Appendix C: Some martingale results in discrete time

This section contains a number of results on martingales and stochastic integrals in discrete time. We formulate them for a general probability space, but point out that any conditions like integrability or boundedness are trivially satisfied whenever the underlying space Ω is finite and the time horizon is finite as well.

We start with a probability space $(\Omega, \mathcal{F}, P)$ and a filtration in discrete time given by $I\!F = (\mathcal{F}_k)_{k=0,1,2,\ldots} = (\mathcal{F}_k)_{k \in I\!N_0}$. We sometimes assume that $\mathcal{F}_0$ is P-trivial, but this is not needed in general. We also sometimes look at processes indexed only by $k = 0, 1, \ldots, T$ for some $T \in I\!N$.

Definition. A stochastic process $M = (M_k)_{k \in I\!N_0}$ is called a *martingale (with respect to P and $I\!F$)* if

(M1) M is adapted to $I\!F$, meaning that M_k is $\mathcal{F}_k$-measurable for all k;

(M2) M is P-integrable, meaning that $E[|M_k|] < \infty$ or $M_k \in L^1(P)$, for all k;

(M3) M satisfies the martingale property that $E[M_\ell \,|\, \mathcal{F}_k] = M_k$ P-a.s. for all $k \le \ell$.

If instead of (M3) we have

$$(\mathrm{M^s3}) \qquad E[M_\ell \,|\, \mathcal{F}_k] \le M_k \qquad P\text{-a.s. for all } k \le \ell,$$

then M is called a *supermartingale*; if instead of (M3) we have

$$(\mathrm{M_s3}) \qquad E[M_\ell \,|\, \mathcal{F}_k] \ge M_k \qquad P\text{-a.s. for all } k \le \ell,$$

then M is called a *submartingale*.

Remarks. 1) The martingale property (M3) is equivalent to

$$E[\Delta M_k \,|\, \mathcal{F}_{k-1}] = 0 \qquad P\text{-a.s. for all } k \in I\!N.$$

If we only look at a martingale $M = (M_k)_{k=0,1,\dots,T}$ in finite discrete time, then (M3) is also equivalent to

$$E[M_T \mid \mathcal{F}_k] = M_k \qquad P\text{-a.s. for all } k = 0, 1, \dots, T.$$

For sub- and supermartingales, the first equivalence also holds, with "=" of course replaced by "$\geq$" and "$\leq$", respectively. However, the second equivalence is specific to the martingale case.

2) An analogous definition can be used for an $I\!R^m$-valued process by simply imposing the conditions on each coordinate.

Example. A first standard example for a martingale is given by *successive partial sums of independent centered random variables*. Suppose that $(Y_j)_{j \in I\!N}$ are independent and integrable random variables with $E[Y_j] \equiv 0$ and define $M_k := \sum_{j=1}^k Y_j$ (with $M_0 = 0$ by the usual convention that an empty sum is zero) as well as $\mathcal{F}_k := \sigma(Y_1, \dots, Y_k) = \sigma(M_0, M_1, \dots, M_k)$ for $k \in I\!N_0$. Then M is a martingale with respect to P and $I\!F$; this follows immediately because $\Delta M_k = Y_k$ is independent of $\mathcal{F}_{k-1}$. In complete analogy, $N_k := \prod_{j=1}^k R_j$ is a martingale (with $N_0 = 1$) if $(R_j)_{j \in I\!N}$ are independent and integrable with $E[R_j] \equiv 1$.

Example. A second standard example is given by *successive predictions*. Suppose we are given a filtration $I\!F$, let Y be an integrable random variable and define $M_k := E[Y \mid \mathcal{F}_k]$ for $k \in I\!N_0$. Using the projectivity of conditional expectations then easily shows that M is a martingale (with respect to P and $I\!F$).

Martingales form a large class of stochastic processes and have many important and useful properties. Our first result shows that a stochastic integral with respect to a martingale is again a martingale if the integrand is sufficiently integrable.

Proposition C.1. *Suppose M is an $I\!R^m$-valued martingale and $H = (H_k)_{k \in I\!N}$ an $I\!R^m$-valued bounded predictable process. Then the stochastic integral $H \cdot M = \int H \, dM$ is again a* *martingale.*

Proof. It is clear that $\int H\,dM$ is adapted and also integrable, because H is bounded. For each k and each coordinate i, H^i_k is bounded and $\mathcal{F}_{k-1}$-measurable; so we have

$$E[H^i_k \Delta M^i_k \,|\, \mathcal{F}_{k-1}] = H^i_k E[\Delta M^i_k \,|\, \mathcal{F}_{k-1}] = 0 \qquad P\text{-a.s.}$$

because M is a martingale, and this implies that P-a.s.,

$$E\left[\Delta\left(\int H\,dM\right)_k \,\middle|\, \mathcal{F}_{k-1}\right] = E[H_k \cdot \Delta M_k \,|\, \mathcal{F}_{k-1}] = \sum_{i=1}^{k} E[H^i_k \Delta M^i_k \,|\, \mathcal{F}_{k-1}] = 0.$$

So $\int H\,dM$ is indeed a martingale as well. **q.e.d.**

Remark. An extension of Proposition C.1 from martingales to sub- or supermartingales is not true in general. However, there is one important exception: If $n = 1$ and M is a sub- or supermartingale, then $\int H\,dM$ is again a sub- or supermartingale, respectively, if H is bounded, predictable and in addition nonnegative. (The proof is left as an exercise.)

Definition. If τ is a stopping time (with respect to $\mathbb{F}$), we call

$$\mathcal{F}_\tau := \big\{A \in \mathcal{F} \,\big|\, A \cap \{\tau \le k\} \in \mathcal{F}_k \text{ for all } k \in \mathbb{N}_0\big\}$$

the σ-field of *events observable up to time* τ.

Remarks. 1) One can (and should) check easily that $\mathcal{F}_\tau$ is indeed a σ-field. One can and should also check that if $\tau \equiv k_0$, then $\mathcal{F}_\tau = \mathcal{F}_{k_0}$ so that there is no abuse of notation.

2) If σ and τ are stopping times with $\sigma \le \tau$, then we also have $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$. Indeed, $\sigma \le \tau$ implies that we have $\{\tau \le k\} = \{\sigma \le k, \tau \le k\}$ for all k; so if $A \in \mathcal{F}_\sigma$ and k is fixed, then

$$A \cap \{\tau \le k\} = (A \cap \{\sigma \le k\}) \cap \{\tau \le k\} \in \mathcal{F}_k$$

and hence also $A \in \mathcal{F}_\tau$.

Definition. Let $Y = (Y_k)_{k \in \mathbb{N}_0}$ be an adapted stochastic process and τ a stopping time. The *value of Y at time τ* is then defined by

$$(Y_\tau)(\omega) := \big(Y_{\tau(\omega)}\big)(\omega),$$

provided that τ is finite-valued (i.e. $\tau < \infty$ P-a.s.). The *process Y stopped in τ* is defined by $Y^\tau = (Y_k^\tau)_{k \in \mathbb{N}_0}$ with

$$Y_k^\tau(\omega) := Y_{k \wedge \tau}(\omega) = \begin{cases} Y_k(\omega) & \text{for } k \leq \tau(\omega) \\ Y_\tau(\omega) & \text{for } k > \tau(\omega) \end{cases}$$

and for $k \in \mathbb{N}_0$.

Note that Y_τ is a random variable, whereas Y^τ is a stochastic process and again adapted to $\mathbb{F}$. The measurability for both of these statements follows from the next result.

Lemma C.2. *Suppose that Y is an $\mathbb{F}$-adapted (real-valued) process and τ a finite-valued stopping time (with respect to $\mathbb{F}$). Then the mapping $Y_\tau : \Omega \to \mathbb{R}$ is $\mathcal{F}_\tau$-measurable.*

Proof. We need to show that for every $c \in \mathbb{R}$, the set $\{Y_\tau \leq c\}$ is in $\mathcal{F}_\tau$. But for any fixed k, we have

$$\{Y_\tau \leq c\} \cap \{\tau \leq k\} = \bigcup_{j=0}^{k} \{Y_\tau \leq c, \tau = j\} = \bigcup_{j=0}^{k} \{Y_j \leq c, \tau = j\},$$

and for every $j \leq k$, we have both $\{Y_j \leq c\} \in \mathcal{F}_j \subseteq \mathcal{F}_k$ because Y is adapted so that Y_j is $\mathcal{F}_j$-measurable, and $\{\tau = j\} \in \mathcal{F}_j \subseteq \mathcal{F}_k$ because τ is a stopping time. Hence we obtain $\{Y_\tau \leq c\} \cap \{\tau \leq k\} \in \mathcal{F}_k$ for all k, and so $\{Y_\tau \leq c\} \in \mathcal{F}_\tau$. **q.e.d.**

Proposition C.3. *If M is a martingale and τ a stopping time, then the stopped process M^τ is again a martingale. The same result is true for sub- and supermartingales.*

Proof. Since we can argue for each coordinate, we may assume without loss of generality that M is real-valued. Define $H_k := I_{\{k \le \tau\}}$ for $k \in I\!N$. Then H is bounded and predictable because $\{\tau \ge k\} = \{\tau \le k-1\}^c \in \mathcal{F}_{k-1}$ since τ is a stopping time. By Proposition C.1, the process

$$\left(\int H \, dM \right)_k = \sum_{j=1}^{k} I_{\{j \le \tau\}} \Delta M_j = \sum_{j=1}^{k \wedge \tau} \Delta M_j = M_{k \wedge \tau} - M_0 = M_k^\tau - M_0, \qquad k \in I\!N,$$

is therefore a martingale, and so is then M^τ. The same argument also works for sub- and supermartingales because H is nonnegative. **q.e.d.**

Definition. A stochastic process $M = (M_k)_{k \in I\!N_0}$ is called a *local martingale (with respect to P and $I\!F$)* if M is $I\!F$-adapted and there exists a sequence $(\tau_n)_{n \in I\!N}$ of stopping times with $\tau_n \nearrow \infty$ P-a.s. and such that for each n, the stopped process $M^{\tau_n} I_{\{\tau_n > 0\}}$ is a martingale. The sequence $(\tau_n)_{n \in I\!N}$ is then called a *localising sequence*.

Remarks. 1) The indicator function $I_{\{\tau_n > 0\}}$ appears because one wants to avoid imposing any integrability conditions on M_0. If $M_0 = 0$ or if M_0 is nonrandom, one can equivalently ask that M^{τ_n} should be a martingale. This applies in particular in the frequently encountered case when $\mathcal{F}_0$ is trivial.

2) If M is indexed by $k = 0, 1, \dots, T$ with some $T \in I\!N$, the requirement for the localising sequence is that $\tau_n \nearrow T$ P-a.s. Because the time index is discrete, this is equivalent to the requirement that the sequence is increasing and $P[\tau_n < T] \to 0$ as $n \to \infty$.

3) Clearly every martingale is a local martingale; it is enough to take $\tau_n \equiv \infty$ (or $\tau_n \equiv T$ in the case of a finite time horizon T).

The notion of a local martingale allows us to extend Proposition C.1 from bounded to arbitrary predictable processes as integrands, at the mere cost of a localisation. Importantly, this result does not generalise to continuous time.

Proposition C.4. *Suppose M is an $\mathbb{R}^m$-valued local martingale and $H = (H_k)_{k\in\mathbb{N}}$ is an $\mathbb{R}^m$-valued predictable process. Then the stochastic integral $H \cdot M = \int H\,dM$ is again a local martingale.*

Proof. Let (τ_n) be a localising sequence for M and set

$$\sigma_n := \inf\big\{ j \in \mathbb{N}_0 \,\big|\, |H_{j+1}| > n \big\}.$$

Then $\{\sigma_n > k\} = \{|H_1| \le n, \dots, |H_{k+1}| \le n\}$ is in $\mathcal{F}_k$ because H is predictable, and so σ_n is a stopping time. Moreover, $\sigma_n \nearrow \infty$ P-a.s. because H is a finite-valued process. Therefore $\varrho_n := \sigma_n \wedge \tau_n$, $n \in \mathbb{N}$, is a sequence of stopping times with $\varrho_n \nearrow \infty$ and

$$\begin{aligned}
\Big(\int H\,dM\Big)^{\varrho_n}_k &= \sum_{j=1}^{k\wedge\varrho_n} H_j \cdot \Delta M_j \\
&= \sum_{j=1}^{k} I_{\{j\le\varrho_n\}} H_j \cdot \Delta M_j^{\varrho_n} I_{\{\varrho_n>0\}} \\
&= \Big(\int (H \centerdot I_{\{\centerdot\le\varrho_n\}})\, d(M^{\varrho_n} I_{\{\varrho_n>0\}})\Big), \qquad k \in \mathbb{N}_0,
\end{aligned}$$

is a martingale by Proposition C.1, because $M^{\varrho_n} I_{\{\varrho_n>0\}} = (M^{\tau_n} I_{\{\tau_n>0\}})^{\sigma_n}$ is a martingale by Proposition C.3 and $H \centerdot I_{\{\centerdot\le\varrho_n\}}$ is predictable and bounded by construction. This gives the result since $\int H\,dM$ is null at 0. **q.e.d.**

The next result is very useful in many applications in mathematical finance in discrete time. We point out that it does not have any analogue in continuous time.

Theorem C.5. *Suppose $L = (L_k)_{k\in\mathbb{N}_0}$ is a real-valued local martingale. If $E[|L_0|] < \infty$ and $E[L_T^-] < \infty$ for some $T \in \mathbb{N}$, then the stopped process $L^T = (L_k)_{k=0,1,\dots,T}$ is a (true) martingale.*

Proof. Let (τ_n) be a localising sequence for L. Then $I_{\{\tau_n > k-1\}} \nearrow 1$ P-a.s. as $n \to \infty$, for every $k \in \mathbb{N}$.

1) We first show inductively that $E[L_k^-] < \infty$ for all $k = 1, \dots, T-1$. (For $k = 0$ and $k = T$, this holds by assumption.) Indeed, because $\{\tau_n > k-1\} \in \mathcal{F}_{k-1}$ and $L^{\tau_n} I_{\{\tau_n > 0\}}$ is a martingale, the inequality $x^- \geq -x$ yields

$$\begin{aligned} E[L_k^- \,|\, \mathcal{F}_{k-1}] I_{\{\tau_n > k-1\}} &= E\big[(L_k^{\tau_n})^- I_{\{\tau_n > 0\}} \,\big|\, \mathcal{F}_{k-1}\big] I_{\{\tau_n > k-1\}} \\ &\geq -L_{k-1}^{\tau_n} I_{\{\tau_n > 0\}} I_{\{\tau_n > k-1\}} \\ &= -L_{k-1} I_{\{\tau_n > k-1\}} \qquad P\text{-a.s.} \end{aligned}$$

Letting $n \to \infty$, we obtain

$$E[L_k^- \,|\, \mathcal{F}_{k-1}] \geq \max(0, -L_{k-1}) = L_{k-1}^- \qquad P\text{-a.s.}$$

and therefore $E[L_{k-1}^-] \leq E[L_k^-]$. This gives the assertion above because $E[L_T^-] < \infty$ by assumption.

2) We next show that $E[|L_k|] < \infty$ for all $k = 1, \dots, T$ so that the stopped process L^T is integrable. Indeed, using $\tau_n \nearrow \infty$, Fatou's lemma and the martingale property of $L^{\tau_n} I_{\{\tau_n > 0\}}$ gives

$$\begin{aligned} E[L_k^+] &= E\Big[\lim_{n\to\infty} L_{k\wedge\tau_n}^+ I_{\{\tau_n > 0\}}\Big] \\ &\leq \liminf_{n\to\infty} E[L_{k\wedge\tau_n}^+ I_{\{\tau_n > 0\}}] \\ &= \liminf_{n\to\infty} E[L_k^{\tau_n} I_{\{\tau_n > 0\}} + L_{k\wedge\tau_n}^- I_{\{\tau_n > 0\}}] \\ &= E[L_0 I_{\{\tau_n > 0\}}] + \liminf_{n\to\infty} E[L_{k\wedge\tau_n}^- I_{\{\tau_n > 0\}}]. \end{aligned}$$

By Step 1), the sum $\sum_{j=0}^T L_j^-$ is an integrable majorant for every $L_{k\wedge\tau_n}^-$ so that we obtain directly

$$E[L_k^+] \leq E[|L_0|] + \sum_{j=0}^T E[L_j^-] < \infty$$

by the assumption and Step 1).

3) To show the martingale property of L, we note that for all $k = 0, 1, \dots, T$ and $n \in I\!N$,

$$|L_k^{\tau_n}| I_{\{\tau_n > 0\}} \le \max_{j=0,1,\dots,T} |L_j| \le \sum_{j=0}^{T} |L_j| \in L^1(P)$$

by Step 2). Moreover, $L^{\tau_n} I_{\{\tau_n>0\}}$ is a martingale so that we obtain by dominated convergence for $k \ge 1$ that

$$\begin{aligned} E[L_k \,|\, \mathcal{F}_{k-1}] &= E\Big[\lim_{n\to\infty} L_{k\wedge\tau_n} I_{\{\tau_n>0\}} \,\Big|\, \mathcal{F}_{k-1}\Big] \\ &= \lim_{n\to\infty} E[L_{k\wedge\tau_n} I_{\{\tau_n>0\}} \,|\, \mathcal{F}_{k-1}] \\ &= \lim_{n\to\infty} L_{(k-1)\wedge\tau_n} I_{\{\tau_n>0\}} \\ &= L_{k-1} \qquad P\text{-a.s.} \end{aligned}$$

This completes the proof. **q.e.d.**

Corollary C.6. **1)** *Suppose L is a real-valued local martingale with $E[|L_0|] < \infty$ and $L \ge -a$ for some $a \ge 0$, in the sense that $L_k \ge -a$ P-a.s. for all $k \in I\!N_0$. Then $L = (L_k)_{k \in I\!N_0}$ is a (true) martingale.*

2) *Suppose M is an $I\!R^m$-valued local martingale. For any $I\!R^m$-valued predictable process $H = (H_k)_{k\in I\!N}$ with $\int H\,dM \ge -a$ for some constant $a \ge 0$, the stochastic integral process $H \cdot M = \int H\,dM$ is a (true) martingale.*

Proof. 1) This follows directly from Theorem C.5 because $L_T^- \in L^1(P)$ for every $T \in I\!N$.

2) We know from Proposition C.4 that $L := H \cdot M$ is a real-valued local martingale. So we can apply part 1) to get the result. **q.e.d.**

Remark. Imposing the (boundedness or integrability) condition on L^- is natural in the context of mathematical finance, as we shall see later. However, from a purely mathematical perspective, we could equally well impose the analogous condition on L^+ and obtain the same conclusion by considering $-L$ instead of L.

The next result is a very convenient characterisation of martingales in finite discrete time.

Lemma C.7. *Let $Y = (Y_k)_{k=0,1,\dots,T}$ be an $\mathbb{R}^m$-valued adapted integrable stochastic process. Then the following are equivalent:*

1) *Y is a martingale.*

2) *$E[(\int H\,dY)_T] = 0$ for all $\mathbb{R}^m$-valued bounded predictable processes $H = (H_k)_{k=1,\dots,T}$.*

3) *$E[Y_\tau] = E[Y_0]$ for all stopping times τ with values in $\{0, 1, \dots, T\}$.*

Proof. "1) $\Rightarrow$ 2)": For every H as in 2), $L := \int H\,dY$ is a martingale by Proposition C.1, and so $E[L_T] = E[L_0]$.

"2) $\Rightarrow$ 3)": From the proof of Proposition C.3, we can see that for each coordinate, $(Y^i)^\tau - Y^i_0 = \int H\,dY$ for some bounded predictable H. So the assertion follows because $(Y^i)^\tau_T = Y^i_\tau$.

"3) $\Rightarrow$ 1)": By arguing separately for each coordinate, we can assume without loss of generality that $n = 1$. We show that $E[Y_T \,|\, \mathcal{F}_k] = Y_k$ for $k = 0, 1, \dots, T$ by choosing a suitable stopping time τ. Fix k and $A \in \mathcal{F}_k$ and define $\tau := kI_A + TI_{A^c}$. Then τ is a stopping time because

$$\{\tau \le \ell\} = (\{k \le \ell\} \cap A) \cup (\{T \le \ell\} \cap A^c) = \begin{cases} \Omega & \text{for } \ell = T \\ A \in \mathcal{F}_k \subseteq \mathcal{F}_\ell & \text{for } k \le \ell < T \\ \emptyset & \text{for } k > \ell \end{cases}$$

is always in $\mathcal{F}_\ell$. Because T is also a stopping time, we obtain

$$E[Y_\tau] = E[Y_0] = E[Y_T]$$

and therefore

$$E[Y_k I_A] = E[Y_T I_A]$$

by the definition of τ. Since this holds for any $A \in \mathcal{F}_k$ and Y_k is $\mathcal{F}_k$-measurable, we obtain

$$Y_k = E[Y_T \,|\, \mathcal{F}_k] \qquad P\text{-a.s.},$$

and this proves the result. **q.e.d.**

If M is a martingale, we have

$$E[M_\ell \,|\, \mathcal{F}_k] = M_k \qquad P\text{-a.s. for } \ell \geq k.$$

We now want to show that this still holds if we replace the deterministic times $k \leq \ell$ by bounded stopping times $\sigma \leq \tau$.

Theorem C.8. (Stopping theorem) *Suppose $M = (M_k)_{k \in \mathbb{N}_0}$ is a martingale and σ, τ are stopping times with $\sigma \leq \tau \leq T$ P-a.s. for some $T \in \mathbb{N}$. Then*

$$E[M_\tau \,|\, \mathcal{F}_\sigma] = M_\sigma \qquad P\text{-a.s.},$$

i.e., the martingale property also holds at (bounded) stopping times.

Proof. By looking at the stopped process M^T and using that σ, τ are bounded by T, we can assume without loss of generality that M is only indexed by $k = 0, 1, \dots, T$. Moreover, both $M_\tau = \sum_{k=0}^T M_k I_{\{\tau = k\}}$ and M_σ are integrable so that the conditional expectation is well defined. Because of $\sigma \leq \tau$, we have $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$, and so it is enough to prove the case where $\tau = T$; indeed, if we consider the pairs (σ, T) and (τ, T), we obtain from the projectivity of conditional expectations that

$$M_\sigma = E[M_T \,|\, \mathcal{F}_\sigma] = E\big[E[M_T \,|\, \mathcal{F}_\tau] \,\big|\, \mathcal{F}_\sigma\big] = E[M_\tau \,|\, \mathcal{F}_\sigma] \qquad P\text{-a.s.}$$

So assume that $\tau = T$. Because M_σ is $\mathcal{F}_\sigma$-measurable by Lemma C.2, we only need to prove that $E[M_T I_A] = E[M_\sigma I_A]$ for any $A \in \mathcal{F}_\sigma$. But if $A \in \mathcal{F}_\sigma$, then $A \cap \{\sigma = k\} \in \mathcal{F}_k$ for all k and therefore, using the martingale property $E[M_T \,|\, \mathcal{F}_k] = M_k$ and that $M_k = M_\sigma$ on $\{\sigma = k\}$, we obtain

$$E[M_T I_A] = \sum_{k=0}^T E\big[M_T I_{A \cap \{\sigma = k\}}\big] = \sum_{k=0}^T E\big[M_k I_{A \cap \{\sigma = k\}}\big] = E[M_\sigma I_A].$$

This completes the proof. **q.e.d.**

Remark. We have imposed the assumption that σ, τ are bounded because we shall mostly work in a setting of finite discrete time. There are other versions of the stopping theorem which obtain the same conclusion under different conditions on τ and/or M. Without any conditions except the martingale and stopping time properties, however, the result is not true.