tau^{J,K_0}(x,y) dy_1 \wedge \cdots \wedge dy_n \right) dx^I \wedge dx^J \\ &= \left(\sum_{|I|=\ell} \omega^I(x) dx^I \right) \wedge \left(\sum_{|J|=k} \left(\int_{\mathbb{R}^n} \tau^{J,K_0}(x,y) dy_1 \wedge \cdots \wedge dy_n \right) dx^J \right) \\ &= \omega \wedge (\pi_*\tau) \end{aligned}$$

Note that all integrals are well-defined, since $y \mapsto \tau^{J,K_0}(x,y)$ has compact support by assumption. This proves the claim locally.