ETH Zurich FS 2018
D-MATH
Prof. Wendelin Werner

Coordinator
Yilin Wang

Brownian Motion and Stochastic Calculus

Exercise Sheet 12

Exercise 12.1 Application of Itô's formula. Give the semimartingale decomposition of the following processes:

(a) $X_t := t^2 B_t^5$;

(b) $Y_t := \exp(tB_t)$;

(c) $Z_t := B_t^3 - 3tB_t$.

Solution 12.1

(a) Let $f(x,y) := x^2 y^5$,

$$X_t = f(t, B_t) = X_0 + \int_0^t \partial_x f(s, B_s) ds + \int_0^t \partial_y f(s, B_s) dB_s + \frac{1}{2} \int_0^t \partial_{yy} f(s, B_s) ds$$
$$= \int_0^t 5s^2 B_s^4 dB_s + \int_0^t (2sB_s^5 + 10s^2 B_s^3) ds,$$

where the first term is the local martingale and the second term is the bounded variation process.

(b) Let $f(x,y) = \exp(xy)$, we have

$$\partial_x f = y \exp(xy); \quad \partial_y f = x \exp(xy)$$
$$\partial_{yy} f = x^2 \exp(xy).$$

The Itô's formula gives

$$Y_t = 1 + \int_0^t s \exp(sB_s) dB_s + \left(\int_0^t B_s \exp(sB_s) ds + \frac{1}{2} \int_0^t s^2 \exp(sB_s) ds \right).$$

(c) Let $f(x,y) = y^3 - 3xy$, we have

$$\partial_x f = -3y; \quad \partial_y f = 3y^2 - 3x$$
$$\partial_{yy} f = 6y.$$

In particular $\partial_x f + \frac{1}{2}\partial_{yy} f = 0$, we have therefore Z_t is itself a local martingale (we have already seen it in Exercise 8.3) and

$$Z_t = \int_0^t \partial_y f(s, B_s) dB_s = \int_0^t 3B_s^2 - 3s dB_s.$$

Exercise 12.2 Geometric Brownian motion. Let $b \in \mathbb{R}$, $\sigma > 0$, B the standard one-dimensional Brownian motion. Let S be the solution to the stochastic differential equation

$$dS_t = bS_t dt + \sigma S_t dB_t, \quad S_0 = 1. \tag{1}$$

(a) Show that $Y_t := e^{-bt} S_t$ solves

$$dY_t = \sigma Y_t dB_t.$$

(b) Show that $S_t = \exp(bt)\exp(\sigma B_t - \sigma^2 t/2)$.

Solution 12.2

(a) Since the function $x \to bx$ and $x \to \sigma x$ are globally Lipschitz, the strong solution to the SDE (1) exists for all $t \geq 0$ and is unique up to 0-probability event.

Applying the Itô's formula to $Y_t = f(t, S_t)$ where $f(x, y) = \exp(-bx)y$, we have

$$dY_t = e^{-bt} dS_t - be^{-bt} S_t dt = e^{-bt}(bS_t dt + \sigma S_t dB_t) - be^{-bt} S_t dt = \sigma Y_t dB_t.$$

(b) The SDE for Y has also a unique strong solution given that $Y_0 = 1$. This equation in fact characterizes the exponential martingale $\mathcal{E}(\sigma B)$, which gives that

$$Y_t = \exp(\sigma B_t - \sigma^2 t/2).$$

We obtain therefore

$$S_t = \exp(bt) S_t = \exp(bt)\exp(\sigma B_t - \sigma^2 t/2).$$

Exercise 12.3 Let D be a bounded domain in $\mathbb{R}^d$ with regular boundary, α be a continuous bounded function in D. Use Itô's formula to show that if U is continuous in $\overline{D}$, C^2 in D,

$$\Delta U(x) = -2\alpha(x), \quad \forall x \in D$$

and U equal to 0 on the boundary of D, then for $x \in \overline{D}$,

$$U(x) = E_x \left(\int_0^T \alpha(B_s) ds \right),$$

where $T = \inf\{t \geq 0, B_t \notin D\}$.

Solution 12.3

Assume that F is a solution to the above similar problem to the Dirichlet problem. We show that it must equal to U. Let T be the hitting time of B at the boundary of D. Consider the process

$$Z_t = F(B_t^T) + \int_0^t \alpha(B_s^T) ds.$$

Since F is C^2, we can apply Itô's formula: for $t < T$,

$$dZ_t = \nabla F(B_t) \cdot dB_t + \alpha(B_t) dt + \frac{1}{2} \Delta F(B_t) dt = \nabla F(B_t) \cdot dB_t.$$

Therefore Z is a continuous local martingale. Moreover,

$$E\left(\sup_{t\geq 0} |Z_t|\right) \leq \|F\|_\infty + \|\alpha\|_\infty E(T) < \infty, \tag{2}$$

it implies that Z is a uniformly integrable martingale. The uniform integrability follows from

$$\sup_{t \geq 0} E\left(|Z_t| 1_{|Z_t|>A}\right) \leq E\left(\sup_{t\geq 0} |Z_t| 1_{\sup_{t\geq 0}|Z_t|>A}\right) \xrightarrow{A\to\infty} 0.$$

It also yields that Z is a martingale. In fact, let (τ_n) be a family of stopping times which converges to ∞ almost surely, such that Z^{τ_n} is a martingale. Then we have for $s \leq t$,

$$E[Z_t^{\tau_n} | \mathcal{F}_s] = Z_s^{\tau_n}.$$

The inequality (2) gives that $(Z_s^{\tau_n})_{n\geq 0}$ is a uniformly integrable family, which converges almost surely and in L^1 to Z_s as $n \to \infty$, therefore $E[Z_t|\mathcal{F}_s] = Z_s$, which says that Z is a (u.i.) martingale. Therefore from the L^1 martingale convergence theorem, Z_t converges in L^1 to

$$Z_\infty = F(B_T) + \int_0^T \alpha(B_s) ds = \int_0^T \alpha(B_s) ds,$$

since $F = 0$ on the boundary. One gets

$$F(x) = E[Z_0] = E[Z_\infty] = E\left(\int_0^T \alpha(B_s) ds\right) = U(x).$$

Exercise 12.4 Feynman-Kac. Let $f \in C_b^2(\mathbb{R}^d)$ and $V \in C_b(\mathbb{R}^d)$. Suppose that $u \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + V \times u \text{ on } \mathbb{R}_+ \times \mathbb{R}^d \text{ and } u(0,\cdot) = f \text{ on } \mathbb{R}^d.$$

Fix $T \in (0,\infty)$, consider the process

$$M_t = u(T-t, B_t)E_t, \text{ where } E_t = \exp\left(\int_0^t V(B_s)ds\right).$$

Show that M is a local martingale, and for all $T \in \mathbb{R}_+$, $x \in \mathbb{R}^d$ we have that

$$u(T,x) = E_x\left[f(B_T)\exp\left(\int_0^T V(B_s)ds\right)\right]$$

where B is a standard Brownian motion.

Solution 12.4 Let $\dot{u}$ denote the derivative of u with respect to the first parameter and $\partial_i u$ denote the derivative of u with respect to the i-th coordinate of its second parameter, B^i the i-th coordinate of B, for $i \leq d$. By Itô's formula, we have that

$$\begin{aligned} dM_t = &\left(\sum_i \partial_i u(T-t,B_t)dB_t^i + \frac{1}{2}\Delta u(T-t,B_t)dt - \dot{u}(T-t,B_t)dt\right)E_t \\ &+ E_t u(T-t,B_t)V(B_t)dt. \end{aligned}$$

Since u is solution to the PDE, the drift term vanishes and M is a local martingale. We have also that M is uniformly bounded on $[0,T]$ as $u \in C_b^{1,2}$ and E_t is bounded by $\exp(T\|V\|_\infty)$ on $[0,T]$. Therefore M is a martingale. Hence

$$u(T,x) = M_0 = E_x(M_T) = E_x(f(B_T)E_T)$$

as desired.

Exercise 12.5 Tanaka's example. When f is a smooth function with $f(0) = 0$, recall that $\int_0^t \operatorname{sgn}(f(s))df(s) = |f(t)|$, where $\operatorname{sgn}(x) = 1_{x\geq 0} - 1_{x<0}$. The goal of this exercise is to show how different things are when one replaces f by Brownian motion:

(a) Show that if X is a (one-dimensional) Brownian motion, then

$$B_t := \int_0^t \operatorname{sgn}(X_s)dX_s$$

is a Brownian motion.

(b) Show that $X_t = \int_0^t \operatorname{sgn}(X_s)dB_s$.

(c) Show that if $Y = -X$, then $Y_t = \int_0^t \operatorname{sgn}(Y_s)dB_s$.

Solution 12.5

(a) The process B_t defined as $\int_0^t \operatorname{sgn}(X_s)dX_s$ is a local martingale. The quadratic variation is given by

$$\langle B \rangle_t = \int_0^t \operatorname{sgn}(X_s)^2 d\langle X \rangle_s = \int_0^t 1ds = t.$$

We conclude that B is a Brownian motion from Lévy's characterization of Brownian motion.

(b) From Exercise 11.2,

$$\operatorname{sgn}(X) \cdot B = \operatorname{sgn}(X) \cdot (\operatorname{sgn}(X) \cdot X) = (\operatorname{sgn}(X))^2 \cdot X = X,$$

which means almost surely for all $t \geq 0$, $X_t = \int_0^t \operatorname{sgn}(X_s)dB_s$.

(c) If $Y = -X$, then

$$Y = -\operatorname{sgn}(X) \cdot B = \operatorname{sgn}(Y) \cdot B - 21_{X=0} \cdot B.$$

The local martingale

$$Z_t := (1_{X=0} \cdot B)_t = \int_0^t 1_{X_s=0}dB_s$$

has quadratic variation $\langle Z \rangle_t = \int_0^t 1_{X_s=0}ds$. Its expected value is given by

$$E(\langle Z \rangle_t) = E\left(\int_0^t 1_{X_s=0}ds\right) = \int_0^t \mathrm{P}(X_s = 0)ds = 0$$

therefore $\langle Z \rangle$ is almost surely constantly zero which implies that $Z = 0$. We conclude that $Y = \operatorname{sgn}(Y) \cdot B$.

Exercise 12.6 Bessel processes. Let B be a d-dimensional Brownian motion started from 0, $d \geq 2$.

(a) Write the semimartingale decomposition of $Z_t := \|B_t\|^2$.

(b) Using Lévy's characterization of Brownian motion, show that there exists a (one-dimensional) Brownian motion β such that

$$Z_t = \int_0^t 2\sqrt{Z_s}\,\mathrm{d}\beta_s + d \times t.$$

This is the squared Bessel SDE, and Z is called the squared Bessel process of dimension d.

(c) Show that if $X = \sqrt{Z} = \|B_t\|$, then

$$X_t = \beta_t + \frac{d-1}{2}\int_0^t \frac{\mathrm{d}s}{X_s}.$$

The solution X is called the Bessel process of dimension d.

Notice that the Bessel SDEs still make sense when d is not integer. Let $d > 2$, and X be the solution of

$$\mathrm{d}X_t = \mathrm{d}\beta_t + \frac{d-1}{2X_t}\,\mathrm{d}t, \quad X_0 = 0.$$

(d) Show that $(M_t := X_{t+1}^{2-d})_{t\geq 0}$ is a local martingale for the filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0} := (\mathcal{F}_{t+1})_{t\geq 0}$.

(e) Show that for $\lambda > 1$, the process

$$\tilde{M}_t := \lambda^{\alpha} M_{\lambda t + \lambda - 1}$$

has the same law as M, where $\alpha = (d-2)/2$.

(f) Assume that $d \geq 3$ is an integer, show that $E(M_0) < \infty$ and deduce that $E(M_t)$ is decreasing with t, and M can not be a martingale.

Solution 12.6

(a) We write $B_t = (B_{1,t}, \cdots, B_{d,t})$. Using Itô's formula,

$$\mathrm{d}Z_t = \mathrm{d}\left(\sum_{i=1}^d B_{i,t}^2\right) = \sum_{i=1}^d (2B_{i,t}\,\mathrm{d}B_{i,t} + \mathrm{d}t).$$

That is

$$Z_t = \sum_{i=1}^d \int_0^t (2B_{i,s}\,\mathrm{d}B_{i,s}) + d \times t.$$

(b) The quadratic variation of $Y_t := \sum_{i=1}^d \int_0^t (2B_{i,s}\,\mathrm{d}B_{i,s})$ equals to

$$\int_0^t \sum_{i=1}^d 4B_{i,s}^2\,\mathrm{d}s = \int_0^t 4Z_s\,\mathrm{d}s = \int_0^t (2\sqrt{Z_s})^2\,\mathrm{d}s.$$

Therefore let $\beta = (2\sqrt{Z})^{-1}\cdot Y$, we have

$$\langle \beta\rangle_t = \left\langle (2\sqrt{Z})^{-1}\cdot Y\right\rangle_t = t,$$

from Lévy's characterization of Brownian motion, we have that β is a Brownian motion and $Y = (2\sqrt{Z})\cdot\beta$. Therefore almost surely,

$$Z_t = Y_t + d\times t = \int_0^t 2\sqrt{Z_s}\,\mathrm{d}\beta_s + d\times t.$$

(c) Let $T = \inf\{t > 0, B_t = 0\}$ which is almost surely ∞. The function $z\mapsto\sqrt{z}$ is C^∞ outside of $\{0\}$. Let $\varepsilon > 0$, from Itô's formula, for $\varepsilon \le t < T$,

$$\begin{aligned}\mathrm{d}X_t = \mathrm{d}\sqrt{Z_t} &= \frac{\mathrm{d}Z_t}{2\sqrt{Z_t}} - \frac{1}{2}\frac{\mathrm{d}\langle Z\rangle_t}{4\sqrt{Z_t}^3}\\ &= \frac{2\sqrt{Z_t}\,\mathrm{d}\beta_t + d\times\,\mathrm{d}t}{2\sqrt{Z_t}} - \frac{1}{2}\frac{\mathrm{d}\langle Z\rangle_t}{4\sqrt{Z_t}^3}.\end{aligned}$$

Since $\mathrm{d}\langle Z\rangle_t = \mathrm{d}\left\langle 2\sqrt{Z}\cdot\beta\right\rangle_t = 4Z_t\,\mathrm{d}t$, we obtain that

$$\mathrm{d}X_t = \mathrm{d}\beta_t + \frac{d-1}{2}\frac{1}{X_t}\,\mathrm{d}t,$$

which implies that almost surely,

$$X_t = X_\varepsilon + \beta_t - \beta_\varepsilon + \frac{d-1}{2}\int_\varepsilon^t \frac{1}{X_s}\,\mathrm{d}s.$$

From the definition of X, as $\varepsilon\to 0$, $X_\varepsilon \to 0$ almost surely, therefore the integral on the right-hand side has a limit almost surely as $\varepsilon\to 0$ and we get that

$$X_t = \beta_t + \frac{d-1}{2}\int_0^t \frac{1}{X_s}\,\mathrm{d}s.$$

(d) We use the Itô's formula to M:

$$\begin{aligned}\mathrm{d}M_t &= (2-d)X_t^{1-d}\,\mathrm{d}X_t + \frac{(2-d)(1-d)}{2}X_t^{-d}\,\mathrm{d}\langle X\rangle_t\\ &= (2-d)X_t^{1-d}\,\mathrm{d}\beta_t + (2-d)X_t^{1-d}\frac{d-1}{2X_t}\,\mathrm{d}t + \frac{(2-d)(1-d)}{2}X_t^{-d}\,\mathrm{d}t\\ &= (2-d)X_t^{1-d}\,\mathrm{d}\beta_t = (2-d)M_t^{\frac{1-d}{2-d}}\,\mathrm{d}\beta_t.\end{aligned}$$

which shows that M is a local martingale.

(e) First notice that the process $\tilde{X}_t := \lambda^{-1/2} X_{\lambda t}$ solves the same equation as X. In fact

$$\mathrm{d}\tilde{X}_t = \lambda^{-1/2}\,\mathrm{d}X_{\lambda t} = \lambda^{-1/2}(\,\mathrm{d}\beta_{\lambda t} + \frac{d-1}{2X_{\lambda t}}\,\mathrm{d}\lambda t) = \lambda^{-1/2}\,\mathrm{d}\beta_{\lambda t} + \frac{d-1}{2\tilde{X}_t}\,\mathrm{d}t = \,\mathrm{d}\tilde{\beta}_t + \frac{d-1}{2\tilde{X}_t}\,\mathrm{d}t,$$

where $\tilde{\beta}_t = \lambda^{-1/2}\beta_{\lambda t}$, and $\tilde{X}_0 = 0$. Therefore

$$\tilde{M}_t = \tilde{X}_{t+1}^{2-d} = \lambda^{(d-2)/2} X_{\lambda t+\lambda}^{2-d} = \lambda^{\alpha} M_{\lambda t+\lambda-1}$$

has the same law as M_t and

$$E(M_t) = E(\tilde{M}_t) = \lambda^{\alpha} E(M_{\lambda t+\lambda-1}).$$

(f) We show the integrabiliby of M_0 when $d \geq 3$ is an integer, so that we have the interpretation of the Bessel process as the norm of the standard Brownian motion B starting from 0, and

$$M_0 = \|B_1\|^{2-d}.$$

As B_1 has a density in $\mathbb{R}^d$, we have for all $u \geq 1$

$$\mathrm{P}(M_0 \geq u) = \mathrm{P}(\|B_1\|^{2-d} \geq u) = \mathrm{P}(\|B_1\| \leq u^{1/(2-d)}) \leq C u^{d/(2-d)} = \frac{C}{u^{d/(d-2)}}.$$

Therefore

$$E(M_0) \leq 1 + \int_1^{\infty} P(M_0 \geq u)\,\mathrm{d}u < \infty.$$

Since $\alpha > 0$, we have $\lambda^{\alpha} > 1$ as $\lambda > 1$. From (e), which shows that for $0 \leq t < s$, and $\lambda = (s+1)/(t+1) > 1$, we have

$$E(M_t) = \lambda^{\alpha} E(M_s) > E(M_s),$$

since similarly $E(M_s)$ is bounded by $E(M_0)$ therefore finite, the strict inequality makes sense. In consequence M is not a martingale.

Exercise 12.7 Let $\sigma : \mathbb{R} \to \mathbb{R}_{>0}$ be a continuous function (not necessarily bounded). The goal of the exercise is to show that any (weak) solution to the one-dimensional stochastic differential equation

$$dX_t = \sigma(X_t)dB_t, \quad X_0 = 0 \tag{3}$$

cannot explode. i.e. $T = \infty$ where T be the explosion time of the solution (here it means that X is defined for all $t \geq 0$).

(a) Show that there exists a Brownian motion β such that $X_t = \beta_{\langle X \rangle_t}$, and that $\langle X \rangle_t \to \infty$ as $t \to T$.

(b) Using $dB_t = (1/\sigma(X_t))dX_t$ and the fact that the one-dimensional Brownian motion is recurrent, show that $T = \infty$ almost surely.

Solution 12.7

(a) If (X, B) is a solution to (3), then X is a local martingale. From Corollary 4.15, there exists such a Brownian motion β. Since $\{X_t, t < T\} = \{\beta_{\langle X \rangle_t}, t < T\}$ is unbounded, therefore $\langle X \rangle_t \to \infty$ as $t \to \infty$.

(b) Since $\sigma(x) > 0$ for all $x \in \mathbb{R}$ and $dB_t = (1/\sigma(X_t))dX_t$,

$$t = \langle B \rangle_t = \int_0^t \frac{1}{\sigma(X_s)^2} d\langle X \rangle_s = \int_0^t \frac{1}{\sigma(\beta_{\langle X \rangle_s})^2} d\langle X \rangle_s = \int_0^{\langle X \rangle_t} \frac{1}{\sigma(\beta_r)^2} dr.$$

From the recurrence of one dimensional Brownian motion (that β visits every real number infinitely many times), it is not hard to see that the right-most integral goes to ∞ as $\langle X \rangle_t \to \langle X \rangle_T = \infty$ almost surely. Therefore $T = \infty$ almost surely.

Hand in before: June 1.

Location: During the exercise class or in the tray outside of HG E 65.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
Al-Fr	Fri 8-9	HG E 21	Martin Stefanik
Ga-Lag	Fri 9-10	HG E 21	Martin Stefanik
Lan-Sche	Fri 8-9	LFW E 13	Mayra Bermudez
Scho-Zim	Fri 12-13	HG E 22	Yilin Wang