## Appendix D: The Kreps–Yan theorem

This section contains an important separation theorem proved independently by D. Kreps and J.-A. Yan around the same time. It is a crucial ingredient for proving most versions of the fundamental theorem of asset pricing and also comes up in the Bichteler–Dellacherie characterisation of semimartingales as good integrators in stochastic analysis.

We begin by recalling some concepts and results from functional analysis. Fix a probability space $(\Omega, \mathcal{F}, P)$. For $p \in [1, \infty)$, the dual of the space $L^p$ is $(L^p)^* = L^q$ with $q$ conjugate to $p$, meaning that $\frac{1}{p} + \frac{1}{q} = 1$. This is not true for $p = \infty$.

If we fix conjugate numbers $p, q$ both in $[1, \infty]$, the dual pairing between $L^p$ and $L^q$ is given by

$$(Y, Z) := E[YZ] \qquad \text{for } Y \in L^p, Z \in L^q.$$

For $p \in [1, \infty)$, the *weak topology* on $L^p$, denoted by $\sigma(L^p, L^q)$, is the coarsest topology on $L^p$ which makes all the linear functionals $Y \mapsto (Y, Z)$ continuous for all $Z \in L^q$. So a sequence $(Y_n)_{n \in I\!N} \subseteq L^p$ converges to $Y$ in $\sigma(L^p, L^q)$ if and only if $\lim_{n \to \infty} E[Y_n Z] = E[YZ]$ for each $Z \in L^q$.

For $p \in (1, \infty]$, the *weak* topology* on $L^p$, denoted by $\sigma(L^p, L^q)$, views $L^p$ as the dual of $L^q$ (which explains why we must take $p > 1$); it is the coarsest topology on $L^p$ which makes all the linear functionals $Y \mapsto (Y, Z)$ continuous for all $Z \in L^q$.

It is clear from the above definitions that for $1 < p < \infty$, the weak and the weak* topology coincide. For $p = 1$, there is only the weak topology on $L^1$, with $Y_n \to Y$ in $\sigma(L^1, L^\infty)$ if and only if $\lim_{n \to \infty} E[Y_n Z] = E[YZ]$ for each $Z \in L^\infty$. For $p = \infty$, there is only the weak* topology on $L^\infty$, with $Z_n \to Z$ in $\sigma(L^\infty, L^1)$ if and only if $\lim_{n \to \infty} E[Y Z_n] = E[YZ]$ for each $Y \in L^1$.

Finally, a convex subset of $L^p$, for $p \in [1, \infty)$, is weakly closed, i.e. closed for the weak topology $\sigma(L^p, L^q)$, if and only if it is (strongly) closed in $L^p$, i.e. for the norm-topology on $L^p$. Note that $p = \infty$ is again not allowed here.

After these preliminaries, we are now in a position to formulate and prove the announced separation result.

**Theorem D.1. (Kreps/Yan)** *Fix conjugate $p, q \in [1, \infty]$ and suppose that $C \subseteq L^p$ is a convex cone with $C \supseteq -L^p_+$ and $C \cap L^p_+ = \{0\}$. If $C$ is closed in $\sigma(L^p, L^q)$ (meaning that it is weak* closed if $p = \infty$), then there exists a probability measure $Q \approx P$ with $\frac{dQ}{dP} \in L^q$ and $E_Q[Y] \le 0$ for all $Y \in C$.*

**Proof.** The proof consists of a combination of a separation argument with an exhaustion argument and goes as follows.

1) For any fixed $x \in L^p_+ \setminus \{0\}$, the assumption gives $x \notin C$. The Hahn–Banach theorem thus allows us to strictly separate $x$ from $C$: there exists some $z_x \in L^q$ with $(x, z_x) > \alpha$ and $(Y, z_x) \le \alpha$ for all $Y \in C$. Because $C$ is a cone, we may take $\alpha = 0$. Choosing $Y := -I_{\{z_x < 0\}}$, which is in $C$ because $C \supseteq -L^p_+$, next gives $-E[z_x I_{\{z_x<0\}}] = (Y, z_x) \le 0$ and therefore $z_x \ge 0$, and because the separation is strict, we must have $z_x \not\equiv 0$ to avoid $(x, z_x) = 0$. So we can and do normalise $z_x$ to have $E[z_x] = 1$, for each $x$.

2) Now consider the family $\mathcal{G}$ of all sets $\Gamma_x := \{z_x > 0\} \in \mathcal{F}$, where $x$ runs through $L^p_+ \setminus \{0\}$. For any set $A \in \mathcal{F}$ with $P[A] > 0$, we have $P[A \cap \Gamma_x] > 0$ for some $\Gamma_x \in \mathcal{G}$; indeed, $I_A \in L^p_+ \setminus \{0\}$ and therefore we can take $x = I_A$ and use that

$$0 < E[I_A z_{I_A}] = E[I_A z_{I_A} I_{\{z_{I_A} > 0\}}] = E[I_A z_{I_A} I_{\Gamma_{I_A}}]$$

to conclude that we must have $P[A \cap \Gamma_{I_A}] > 0$. By Lemma D.2 below, this implies that the family $\mathcal{G}$ contains a countable subfamily of sets whose union has probability 1. So there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $L^p_+$ such that

$$P\left[\bigcup_{n=1}^{\infty} \Gamma_{x_n}\right] = P\left[\bigcup_{n=1}^{\infty} \{z_{x_n} > 0\}\right] = 1.$$

Defining $z := \sum_{n=1}^{\infty} 2^{-n} z_{x_n}$ therefore yields a random variable $z > 0$ $P$-a.s. which is in $L^q$ like all the $z_{x_n}$, and we also have $E[Yz] = \sum_{n=1}^{\infty} 2^{-n} E[Y z_{x_n}] \le 0$ for all $Y \in C$. Finally, monotone integration gives $E[z] = \sum_{n=1}^{\infty} 2^{-n} E[z_{x_n}] = 1$ so that $dQ := z\,dP$ gives the desired probability measure. **q.e.d.**

The following abstract result provides the missing step in the proof of Theorem D.1.

**Lemma D.2.** *Let $\Lambda \neq \emptyset$ be an index family and $\mathcal{G} = (\Gamma_\lambda)_{\lambda \in \Lambda}$ a family of sets in $\mathcal{F}$ such that any set $A \in \mathcal{F}$ with $P[A] > 0$ has a nontrivial intersection with some $\Gamma_\lambda \in \mathcal{G}$, meaning that $P[A \cap \Gamma_\lambda] > 0$. Then there exists an at most countable subfamily $(\Gamma_{\lambda_n})_{n \in I\!N}$ of sets in $\mathcal{G}$ whose union has probability 1.*

**Proof.** Suppose first that $\mathcal{G}$ is closed under countable unions. Then $\sup_{\lambda \in \Lambda} P[A_\lambda]$ is attained in some $\Gamma_{\lambda^*} \in \mathcal{G}$, because we can approximate the supremum along a sequence $(\Gamma_{\lambda_m})_{m \in I\!N}$ and take $\Gamma_{\lambda^*} := \bigcup_{m=1}^{\infty} \Gamma_{\lambda_m}$, which is in $\mathcal{G}$ by the above closedness assumption. If we had $P[\Gamma_{\lambda^*}^c] > 0$, we could find a set $\Gamma_\lambda \in \mathcal{G}$ with $P[\Gamma_{\lambda^*}^c \cap \Gamma_\lambda] > 0$ by the assumption on $\mathcal{G}$, and so we should get $P[\Gamma_\lambda \cup \Gamma_{\lambda^*}] > P[\Gamma_{\lambda^*}]$, contradicting the maximality of $\Gamma_{\lambda^*}$. So $\Gamma_{\lambda^*}$ has probability 1 and we can take the family consisting of this single set.

In general, we consider the family $\mathcal{G}'$ formed by all countable unions of sets from $\mathcal{G}$; this family satisfies the same assumption as $\mathcal{G}$. Applying the above argument to $\mathcal{G}'$ then gives the assertion. **q.e.d.**