REPRESENTATIONS OF GENERAL LINEAR GROUPS OVER $p-$ADIC FIELD

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Start of lecture 1

1. Introduction

The goal of this course is to present the theory of complex representations of general linear group over $p-$adic field. More precisely, denote

$$G = \mathrm{GL}_n, \text{ the invertible elements in matrix algebra } M_{n\times n},$$
$$\mathbb{F} = \ p\text{−adic field which is a finite field extension of } \mathbb{Q}_p,$$
$$\pi : G(\mathbb{F}) \to \mathrm{GL}(V), \text{ where } V \text{ is vector space over complex field } \mathbb{C}, \text{ most of time infinite dimensional.}$$

We want to classify those π's which are irreducible, smooth, admissible. We shall see later on what the latter two words mean.

1.1. **Motivation.** Apart from pure representation theory interest, one of the main motivation and source of applications for this study is its relation to the theory of modular forms and automorphic forms. They are related in the following way.

(1.1)
$$\left\{\begin{array}{c}\text{Modular forms}\\ \text{holomorphic of weight } k\\ \text{Level } N\end{array}\right\} \to \left\{\begin{array}{c}\text{Automorphic forms}\\ \text{Automorphic representations}\end{array}\right\} \to \left\{\begin{array}{c}\text{Representations of } \mathrm{GL}_2(\mathbb{R})\\ \text{and } \mathrm{GL}_2(\mathbb{Q}_p) \text{ for all primes } p\end{array}\right\}$$

Better knowledge of representations of $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ will give more insights into the left two areas.

Example 1.1.

$$\text{holomorphic weight } k \longleftrightarrow \text{Discrete series representation of weight } k \text{ of } \mathrm{GL}_2(\mathbb{R}),$$
$$\text{Level } N = \prod_i p_i^{c_i} \longleftrightarrow \text{ Representations of } \mathrm{GL}_2(\mathbb{Q}_{p_i}) \text{ of level } p_i^{c_i},$$
$$\text{Fourier expansion of modular forms } \longleftrightarrow \text{Whittaker model of representations of } \mathrm{GL}_2,$$
$$\text{Certain integrals of modular forms} \longleftrightarrow \begin{array}{c}\text{product of local integrals on } \mathrm{GL}_2\\ \text{involving matrix coefficient and Whittaker model.}\end{array}$$

It is also directly related to local Langlands correspondence, which relates π to certain $n-$dimensional Galois representations. It is has been a main topic in math for several decades and has great influences.

1.2. **Plan for the course.** The main tool in this course to study representations of $G(\mathbb{F})$ is the induction of an irreducible representation σ from a subgroup H

$$\mathrm{Ind}_H^G \sigma,$$

By general reciprocity, one can show that any irreducible smooth π is a subrepresentation of $\mathrm{Ind}_H^G \sigma$ for some H and σ. But this is almost useless as we lack detailed information. So our task is basically threefold

(1) Specify H and σ (and also the type of induction) with explicit parametrization.
(2) Show that $\mathrm{Ind}_H^G \sigma$ is nice. This means it is irreducible if possible, or otherwise one can uniquely identify π from it.
(3) Show that all π's we care about occur in this way.

There are two main types of induction in this course, the parabolic induction and the compact induction. These two construction methods gives a dichotomy of representations: non-supercuspidal representations and supercuspidal ones. The parabolic induction is parallel to what one can do for $\mathrm{GL}_n(\mathbb{R})$, while the compact induction is special for $p-$adic setting. We will cover these two methods following the historical order.

We will assume $p \neq 2$ to avoid a lot of technical problems. Some of the results will not hold when $p = 2$.

If time allows, we will cover more topics. Priority will be given to Whittaker model/Kirillov model, level and newform theory. Further more we can talk about Langlands correspondence, L-functions, etc,.

The main reference is [1] and [2]

2. $p-$ADIC FIELD

The easiest case of a $p-$adic field is $\mathbb{F} = \mathbb{Q}_p$. It is the completion of the rational field $\mathbb{Q}$ WRT(with respect to) $p-$adic norm $\|\cdot\|_p$. For $x = p^i \cdot q \in \mathbb{Q}$, where q is rational of form $\frac{m}{n} \in \mathbb{Q}$ with $(mn, p) = 1$ and $m, n, i \in \mathbb{Z}$, we define the $p-$adic evaluation

$$v_p(x) = i, \tag{2.1}$$

and $p-$adic norm

$$\|x\|_p = p^{-v_p(x)} = p^{-i}. \tag{2.2}$$

Define $v_p(0) = +\infty$ and $\|0\|_p = 0$. The $p-$adic valuation satisfies the following properties

$$\begin{cases} v_p(xy) = v_p(x) + v_p(y) \\ v_p(x+y) \geq \min\{v_p(x), v_p(y)\}. \end{cases} \tag{2.3}$$

Correspondingly

$$\begin{cases} \|x\|_p \geq 0 \text{ with equality iff } x = 0, \\ \|xy\|_p = \|x\|_p \|y\|_p, \\ \|x+y\|_p \leq \max\{\|x\|_p, \|y\|_p\}. \end{cases} \tag{2.4}$$

The last property is called strong triangle inequality, as the usual triangle inequality is $||x + y||_p \leq ||x||_p + ||y||_p$ and $\max\{||x||_p, ||y||_p\} \leq ||x||_p + ||y||_p$.

Exercise 2.1. Check the strong triangle inequality for $|| \cdot ||_p$.

In particular $|| \cdot ||_p$ is a norm. Just like $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the standard absolute value norm, p-adic field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|| \cdot ||_p$.

2.1. **p-adic digits.** One way to present a $p-$adic number is to use p-adic digits. A real number can be written like

$$x = 1.234\cdots = 1 \times 10^0 + 2 \times 10^{-1} + 3 \times 10^{-2} + 4 \times 10^{-3} + \cdots$$

With a proximation 1.234.

A p-adic number can be written as

$$x = 1 \times p^0 + 2 \times p^1 + 3 \times p^2 + \cdots$$

with a proximation $1 \times p^0 + 2 \times p^1 + 3 \times p^2$. Higher powers= smaller error.

Exercise 2.2. For $x = 1 \times p^0 + 2 \times p^1 + 3 \times p^2$, compute the first three digits for x^{-1} with general p.

For any $x \in \mathbb{Q}_p$ we can write $x = \sum_n a_n p^n$, $a_n = 0$ for n negative enough. To make the expression unique, we can also require that $0 \leq a_n < p$, here $0 \cdots p - 1$ are fixed lifts of $\mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}$.

The ring of integers is $\mathbb{Z}_p = \{x | a_n = 0 \text{ for } n < 0\}$. It has a unique maximal prime ideal $p\mathbb{Z}_p$, generated by one element.

In general, we can have F a finite field extension of $\mathbb{Q}$, and extend $p-$adic norm and valuation onto F and $\mathbb{F}$ is the completion of F WRT the extended $p-$adic norm. In particular the $p-$adic norm on F the the composition of $|| \cdot ||_p$ with the field norm from F to $\mathbb{Q}$.

We make the following definitions

Definition 2.3. The ring of integers is $O_{\mathbb{F}} = \{x \in \mathbb{F}, ||x||_{\mathbb{F}} \leq 1\}$. This is a P.I.D. (Think about $\mathbb{Z}_p$, or p-adic numbers starting with p^0 digits.)

$O_{\mathbb{F}}$ has a unique maximal ideal $\mathcal{P} = \{x \in \mathbb{F}, ||x||_{\mathbb{F}} < 1\}$, which is generated by one element $\mathcal{P} = \varpi O_{\mathbb{F}}$. Usually we fix ϖ and call it a uniformizer. (Think about $p\mathbb{Z}_p$, with the uniformizer just p.)

The residue field is $k = O_{\mathbb{F}}/\mathcal{P}$. (Think about F_p, the finite field with p elements.) Let $q = |k|$. Then $q = p^f$ where f is the inertial degree of $\mathbb{F}/\mathbb{Q}_p$.

The group of units is $U = O_{\mathbb{F}}^*$. It has subgroups $U_{\mathbb{F}}(n) = 1 + \varpi^n O_{\mathbb{F}}$.

Remark 2.4. In general one can still write elements in $\mathbb{F}$ in digits.

$$x = \sum_{n \in \mathbb{Z}} a_n \varpi^n,$$

where $a_n \in \tilde{k}$, the set of fixed lifts of elements from k to $O_{\mathbb{F}}$. $p-$adic valuation $v_{\mathbb{F}}(x)$ is such that $||x||_{\mathbb{F}} = \min\{n, a_n \neq 0\}$. In particular $v_{\mathbb{F}}(\varpi) = 1$. A not so trivial fact is that

$$||x||_{\mathbb{F}} = |k|^{-v_{\mathbb{F}}(x)}. \tag{2.5}$$

2.2. **Totally disconnected topology for $\mathbb{F}$.** Just as for $\mathbb{R}$, we think of the norm map to be continuous, and consider the preimage of open/closed subset of $\mathbb{R}$ (the image of norm map) to be open/closed. For example we define the open balls in $\mathbb{F}$ to be

$$B_r(x_0) = \{x \in \mathbb{F}, \|x - x_0\|_{\mathbb{F}} < r\}. \tag{2.6}$$

In particular any set of form $x_0 + \varpi^i O_{\mathbb{F}}$ or $x_0 U_{\mathbb{F}}(i)$ is an open set. Note that when $i > v_{\mathbb{F}}(x_0)$, $x_0 + \varpi^i O_{\mathbb{F}} = x_0 U_{\mathbb{F}}(i - v_{\mathbb{F}}(x_0))$. They actually form a topological basis.

But for $p-$adic fields, this set is also closed, because the image of $\|\cdot\|_{\mathbb{F}}$ is discrete in $\mathbb{R}$. (For example when $\mathbb{F} = \mathbb{Q}_p$, it is $\{\frac{1}{p^i}\}$.) So

$$\{x \in \mathbb{F}, \|x - x_0\|_{\mathbb{F}} < r\} = \{x \in \mathbb{F}, \|x - x_0\|_{\mathbb{F}} \leq r - \delta\} \tag{2.7}$$

is open and closed. As a result, $\mathbb{F}$ is totally disconnected.

2.3. **Hensel's lemma.**

Definition 2.5. $x \equiv y \mod \varpi^n$ iff $x - y \in \varpi^n O_{\mathbb{F}}$.

Theorem 2.6. *Let $f \in O_{\mathbb{F}}[x]$. If there exists $x \in O_{\mathbb{F}}$ such that $f(x) \equiv 0 \mod \varpi$ and its derivative $f'(x) \not\equiv 0 \mod \varpi$, then there exists a unique $y \in O_{\mathbb{F}}$ such that $y \equiv x \mod \varpi$ and $f(y) = 0$.*

This theorem is about uniquely lifting solution of polynomial equation from residue field to $\mathbb{F}$.

Proof. Let's work with $\mathbb{Q}_p$. We shall prove by induction the following: If there exist $x_i \in \mathbb{Z}_p$ such that $f(x_i) \equiv 0 \mod p^i$ and $f'(x_i) \not\equiv 0 \mod p$, then there exists $x_{i+1} \in \mathbb{Z}_p$ such that $x_{i+1} \equiv x_i \mod p^{i-1}$, $f(x_{i+1}) \equiv 0 \mod p^{i+1}$, $f'(x_{i+1}) \not\equiv 0 \mod p$. By the condition, we can assume that $f(x_i) \equiv ap^i \mod p^{i+1}$ for some integer a. The basic tool is the Taylor expansion,

$$f(x + up^i) = f(x) + f'(x)up^i + \cdots \equiv 0 \mod p^{i+1} \tag{2.8}$$

as higher order terms will have larger $p-$powers. Since $f'(x) \not\equiv 0 \mod p$, we can find proper $u \in \mathbb{Z}$ such that $f'(x)u \equiv -a \mod p$. Then $x_{i+1} = x_i + up^i$ has the required properties.

Note that $\|x_{i+1} - x_i\|_p = p^{-i}$. Using the completeness of $\mathbb{F}$, there exists $y \in \mathbb{Z}_p$ which is the limit of $\{x_i\}$. Then by Taylor expansion again $f(y) \equiv 0 \mod p^i$ for any i, thus $f(y) = 0$.

□

Exercise 2.7. Check uniqueness. Hint: use Taylor expansion again.

Remark 2.8. Essentially we are figuring out y digit by digit.

With the help of Hensel's lemma, we have the following structure of $\mathbb{F}^*$.

Lemma 2.9.

$$\mathbb{F}^* \simeq \mathbb{Z} \times k^* \times U_{\mathbb{F}}(1). \tag{2.9}$$

Proof. For any $x \in \mathbb{F}^*$, we can first write it as $x = \varpi^i u$ with $i = v_{\mathbb{F}}(x)$ and $u \in O_{\mathbb{F}}^*$. Thus

$$\mathbb{F}^* \simeq \mathbb{Z} \times O_{\mathbb{F}}^*,$$

with $\mathbb{Z}$ identified with $\{\varpi^i\}$. We also have a short exact sequence

$$1 \to U_{\mathbb{F}}(1) \xrightarrow{\iota} O_{\mathbb{F}}^* \xrightarrow{pr} k^* \to 1$$

where ι is the natural inclusion and pr is the quotient map which ignores higher digits. The main point of this Lemma is that this exact sequence splits, i.e., there exists an injective group homomorphism $f : k^* \to O_{\mathbb{F}}^*$ such that $pr \circ f$ is the identity map.

We note that k^* is a cyclic group, so any $a \in k^*$ satisfies $a^{q-1} = 1$ in k, where $q = |k|$ is a power of p. This implies that the polynomial equation $x^{q-1} - 1$ has a solution $a \in k^*$, with its derivative $(q-1)x^{q-2} \not\equiv 0 \mod p$ as $(p^i - 1, p) = 1$. Thus we can apply Hensel's lemma and obtain $\tilde{a} \in O_{\mathbb{F}}^*$ such that $\tilde{a} \equiv a \mod \varpi$, $\tilde{a}^{q-1} = 1$. The map $f : a \mapsto \tilde{a}$ is then an injection. $pr \circ f$ is identity as $\tilde{a} \equiv a \mod \varpi$. It is a group homomorphism because of the following. $\widetilde{a_1 a_2}$, $\tilde{a}_1\tilde{a}_2$ both satisfy the equation $x^{q-1} = 1$ and have same image in the residue field, so they must be equal using uniqueness from Hensel's lemma. □

2.4. **Classify quadratic extensions of $\mathbb{F}$ when $p \neq 2$.** First of all, quadratic extensions $\mathbb{E}$ over $\mathbb{F}$ are parametrized by $\mathbb{F}^*/(\mathbb{F}^*)^2$. Here $(\mathbb{F}^*)^2 = \{x^2, x \in \mathbb{F}^*\}$. This is because in general we can write $\mathbb{E} = \mathbb{F}(\sqrt{D}) = \mathbb{F}[x]/(x^2 - D)$ for $D \in \mathbb{F}^*$ a non-square, and D, d^2D will give the same quadratic extension.

Now we use Lemma 2.9, and get

$$\mathbb{F}^*/(\mathbb{F}^*)^2 = \mathbb{Z}/2\mathbb{Z} \times k^*/(k^*)^2 \times U_{\mathbb{F}}(1)/U_{\mathbb{F}}(1)^2 \tag{2.10}$$

We can pick representatives $\{1, \varpi\}$ for $\mathbb{Z}/2\mathbb{Z}$.

$k^*/(k^*)^2$ is not trivial as $2|(q-1) = |k^*|$, and it has at most two elements. Pick for it representatives $\{1, \xi\}$ for some $\xi \in \mathbb{F}^*$ which is a lift of a non-square element in k^*.

$U_{\mathbb{F}}(1)/U_{\mathbb{F}}(1)^2$ is trivial. To prove this, we need to show that for any $a \in U_{\mathbb{F}}(1)$, the equation $x^2 - a = 0$ has solution in $U_{\mathbb{F}}(1)$. This is true because $x^2 - a \equiv x^2 - 1 \equiv 0 \mod \varpi$ has a solution $x \equiv 1 \mod \varpi$. Then we can use Hensel's Lemma.

To summarise, we have the following quadratic field extentions of $\mathbb{F}$:

(1) $\mathbb{E} = \mathbb{F}(\sqrt{\xi})$. ϖ is also a uniformizer for $\mathbb{E}$. The residue field $k_{\mathbb{E}}$ is a quadratic field extension of k. In this case $\mathbb{E}$ is called an inert quadratic extension over $\mathbb{F}$.
(2) $\mathbb{E} = \mathbb{F}(\sqrt{\varpi})$, or $\mathbb{F}(\sqrt{\varpi\xi})$. The uniformizer $\varpi_{\mathbb{E}}$ can be chosen (though not necessary) so that $\varpi_{\mathbb{E}}^2 = \varpi$. The residue field $k_{\mathbb{E}} = k$. In this case $\mathbb{E}$ is called a ramified quadratic extension over $\mathbb{F}$.
(3)* $\mathbb{E} = \mathbb{F}(1) \simeq \mathbb{F} \times \mathbb{F}$. $\mathbb{E}$ is not a field in this case. But we still call it split quadratic extension over $\mathbb{F}$.

Start of lecture 2

2.5. **Additive characters on $\mathbb{F}$.**

Definition 2.10. Let $\mathbb{C}^1$ be the set of complex numbers with absolute value 1. $\psi : \mathbb{F} \to \mathbb{C}^1$ is called an additive character on $\mathbb{F}$ if it is continuous and satisfies

$$\psi(x + y) = \psi(x) \cdot \psi(y). \tag{2.11}$$

Lemma 2.11. *If ψ is an additive character over $\mathbb{F}$, then it is locally constant, i.e., there exists $n \in \mathbb{Z}$ such that for any $y \in \varpi^n O_{\mathbb{F}}$, $\psi(y) = 1$. Then $\psi(x + y) = \psi(x)$ for any $x \in \mathbb{F}$.*

Proof. Let $\gamma = \{z \in \mathbb{C}^1, arg(z) < \frac{2\pi}{p}\}$ be an open arc in $\mathbb{C}^1$. Since ψ is continuous, the preimage of γ is open, and in particular contains $\varpi^n O_{\mathbb{F}}$ for some n. Then we claim that $\psi(\varpi^n O_{\mathbb{F}}) = 1$.

Suppose that $\psi(y) \neq 1$ for some $y \in \varpi^n O_{\mathbb{F}}$, then by (2.11)

$$\psi(p^i y) = \psi(y)^{p^i}$$

for any $i \in \mathbb{Z}_{>0}$. On the left hand side $p^i y$ is always in $\varpi^n O_{\mathbb{F}}$, so $\psi(p^i y)$ should still be inside γ. But on the other side $\psi(y)^{p^i}$ will eventually leave γ. Contradiction.

Now the lemma follows immediately by using (2.11). □

Corollary 2.12. *The images of ψ are roots of unity. In particular we can think of ψ as an element of $\hat{\mathbb{F}} = \mathrm{Hom}(\mathbb{F}, \mathbb{Q}/\mathbb{Z})$, the Pontryagin dual.*

Remark 2.13. We can think that the topology on $p-$adic field is so different from the complex topology, that requiring continuity is as strong as requiring locally constant. This also happens for the representations of G. In this course we shall not distinguish the following notions: continuous, smooth, locally constant.

Definition 2.14. Suppose that ψ is not the trivial charactrer. The level $c(\psi)$ of ψ is defined to be the smallest integer c such that $\psi(\varpi^c O_{\mathbb{F}}) = 1$.

Example 2.15. For $\mathbb{Q}_p$, we can define $\psi_0(x) = e^{2\pi i x}$. $c(\psi) = 0$. In general for $\mathbb{F}$ a finite extension of $\mathbb{Q}_p$, we can define an additive character $\psi_0(x) = e^{2\pi i \mathrm{Tr}(x)}$ where Tr is the trace map from $\mathbb{F}$ to $\mathbb{Q}_p$.

Further more for any $a \in \mathbb{F}$, $\psi_a(x) = \psi_0(ax)$ is also an additive character on $\mathbb{F}$.

Proposition 2.16. *Any additive character on $\mathbb{F}$ is of form ψ_a for some $a \in \mathbb{F}$.*

Remark 2.17. I.e., $\hat{\mathbb{F}} = \mathbb{F}$.

Sketch of proof. If ψ is trivial, we can pick $a = 0$. Otherwise, let $c = c(\psi)$, $c_0 = c(\psi_0)$. Then ψ is a nontrivial character on $\varpi^i O_{\mathbb{F}}/\varpi^c O_{\mathbb{F}} \simeq k$ with $i < n$, which is a finite group. Then we need the following lemmas

Lemma 2.18. *For any finite abelian groups H, its Pontryagin dual $\hat{H} \simeq H$.*

The proof of this lemma amounts to checking for cyclic groups and the using that all finite abelian groups are direct product of cyclic ones.

Lemma 2.19. *Suppose that $c = c(\psi)$. Then we have the following identification*

$$\varpi^{-n} O_{\mathbb{F}}/\varpi^{-m} O_{\mathbb{F}} \xrightarrow{\simeq} \varpi^{m+c}\widehat{O_{\mathbb{F}}/\varpi^{n+c}} O_{\mathbb{F}} \qquad (2.12)$$
$$a \mapsto \psi_a$$

To prove this, one need to show that the map given above is injective, and then do a counting on both sides using the previous lemma.

By this result, we have $\psi|_{\varpi^i O_{\mathbb{F}}/\varpi^c O_{\mathbb{F}}}(x) = \psi_0(a_i x)$ for some $a_i \in \mathbb{F}$ with $v_{\mathbb{F}}(a_i) = -c + c_0$. One can show that $\{a_i\}_{i \to -\infty}$ is a convergent sequence, and its limit a is the require element in the lemma. □

Exercise 2.20. Fill in the details for this proof.

2.6. **Multiplicative character on $\mathbb{F}^*$.**

Definition 2.21. A multiplicative character χ on $\mathbb{F}^*$ is a continuous function $\chi : \mathbb{F}^* \to \mathbb{C}^1$ such that

$$\chi(xy) = \chi(x) \cdot \chi(y). \qquad (2.13)$$

One can expect that the continuity implies that χ is locally constant.

Definition 2.22. When χ is nontrivial, the level $c = c(\chi)$ is the smallest integer such that $\chi(y) = 1$ for any $y \in U_{\mathbb{F}}(c)$. Then $\chi(xy) = \chi(x)$ for any $x \in \mathbb{F}^*$.

Definition 2.23. For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer $n \leq x$, $\lceil x \rceil$ denote the smallest integer $n \geq x$.

Lemma 2.24. *Let χ be nontrivial with $c = c(\chi) \geq 2$. Then there exists $\alpha_\chi \in \mathbb{F}^*$ such that*

$$\chi(1 + x) = \psi_{\alpha_\chi}(x) \tag{2.14}$$

for any $x \in \varpi^{\lceil c/2 \rceil} O_{\mathbb{F}}$.

Proof. For any $x_1, x_2 \in \varpi^{\lceil c/2 \rceil} O_{\mathbb{F}}$,

$$\chi((1 + x_1)(1 + x_2)) = \chi(1 + x_1 + x_2 + x_1 x_2) = \chi(1 + x_1 + x_2). \tag{2.15}$$

The last equality follows from that $c(\chi) = c \leq v_{\mathbb{F}}(x_1 x_2)$. This means that $x \mapsto \chi(1+x)$ is an additive character on $\varpi^{\lceil c/2 \rceil} O_{\mathbb{F}}$. Thus there exists α_χ with required property. □

3. Structure of GL_n

3.1. **Subgroups.** Here we consider the group $G = \mathrm{GL}_n$. One can define the determinant and trace for elements in G as usual.

The center of GL_n is

$$Z = \{g, gh = hg \text{ for any } h \in \mathrm{GL}_n\} = \{aI, a \in \mathbb{F}^*\}$$

.

The diagonal torus is

$$T = \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}.$$

The parabolic subgroup we care about is an upper triangular block matrices associated to a partition $n = n_1 + n_2 + \cdots n_k$. For $\underline{n} = (n_1, \cdots, n_k)$,

$$P_{\underline{n}} = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}.$$

It has a unipotent subgroup

$$N_{\underline{n}} = \begin{pmatrix} I & * & \cdots & * \\ 0 & I & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{pmatrix},$$

and a Levi subgroup

$$M_{\underline{n}} = \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}.$$

Note that when all $n_i = 1$, $B = P_{\underline{n}}$ is also called Borel subgroup, and $T = M_{\underline{n}}$ in this case.

The Weyl group is defined to be $W = N_G(T)/Z(T)$, where $N_G(T)$ is the normalizer of T in G, and $Z(T)$ is the centralizer of T (in this case is T itself). In the case $G = \mathrm{GL}_n$, $W \simeq S_n$, the permutation group of n elements, and a set of representatives can be chosen as a permutation of rows for the identity matrix.

Example 3.1. When $n = 2$, $W = \{I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$.

3.2. **Compact open subgroups.** Up to this point, all constructions are parallel to those for $\mathrm{GL}_n(\mathbb{R})$. Now specific for $p-$adic field, we have a standard maximal compact open subgroup

$$K = \mathrm{GL}_n(O_{\mathbb{F}}) = \{g \in M_{n\times n}(O_{\mathbb{F}}), \det g \in O_{\mathbb{F}}^*\}.$$

Note that $W \subset K$.

K has a filtration of normal compact open subgroups $K_I(n) = I + \varpi^n M_{n\times n}(O_{\mathbb{F}})$. $gK_I(n)$'s provide a topological basis for GL_n.

Later on we will introduce more compact open subgroups.

Proposition 3.2. *Any compact open subgroup of GL_n is a subgroup of $g^{-1}Kg$ for some $g \in GL_n$.*

This result follows from the following results. The basic idea is to work over a vector space on which the groups act, and change the problem of conjugating into that of choosing proper basis.

Definition 3.3. Let V be a $n-$dimensional vector space over $\mathbb{F}$. An $O_{\mathbb{F}}-$lattice L in V is an $O_{\mathbb{F}}$ module such that

(1) L is a finitely generated $O_{\mathbb{F}}-$module,
(2) $L \otimes_{O_{\mathbb{F}}} \mathbb{F} = V$.

Lemma 3.4. *There exists an $O_{\mathbb{F}}$ generators $\{v_1, \cdots, v_n\}$ which is also a basis for V.*

Proof. As L is finitely generated $O_{\mathbb{F}}-$module, we can choose a set of generators $\{v_1, \cdots, v_k\}$such that k is minimal. It's easy to see that $k \geq n$ by (2) above. Suppose that $k > n$. Then they are linearly dependent on V, i.e., there exists coefficients a_i such that

$$\sum_{1\leq i\leq k} a_i v_i = 0. \tag{3.1}$$

By multiplying the whole equation with proper ϖ^n, we can assume that $a_i \in O_{\mathbb{F}}$ and say, $a_1 \in O_{\mathbb{F}}^*$. Then we can write

$$v_1 = -\sum_{2\leq i\leq k} a_1^{-1} a_i v_i, \tag{3.2}$$

with all coefficients in $O_{\mathbb{F}}$, contradicting the minimality of k. Thus $k = n$. □

Lemma 3.5. *For any $O_{\mathbb{F}}-$lattice L, its stabilizer $Stab_G L = \{g, gL = L\}$ (consequently, $g^{-1}L = L$) is a compact open subgroup, conjugated to K.*

Proof. By the previous lemma, we can pick an $O_{\mathbb{F}}$ basis $\{v_1, \cdots, v_n\}$. One can easily check that under this basis, $\mathrm{Stab}_G L$ is exactly as K, which is compact and open. Changing basis amount to a conjugation, thus the conclusion. □

Lemma 3.6. *Any compact open subgroup $H \subset Stab_G L$ for some $O_{\mathbb{F}}-$lattice L.*

Proof. We first construct an $O_{\mathbb{F}}$ module with possibly infinite many generators

$$L = O_{\mathbb{F}} \text{ span of } \{\text{all column vectors for any } k \in H\}. \tag{3.3}$$

Then $L \otimes_{O_{\mathbb{F}}} \mathbb{F} = V$ as the identity matrix $I \in H$ whose column vectors already span V over $\mathbb{F}$. Now we check that L is actually finitely generated by using compactness. For any $g = (g_1, \cdots, g_n) \in H$ with g_i being column vectors, it gives arise to a finitely generated $O_{\mathbb{F}}-$module

$$g \mapsto O_{\mathbb{F}} g_1 \oplus \cdots \oplus O_{\mathbb{F}} g_n. \tag{3.4}$$

Then any $g' \in g \cdot (\mathrm{GL}_n(O_{\mathbb{F}}) \cap H)$ give rise to the same $O_{\mathbb{F}}$ module, because multiplication on right by an element in $\mathrm{GL}_n(O_{\mathbb{F}})$ will give $O_{\mathbb{F}}$ liner combinations of g_i's. Each $g \cdot (\mathrm{GL}_n(O_{\mathbb{F}}) \cap H)$ is an open subset of H and covers H, thus by compactness, we can just choose a finite number of g to cover H. So L is finitely generated.

Now we show that $H \subset \mathrm{Stab}_G L$. By definition for any $h \in H$, we need to show that $hL \subset L$ and $h^{-1}L \subset L$. But as L is generated by column vectors of $g \in H$, this is equivalent to that

$$hg \in H, h^{-1}g \in H, \tag{3.5}$$

which is clearly true. □

Start of lecture 3

3.3. **Decomposition results.**

Theorem 3.7 (Bruhat decomposition)**.**

$$GL_n = \coprod_{w \in W} BwB.$$

Proof. This is essentially reducing to echelon form using Gauss elimination and row ordering in linear algebra. We briefly show for $n = 2$ case. If $g \in \mathrm{GL}_2$ is already in B, done. Otherwise $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$ with $g_3 \neq 0$. Then

$$\begin{pmatrix} 1 & -g_1 g_3^{-1} \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} 0 & -\det(g)/g_3 \\ g_3 & g_4 \end{pmatrix}, \tag{3.6}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -g_1 g_3^{-1} \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} g_3 & g_4 \\ 0 & -\det(g)/g_3 \end{pmatrix}. \tag{3.7}$$

That is

$$g = \begin{pmatrix} 1 & g_1 g_3^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g_3 & g_4 \\ 0 & -\det(g)/g_3 \end{pmatrix} \tag{3.8}$$

□

Theorem 3.8 (Iwasawa decomposition)**.**

$$GL_n = BK.$$

Proof. Proof for $n = 2$ case. Let $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$. As $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K$ and

$$g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} g_2 & g_1 \\ g_4 & g_3 \end{pmatrix}, \tag{3.9}$$

we can assume that $v(g_4) \leq v(g_3)$. In particular $g_4 \neq 0$. Then

$$g \begin{pmatrix} 1 & 0 \\ -g_3 g_4^{-1} & 1 \end{pmatrix} = \begin{pmatrix} \det(g) g_4^{-1} & g_2 \\ 0 & g_4 \end{pmatrix} \in B. \tag{3.10}$$

□

Exercise 3.9. Prove the theorem for general n using induction.

Theorem 3.10 (Cartan decomposition). *Let $I = (i_1, i_2, \cdots, i_n)$ such that $i_j \in \mathbb{Z}_{\geq 0}$ and $i_j \geq i_{j+1}$. Denote*

$$diag(\varpi^I) = \begin{pmatrix} \varpi^{i_1} & 0 & \cdots & 0 \\ 0 & \varpi^{i_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varpi^{i_n} \end{pmatrix}.$$

Then

$$GL_n = \coprod_I K diag(\varpi^I) K. \tag{3.11}$$

Proof. Consider the case $n = 2$. By permuting rows and columns, we can assume that g_4 has lowest valuation in g. Then

$$\begin{pmatrix} 1 & -g_2 g_4^{-1} \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ -g_3 g_4^{-1} & 1 \end{pmatrix} = \begin{pmatrix} \det(g) g_4^{-1} & 0 \\ 0 & g_4 \end{pmatrix} \tag{3.12}$$

Further we have

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \varpi^{v(a)-v(b)} & 0 \\ 0 & \varpi^{v(b)} \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & b_0 \end{pmatrix}. \tag{3.13}$$

Here $a = a_0 \varpi^{v(a)}$ and $b = b_0 \varpi^{v(b)}$. One just have to note that $v(\det(g) g_4^{-1}) \geq v(g_4)$. □

3.4. **Embedding of field.** Let $\mathbb{E} = \mathbb{F}(\sqrt{D})$ be a quadratic field extension over $\mathbb{F}$. Then it can be embedded into $M_{2\times 2}$ by

$$\iota : a + b\sqrt{D} \mapsto \begin{pmatrix} a & b \\ bD & a \end{pmatrix}. \tag{3.14}$$

Exercise 3.11. Prove that all other embeddings will differ from this by a conjugation. Hint: it suffice to show this for a generator, eg. $\sqrt{D}$. To do this, start with a general embedding and work with the 2−dimensional vector space on which the matrices act. Show that after choosing proper basis, the action of $\sqrt{D}$ is given as above.

One can easily check that this embedding is consistent with norm and trace maps, i.e.,

$$\mathrm{Tr}_{M_{2\times 2}} \circ \iota = \mathrm{Tr}_{\mathbb{E}/\mathbb{F}}, \tag{3.15}$$

$$\det \circ \iota = \mathrm{Nm}_{\mathbb{E}/\mathbb{F}}. \tag{3.16}$$

4. Haar measure

Let G be a group over p−adic field. Let $C_c^\infty(G)$ be the space of functions $f : G \to \mathbb{C}$ which are locally constant and compactly supported. G acts on $C_c^\infty(G)$ by left and right translations.

Definition 4.1. A left Haar measure on G is a non-negative measure such that

$$\int f(x) d_L x = \int f(g^{-1}x) d_L x, \tag{4.1}$$

for any $f \in C_c^\infty(G)$ and $g \in G$. A right Haar measure on G is defined similarly to have the property

$$\int f(x) d_R x = \int f(xg) d_R x. \tag{4.2}$$

It basically is saying that we can do change of variable for integrals.

Lemma 4.2 (Without proof)**.** *Left/right Haar measures exist and are always* $1-$*dimensional.*

Definition 4.3. Let $d_L x$ be a left Haar measure. Modular character $\Delta_G(g) : G \to \mathbb{R}_{>0}$ is such that

$$\Delta_G(g) \int f(xg) d_L x = \int f(x) d_L x. \tag{4.3}$$

G is called unimodular if any left Haar measure is also a right Haar measure, or equivalently $\Delta_G = 1$.

Formally we can write

$$\Delta_G(g) d_L(xg^{-1}) = d_L x, \text{ or } d_L(xg) = \Delta_G(g) d_L x. \tag{4.4}$$

Note that it's indeed a character as

$$\Delta_G(g_1 g_2) d_L x = d_L(x g_1 g_2) = \Delta_G(g_2) d_L(x g_1) = \Delta_G(g_1)\Delta_G(g_2) d_L x. \tag{4.5}$$

Proposition 4.4. *Any finite group is unimodular. Any compact open subgroup of* $GL_n(\mathbb{F})$ *is also unimodular.*

Proof. When G is finite, the counting measure

$$\int f dx = \sum_{g \in G} f(g) \tag{4.6}$$

is automatically left and right Haar measure.

When G is a compact open subgroup of $\mathrm{GL}_n(\mathbb{F})$, we define a measure μ on G such that $\mathrm{Vol}(G, dx) = 1$ and $\mathrm{Vol}(H, dx) = \frac{1}{[H:G]}$ for any compact open subgroup H. $f \in C_c^\infty(G)$ is locally constant so there exists a normal compact subgroup H such that

$$f(xh) = f(x) = f(hx) \tag{4.7}$$

for any $x \in G$. Then

$$\int f dx = \frac{1}{[H:G]} \sum_{x \in G/H} f(x). \tag{4.8}$$

We have

$$\int f(gx) dx = \int f(xg) dx = \int f(x) dx \tag{4.9}$$

because multiplication by g on left or right just permutes elements in G/H. □

The second part of the proof used the key feature of $p-$adic analysis, that is, the continuous functions are always locally constant. So checking the left/right-invariance of a measure is equivalent to checking how the volume of an open set behaves under left and right actions.

4.1. Measures on $\mathbb{F}$ and $\mathbb{F}^*$.

Abelian groups are always unimodular since the left translation is the same as right translation.

We normalise the Haar measure dx on $\mathbb{F}$ so that

$$\mathrm{Vol}(O_{\mathbb{F}}, dx) = 1. \tag{4.10}$$

Then automatically we have

$$\mathrm{Vol}(\varpi^n O_{\mathbb{F}}, dx) = \frac{1}{q^n}. \tag{4.11}$$

This is because $O_{\mathbb{F}}$ can be written as

$$O_{\mathbb{F}} = \coprod_{a_i} \left(\sum_{0 \le i < n} a_i \varpi^i + \varpi^n O_{\mathbb{F}} \right), \tag{4.12}$$

and every piece $\sum_{0 \le i < n} a_i \varpi^i + \varpi^n O_{\mathbb{F}}$ should have same volume.

Recall in the real case, $d^*x = |x|^{-1}dx$. Here we claim that $d^*x = |x|_{\mathbb{F}}^{-1}dx$ is a Haar measure on $\mathbb{F}^*$. Note that

$$\mathrm{Vol}(y\varpi^n O_{\mathbb{F}}, dx) = \frac{1}{q^{v_{\mathbb{F}}(y)+n}} = |y|_{\mathbb{F}} \mathrm{Vol}(\varpi^n O_{\mathbb{F}}, dx).$$

One can formally write $d(yx) = |y|_{\mathbb{F}} dx$, and formally verify that

$$|yx|_{\mathbb{F}}^{-1} d(yx) = |yx|_{\mathbb{F}}^{-1} |y|_{\mathbb{F}} dx = |x|_{\mathbb{F}}^{-1} dx. \tag{4.13}$$

Thus d^*x defined above is a Haar measure on $\mathbb{F}^*$.

We normalise the Haar measure d^*x on $\mathbb{F}^*$ so that

$$\mathrm{Vol}(O_{\mathbb{F}}^*, d^*x) = 1. \tag{4.14}$$

Then for a similar reason we have

$$\mathrm{Vol}(U_{\mathbb{F}}(n), d^*x) = \frac{1}{(q-1)q^{n-1}}. \tag{4.15}$$

Exercise 4.5. Check this result.

4.2. **Measures on $\mathbf{GL}_n$ and P.** The discussion here is a direct analogue of real case, with $p-$adic norm in place of absolute value.

Let dA denote the measure on $M_{n\times n}$ which is a product of Haar measures on $\mathbb{F}$.

Lemma 4.6. *$G = GL_n(\mathbb{F})$ is unimodular, with the Haar measure given by $dg = |\det(g)|_{\mathbb{F}}^{-n} dA$.*

Proof. We formally check that this is a left Haar measure. For any $h \in G$, and $x \in \mathbb{F}^n$ consider as column vector, with dx a product of Lebesgue measures,

$$d(hx) = |\det(h)|_{\mathbb{F}} dx. \tag{4.16}$$

Consider $A \in M_{n\times n}$ as n column vectors. Then

$$d(hA) = |\det(h)|_{\mathbb{F}}^{n} dA. \tag{4.17}$$

Thus

$$d(hg) = |\det(hg)|_{\mathbb{F}}^{-n} d(hA) = |\det(g)|_{\mathbb{F}}^{-n} dA = dg. \tag{4.18}$$

Checking it's right Haar measure is similar. □

We shall normalise the Haar measure on G so that $\mathrm{Vol}(K) = 1$.

Lemma 4.7. *For $\underline{n} = (n_1, \cdots, n_k)$ and*

$$p = \begin{pmatrix} M_{1,1} & N_{1,2} & \cdots & N_{1,k} \\ 0 & M_{2,2} & \cdots & N_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k,k} \end{pmatrix} \in P_{\underline{n}}$$

$$\Delta_{P_{\underline{n}}}(p) = \prod_{1 \le i \le t} |\det(M_{i,i})|_{\mathbb{F}}^{\sum_{j<i} n_j - \sum_{j>i} n_j} \tag{4.19}$$

Example 4.8. When $\underline{n} = (1,1)$,

$$\Delta_{P_{\underline{n}}}(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = |\frac{b}{a}|_{\mathbb{F}}. \tag{4.20}$$

Proof. We shall prove the case $\underline{n} = (1,1)$.

Recall that formally

$$d_L(gh) = \Delta_P(h) d_L g. \tag{4.21}$$

First we need to figure out $d_L g$.

Writing $h = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$, $g = \begin{pmatrix} x & n \\ 0 & y \end{pmatrix}$, we have that

$$hg = \begin{pmatrix} x' & n' \\ 0 & y' \end{pmatrix} = \begin{pmatrix} ax & an + ym \\ 0 & by \end{pmatrix}.$$

So $d^*x'd^*y'dn' = |a|_{\mathbb{F}} d^*xd^*ydn$, and $d_L g = |x|_{\mathbb{F}}^{-1} d^*xd^*ydn$ is a left Haar measure.

Then when we write

$$gh = \begin{pmatrix} x' & n' \\ 0 & y' \end{pmatrix} = \begin{pmatrix} ax & xm + nb \\ 0 & by \end{pmatrix},$$

$$d_L(gh) = |ax|_{\mathbb{F}}^{-1} |b|_{\mathbb{F}} d^*xd^*ydn = |\frac{b}{a}|_{\mathbb{F}} d_L g. \tag{4.22}$$

Thus $\Delta_G(h) = |\frac{b}{a}|_{\mathbb{F}}$. □

Exercise 4.9. Prove the general case. Hint: In general let $dA_{i,j}$ be the product Haar measure on $N_{i,j}$ and dg_i be the Haar measure on $M_{i,i}$. Then

$$dg = \prod_i |\det(M_{i,i})|_{\mathbb{F}}^{-\sum_{j>i} n_j} dg_i \prod_{i,j} dA_{i,j} \tag{4.23}$$

is a left Haar measure.

Remark 4.10. In practice the Haar measure on GL_n given above is not so convenient to use. Using Iwasawa decomposition $\mathrm{GL}_n = BK$, one can expect that $dg = d_L b dk$ for $b \in B, k \in K$. There is an ambiguity, as $B \cap K = B(O_{\mathbb{F}})$ is not empty. But we can normalise the Haar measures so that $\mathrm{Vol}(B(O_{\mathbb{F}})) = 1$ and the redundant integrals on the common part does not matter. This decomposition of integral is very useful if we further know that the function f to integrate is also $K-$invariant (in which case f is called spherical), then we can essentially reduce the integral to one only on B.

Start of lecture 4

5. Basic representation theory

5.1. **Basic representation definitions.** A representation ρ of group G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ where V is a vector space over $\mathbb{C}$.

A subrepresentation U of V is a subspace which is closed under the action of G ($\rho(G)U \subset U$). When such subrepresentation exists, one can also define the quotient representation of G on V/U.

ρ is called irreducible if V has no proper subrepresentations(i.e., $U = \{0\}$ or V for any subrepresentations).

A representation is called semisimple if it is a direct sum of irreducible representations.

In general there exists a filtration of V $0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subset V$ such that V_i are closed under the action of G and V_{i+1}/V_i are irreducible representations. If there exists $V_n = V$, we say V is of finite length and n is the length of the representation ρ.

We also say V is glued together from V_{i+1}/V_i and $V_{ss} = \oplus V_{i+1}/V_i$.

Lemma 5.1 (Schur's lemma/Without proof)**.** *Let ρ_1, ρ_2 be irreducible representations of G. Then* $\mathrm{Hom}_G(\rho_1, \rho_2) = 0$ *or* $\mathbb{C}$*, with the latter iff* $\rho_1 \simeq \rho_2$.

Lemma 5.2. *Let ρ be an irreducible representation of an abelian group G. Then ρ is $1-$dimensional.*

Proof. For any $x, y \in G$, $\rho(x)\rho(y) = \rho(y)\rho(x)$ implies that $\rho(x) \in \mathrm{Hom}_G(\rho, \rho)$ for any x. By Schur's lemma, $\mathrm{Hom}_G(\rho, \rho)$ is $1-$dimensional. So $\rho(x)$ is a constant multiple of identity map for any x. Then ρ must be $1-$dimensional to be irreducible. □

Now if (π, V) is a representation of $G = \mathrm{GL}_n$, and $Z \simeq \mathbb{F}^*$ is the group of centers of G, then $\pi|_Z = \oplus\chi$ is a direct sum of multiplicative characters. Let V^χ be the subspace of V on which Z acts by χ. Then each V^χ is a subrepresentation of V, as $\pi(g)\pi(z) = \pi(z)\pi(g) = \chi(z)\pi(g)$. In particular if π is irreducible, then $\pi|_Z$ will be a single character. Denote it by w_π, called central character of π.

Lemma 5.3. *Any representation (ρ, V) of a finite group is semisimple.*

Proof. Let $U \subset V$ be any subrepresentation. Let $\varphi : V \to U$ be any projection map, meaning that $\varphi|_U$ is the identity map. Define

$$\tilde{\varphi}(v) = \frac{1}{\#G} \sum_{g \in G} \pi(g)\varphi(\pi(g^{-1})v). \tag{5.1}$$

Then $\tilde{\varphi}|_U$ is still the identity map, and $\tilde{\varphi}$ is further a homomorphism of $G-$representations. Then $V = U \oplus \ker\varphi$ is a decomposition of $G-$representations. □

Corollary 5.4 (Inverse Schur's lemma)**.** *When ρ is semisimple, then*

$$\rho \text{ is irreducible} \Leftrightarrow \mathrm{Hom}_G(\rho, \rho) = \mathbb{C}. \tag{5.2}$$

Beware that in general representations of $\mathrm{GL}_n(\mathbb{F})$ are not necessarily semisimple.

5.2. **Smooth and admissible.** For groups over $p-$adic field, we care about the following types of representations.

Definition 5.5. (ρ, V) of G is called smooth if for any $v \in V$, there exists a compact open subgroup K of G such that $\rho(K)v = v$.

This is the direct generalization of continuity/smoothness of additive/multiplicative characters.

Equivalently, let V^K denote the subspace of V which is $\rho(K)-$invariant, then ρ is smooth if and only if

$$V = \bigcup_K V^K. \tag{5.3}$$

Definition 5.6. ρ is called admissible if V^K is finite dimensional for any compact open subgroup K.

In this course we will classify irreducible smooth admissible representations of GL_n. For simplicity denote $\mathrm{Irr}(G)$ to be the set of such representations. One don't have to worry about being admissible too much, because of the following result.

Proposition 5.7 (Partial proof later on)**.** *Any smooth irreducible representation of GL_n is admissible.*

This result follows from the classification result for representations of GL_n. But to start with, we impose this condition. We will soon see that it is related to contragredient representation.

Lemma 5.8. *Let (ρ, V) be a smooth representation of a compact open subgroup K. Then ρ is semisimple.*

Sketch. As in the proof of Lemma 5.3, we can for any projection map $\varphi : V \to U$ on to a subrepresentation, we can define

$$\tilde{\varphi}(v) = \frac{1}{\mathrm{Vol}(K)} \int_K \pi(k)\varphi(\pi(k^{-1})v)dk. \tag{5.4}$$

The integral is essentially a finite sum by the representation being smooth. ☐

5.3. **Induction and compact induction.**

Definition 5.9. Let H be a subgroup of G and (σ, W) be an irreducible smooth representation of H. Define the *smooth induced representation* $\mathrm{Ind}_H^G \sigma$ to be the space of smooth functions $f : G \to W$ such that

$$f(hg) = \sigma(h)f(g) \text{ for any } h \in H, g \in G.$$

The action of G on this space is by right translation $\pi(g)f(x) = f(xg)$.

Define the *compactly induced representation* $c - \mathrm{Ind}_H^G \sigma$ to be the space of functions f as above with additional condition that the support of f is compact in $H\backslash G$. It is naturally a subrepresentation of $\mathrm{Ind}_H^G \sigma$ (though not necessarily different).

Remark 5.10. These two functors satisfy many nice properties one would expect. For example they send short exact sequeces(s.e.s) to s.e.s. They are transitive.

The compact induction is most useful when H is an open subgroup. In that case H is open and closed (with complement being union of H cosets), and $H\backslash G$ has discrete topology. Then any $f \in c - \mathrm{Ind}_H^G \sigma$ having compact support is actually supported on a finite number of $H-$cosets. We can give a explicit basis for $c - \mathrm{Ind}_H^G \sigma$ as follows. Fix a set of representative $g_i \in H\backslash G$ and a basis $w_j \in W$, let

$$f_{g_i,w_j}(g) = \begin{cases} \sigma(h)w_j, & \text{if } g = hg_i, \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

Lemma 5.11. *If H is an open subgroup of G, then $\{f_{g_i,w_j}\}_{i,j}$ form a basis for $c - \mathrm{Ind}_H^G \sigma$.*

5.4. **Frobenius reciprocity.**

Thus we have two functors $\mathrm{Ind}, c - \mathrm{Ind} : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$. There is naturally a functor in the opposite direction, that is, the restriction of a representation of G to H (forgetting how the other elements act). We shall simply write $\pi|_H$ for the restriction. The following two lemmas tell us how inductions are related to the functor of restriction.

Lemma 5.12 (Frobenius Reciprocity 1)**.**

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) = \mathrm{Hom}_H(\pi|_H, \sigma). \tag{5.6}$$

Proof. There is a canonical $H-$homomorphism

$$\begin{aligned} \alpha_\sigma : \operatorname{Ind}_H^G \sigma &\to W \\ f &\mapsto f(1) \\ \pi(h)f &\mapsto f(1 \cdot h) = f(h) = \sigma(h)f(1) \end{aligned} \tag{5.7}$$

For any $\phi \in \operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \sigma)$, we associate $\tilde{\phi} \in \operatorname{Hom}_H(\pi|_H, \sigma)$ by the following. For any $v \in \pi$, $\phi(v) \in \operatorname{Ind}_H^G \sigma$ and

$$\tilde{\phi}(v) = \alpha_\sigma \circ \phi(v). \tag{5.8}$$

For the inverse direction, let $\tilde{\psi} : V \to W$ be an $H-$homomorphism. Then we associate $\psi \in \operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \sigma)$ satisfying

$$\psi(v) : g \mapsto \tilde{\psi}(\pi(g)v).$$

□

Exercise 5.13. Check that $\psi(v) \in \operatorname{Ind}_H^G \sigma$, ψ is a $G-$homomorphism, and the association $\psi \leftarrow \tilde{\psi}$ is inverse to $\phi \to \tilde{\phi}$.

Lemma 5.14 (Frobenius Reciprocity 2)**.** *Suppose that H is an open subgroup of G.*

$$\operatorname{Hom}_G(c - \operatorname{Ind}_H^G \sigma, \pi) = \operatorname{Hom}_H(\sigma, \pi|_H). \tag{5.9}$$

Proof. There exists a canonical $H-$homomorphism

$$\begin{aligned} \beta_\sigma : W &\to c - \operatorname{Ind} \sigma \\ w &\mapsto f_{1,w}, \end{aligned} \tag{5.10}$$

where $f_{1,w}$ is similarly defined as in (5.5). Now for $\phi \in \operatorname{Hom}_G(c - \operatorname{Ind}_H^G \sigma, \pi)$, we associate $\tilde{\phi} \in \operatorname{Hom}_H(\sigma, \pi|_H)$ by

$$\tilde{\phi}(w) = \phi \circ \beta_\sigma(w). \tag{5.11}$$

For $\tilde{\psi} \in \operatorname{Hom}_H(\sigma, \pi|_H)$, we associate $\psi : c - \operatorname{Ind}_H^G \sigma \to V$ satisfying

$$\psi(f_{1,w}) = \tilde{\psi}(w). \tag{5.12}$$

Then it extends uniquely to be a $G-$ homomorphism. One can easily check that $\psi \leftarrow \tilde{\psi}$ is inverse to $\phi \to \tilde{\phi}$. □

Corollary 5.15 (Useless)**.** *Any irreducible representation π can be realised as a subrepresentation of $\operatorname{Ind}_H^G \sigma$ or a quotient representation of $c - \operatorname{Ind}_H^G \sigma$ for some subgroup H and irreducible σ.*

5.5. **Mackey theory.** For simplicity suppose that G is finite. Let H, K be two subgroups of G, (σ, W) be an irreducible representation of H, and $\rho = \operatorname{Ind}_H^G \sigma$.

Theorem 5.16 (Mackey)**.**

$$\rho|_K \simeq \oplus_{g \in H\backslash G/K} \operatorname{Ind}_{K \cap H^g}^K \sigma^g|_{K \cap H^g}, \tag{5.13}$$

here $H^g = g^{-1}Hg$ and σ^g is the representation of H^g which is realised on W and defined by

$$\sigma^g(h^g)w = \sigma(h)w. \tag{5.14}$$

Proof. For any $g \in H\backslash G/K$, we shall find a subspace of V which is $K-$isomorphic to $\sigma_{g,K} := \mathrm{Ind}_{K\cap H^g}^K \sigma^g|_{K\cap H^g}$. As G is finite, we can explicitly describe a basis $\{f_{g_i,w_j}\}$ as in (5.5) for g_i being representatives of $H\backslash G$ and $\{w_j\}$ basis of W. When $Hg_i \subset HgK$, we can further assume that $g_i = gk_i$. Then to g we associate a subspace

$$V_g = \oplus_{Hgk_i\subset HgK} \oplus_j \mathbb{C}f_{gk_i,w_j}. \tag{5.15}$$

Note that we have a bijection

$$H\backslash HgK \xrightarrow{\simeq} (H^g \cap K)\backslash K \tag{5.16}$$

with elements gk_i sent to k_i.

On the other hand, $\sigma_{g,K}$ has a basis f'_{k_i,w_j}. We define a linear map from V_g to $\sigma_{g,K}$, sending f_{gk_i,w_j} to f'_{k_i,w_j}. We need to check that it's a K-isomorphism. For any $k \in K$, suppose that

$$gk_ik = hgk_j. \tag{5.17}$$

Here $h = h_{k,i,j} \in H$ depends on all parameters. Then

$$\rho(k^{-1})f_{gk_i,w_j} = f_{gk_j,\sigma(h^{-1})w_j}. \tag{5.18}$$

On the other hand, (5.17) can be rewritten as

$$k_ik = g^{-1}hgk_j. \tag{5.19}$$

$$\sigma_{g,K}(k)f'_{k_i,w_j} = f'_{k_j,\sigma^g(g^{-1}h^{-1}g)w_j} = f'_{k_j,\sigma(h^{-1})w_j}. \tag{5.20}$$

So it's indeed $K-$isomorphism. □

Corollary 5.17. *Let σ be an irreducible representation of H and $\pi = \mathrm{Ind}_H^G \sigma$ be semisimple. Then π is irreducible if and only if*

$$\mathrm{Hom}_H(\sigma, \mathrm{Ind}_{H\cap H^g}^H \sigma^g) \neq 0 \Leftrightarrow g \in H. \tag{5.21}$$

Exercise 5.18. Prove this corollary. Hint: use Corollary 5.4, Frobenius reciprocity and the theorem above.

This result is not so useful in the course as π is not necessarily semisimple.
Start of lecture 5

5.6. **Contragredient representation.** In general for a representation (π, V), we can define its dual representation on $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ via

$$< \pi^*(g)v^*, v >=< v^*, \pi(g^{-1})v > . \tag{5.22}$$

Here $< \cdot, \cdot >$ is the natural pairing between V and V^*. This representation in our setting is usually too large and not smooth. We shall consider the smooth part of it

$$\check{V} = \bigcup_K (V^*)^K, \tag{5.23}$$

which is $G-$invariant as conjugation by $g \in G$ gives another compact open subgroup. Let

$$\check{\pi} = \pi^*|_{\check{V}}. \tag{5.24}$$

Definition 5.19. The representation $(\check{\pi}, \check{V})$ is called the contragredient representation, or smooth dual of (π, V).

Example 5.20. Let χ be a character of $\mathbb{F}^* = \mathrm{GL}_1(\mathbb{F})$. Then associated V, $V^* = \check{V}$ are $1-$dimensional, and

$$\check{\chi}(x) = \chi(x^{-1}) = \chi^{-1}(x). \tag{5.25}$$

Lemma 5.21. *G/K is countable. As a result, smooth irreducible representation π of G has countable dimension.*

Proof. There are different ways to show the first claim. For example by Iwasawa decomposition

$$G/K \simeq B/(B \cap K).$$

The right hand side is countable. For simplicity consider $G = \mathrm{GL}_2$, in which case

$$B/(B \cap K) = \begin{pmatrix} \mathbb{F}^* & \mathbb{F} \\ 0 & \mathbb{F}^* \end{pmatrix} / \begin{pmatrix} O_{\mathbb{F}}^* & O_{\mathbb{F}} \\ 0 & O_{\mathbb{F}}^* \end{pmatrix}. \tag{5.26}$$

Note that $\mathbb{F}^*/O_{\mathbb{F}}^* \simeq \varpi^{\mathbb{Z}}$, $\mathbb{F}/O_{\mathbb{F}}$ can be represented by elements from finite field extension of $\mathbb{Q}$. Both are countable and so is $B/(B \cap K)$.

Any vector $v \in \pi$ is fixed by some K' by smoothness. Then by irreducibility π is spanned by $\pi(g)v, g \in G/K'$, which is also countable as $[K : K'] < \infty$. □

We note that if $\dim V = \infty$ but countable, then V^* is not countable. (Think about digits for real numbers.)

Lemma 5.22. *Restricting to V^K induces isomorphism $\check{V}^K \simeq (V^K)^*$*

Proof. To prove that the map $\check{V}^K \to (V^K)^*$ is injective, let $f_1, f_2 \in \check{V}^K$ such that

$$< V^K, f_1 - f_2 >= 0. \tag{5.27}$$

For any $v \in V$, as $f_1 - f_2$ is $K-$invariant, we have that

$$< v, f_1 - f_2 >= \frac{1}{\mathrm{Vol}(K)} \int < v, \check{\pi}(f_1 - f_2) > dk =< \frac{1}{\mathrm{Vol}(K)} \int \pi(k^{-1}) v dk, (f_1 - f_2) >= 0 \tag{5.28}$$

as $\int \pi(k^{-1}) v \in V^K$. Thus $f_1 = f_2$.

On the other hand, define

$$V(K) = \{v \in V, \int_K \pi(k) v dk = 0\}.$$

Then $V = V^K \oplus V(K)$, as

$$v = \frac{1}{\mathrm{Vol}(K)} \int_K \pi(k) v dk + (v - \frac{1}{\mathrm{Vol}(K)} \int_K \pi(k) v dk)$$

where $\frac{1}{\mathrm{Vol}(K)} \int_K \pi(k) v dk \in V^K$ and $(v - \frac{1}{\mathrm{Vol}(K)} \int_K \pi(k) v dk) \in V(K)$.

Then for any linear functional on V^K, we can extend it to be a linear functional on V by taking 0 on $V(K)$. It will be smooth as it's fixed by K. □

In the following lemma we show how admissibility comes into play.

Lemma 5.23. *Let π be smooth. $\pi \to \check{\check{\pi}}$ is isomorphism if and only if π is admissible.*

Proof. It is clear that $\pi \hookrightarrow \check{\check{\pi}}$. To see that the injection is also surjective, we use that both sides are smooth, so it suffice to show that

$$V^K \hookrightarrow \check{\check{V}}^K = (\check{V}^K)^* = (V^K)^{**} \tag{5.29}$$

is surjection for any compact open subgroup K. Here we have used the lemma above for the right hand side. If π is admissible, then V^K is finite dimensional and the map is surjective for finite dimensional dual spaces. On the other hand, if V^K is infinite dimensional but countable, $V^K \to (V^K)^{**}$ is not surjective as $(V^K)^{**}$ is not countable. □

Lemma 5.24. *Let $\pi \in Irr(G)$. Then $\check{\pi} \in Irr(G)$.*

Proof. We just need to prove that $\check{\pi}$ is irreducible. If not, there is a s.e.s of nontrivial representations

$$0 \to V_1 \to \check{\pi} \to V_2 \to 0.$$

By taking smooth dual, we get

$$0 \to \check{V}_2 \to \pi \to \check{V}_1 \to 0,$$

contradicting the condition on π. □

Let $v \in (\pi, V)$ and $v' \in (\check{\pi}, \check{V})$.

Definition 5.25. The matrix coefficient associated to v and v' is defined as the following function on G.

$$\Phi_{v,v'}(g) = < \pi(g)v, v' > . \tag{5.30}$$

6. Parabolic induction theory

Before we start, first note that there is a simple way to product new representation for matrix groups, which is twisting by a character. More precisely, let (π, V) be a representation of GL_n and χ be a multiplicative character of $\mathbb{F}^*$. Then the twisted representation $(\pi \otimes \chi, V)$ is defined on the same space, with the action

$$(\pi \otimes \chi)(g)v = \chi(\det(g))\pi(g)v. \tag{6.1}$$

6.1. **Parabolic induction and Jacquet functor.** Here we develop a variant of induced representation and restriction functor related to parabolic subgroups. Let $G = \mathrm{GL}_n$, $P = P_{\underline{n}}$, $M = M_{\underline{n}}$, $N = N_{\underline{n}}$ as defined in Section 3.1. Note that $P = MN$ and N is a normal subgroup of P.

In particular $M \simeq \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$. Let $(\sigma_i, V_i) \in \mathrm{Irr}(\mathrm{GL}_{n_i})$ and $(\sigma, V) = \otimes_i(\sigma_i, V_i) \in \mathrm{Irr}(M)$. On the other hand let θ be a character of N which is normalised by P, meaning that $\theta(n) = \theta(p^{-1}np)$.

Define a representation $\sigma\theta$ of P on V by

$$(\sigma\theta)(mn) = \sigma(m)\theta(n), \ m \in M, n \in N. \tag{6.2}$$

It is well defined by our assumption on θ.

We first define the parabolic induction to be $\pi = \mathrm{Ind}_P^G \sigma\theta$. Later on we will make slight refinement for this definition.

The analogue of the restriction functor in this case is called Jacquet functor, defined as follows. For $(\pi, V) \in \mathrm{Irr}(G)$, let $V(N, \theta) = \{\theta(n)v - \pi(n)v, n \in N, v \in V\}$, which is closed under the action of P. Let $\pi_{N,\theta}$ be the representation of P on the space $V_{N,\theta} = V/V(N, \theta)$ by restriction and quotient. $\pi_{N,\theta}$ is called Jacquet module associated to (N, θ). It is the maximal quotient of π on which N acts by θ.

Recall that there is a Haar measure on N (which is the product of Lebesgue measures).

Lemma 6.1. *Let $v \in V$. Then $v \in V(N,\theta)$ if and only if there exists a compact open subgroup N_0 of N such that*

$$\int_{N'} \theta^{-1}(n)\pi(n)v dn = 0 \tag{6.3}$$

for any $N_0 < N'$.

Proof. We shall prove for $\theta = 1$ here. Suppose first that $v = \sum_i \pi(n_i)v_i - v_i \in V(N)$. Then there exists a compact open subgroup N_0 containing all n_i, and

$$\int_{N'} \pi(n)v dn = \sum_i \int_{N'} \pi(nn_i)v_i dn - \sum_i \int_{N'} \pi(n)v_i dn = 0. \tag{6.4}$$

On the other hand for any $v \in V$, it is fixed by some compact open subgroup N_1 of N_0. So $v \in V^{N_1}$ and the finite group N_0/N_1 acts on V^{N_1}. Then similar to the Jacquet module, we have

$$V^{N_1}_{N_0} = V^{N_1}/V^{N_1}(N_0/N_1) \tag{6.5}$$

is the maximal quotient of V^{N_1} on which N_0/N_1 acts trivially. But as N_0/N_1 is finite, so any of its representation is semisimple and we have

$$V^{N_1} = V^{N_1}(N_0/N_1) \oplus V^{N_0}. \tag{6.6}$$

The N_0-projection map onto V^{N_0} is given by

$$\varphi : w \mapsto \mathrm{Vol}(N_0)^{-1} \int_{N_0} \pi(n)w dn. \tag{6.7}$$

Then the condition on v is saying that $v \in \ker\varphi = V^{N_1}(N_0/N_1) \subset V(N)$. □

Exercise 6.2. Prove the lemma for general θ. Hint: consider the twist $\pi|_N \otimes \theta^{-1}$.

Lemma 6.3. *Let σ be admissible. Then $\pi = \mathrm{Ind}_P^G \sigma\theta$ is admissible.*

Proof. Let H be any compact open subgroup of K (replace H by $H \cap K$ if necessary). We shall show that π^H is finite dimensional. By Iwasawa decomposition $G = BK$, there are finite number of double $P - H$cosets in G. Let $\{g_i\}_{1 \le i \le k}$ be the collection of representatives and we can further assume that $g_i \in K$. Let $J \subset H$ be a normal compact open subgroup. Note that $J \cap M$ is a compact open subgroup of M.

By the definition of induced representation,

$$\pi = \{f : G \to V, f(pg) = \sigma(p)f(g)\}. \tag{6.8}$$

Then elements f in π^H are uniquely determined by the values $f(g_i) \in V$. Further more for $j \in J \cap M$

$$\pi(j)f(g_i) = f(jg_i) = f(g_i(g_i^{-1}jg_i)) = f(g_i) \tag{6.9}$$

as $g_i^{-1}jg_i \in J$ and f is $J-$invariant. Thus the space π^H is spanned by functions f supporting on Pg_iH such that $f(g_i) \in V^{J\cap M}$. So π^H is finite dimensional as (σ, V) is also admissible.

□

The parabolic induction and Jacquet functor has usual good properties, like transitivity and sending s.e.s to s.e.s.

In the following we will only be interested in the situation when $\theta = 1$. Then we simply omit θ in the notations above.

6.2. **Contragredient representation and normalised induction.**

Proposition 6.4. *Let $\check{\sigma}_i$ be the contragredient representation of σ_i, and $\check{\sigma} = \otimes\check{\sigma}_i$. Then the contragredient representation of* $\mathrm{Ind}_P^G \sigma$ *is* $\mathrm{Ind}_P^G(\check{\sigma} \otimes \Delta_P^{-1})$

Partial proof. We shall only show how to establish a $G-$invariant pairing between $f \in \mathrm{Ind}_P^G \sigma$ and $f' \in \mathrm{Ind}_P^G(\check{\sigma} \otimes \Delta_P^{-1})$. Let $[\cdot, \cdot]$ be the pairing between σ and $\check{\sigma}$. Define the pairing by the integral

$$< f, f' >= \int_K [f(k), f'(k)]dk \tag{6.10}$$

We need to show that this integral is $G-$invariant, i.e. for $\Phi(g) := [f(g), f'(g)]$ and any $h \in G$

$$\int_K \Phi(k)dk = \int_K \Phi(kh)dk. \tag{6.11}$$

Note that the function $\Phi(g)$ satisfies for $p \in P$ that

$$\Phi(pg) = [f(pg), f'(pg)] = [\sigma(p)f(g), \Delta_P^{-1}(p)\check{\sigma}(p)f'(g)] = \Delta_P^{-1}(p)\Phi(g). \tag{6.12}$$

Let $C^\infty(G, \Delta_P^{-1})$ denote the space of functions on G satisfying $\Phi(pg) = \Delta_P^{-1}(p)\Phi(g)$.

On the other hand for $\varphi \in C_c^\infty(G)$, we have

$$\int_G \varphi(gh)dg = \int_G \varphi(g)dg. \tag{6.13}$$

By Iwasawa decomposition and properly normalised Haar measures,

$$\int_G \varphi(g)dg = \int_K \int_P \varphi(pk)d_L p dk, \tag{6.14}$$

with $\tilde{\varphi}(g) := \int\limits_P \varphi(pk)d_L p$ satisfying

$$\tilde{\varphi}(p'g) = \int\limits_B \varphi(pp'k)d_L p = \Delta_P(p'^{-1}) \int\limits_B \varphi(pk)d_L p = \Delta_P^{-1}(p)\tilde{\varphi}(g). \tag{6.15}$$

So if the map

$$\begin{gathered} C_c^\infty(G) \to C^\infty(G, \Delta_P^{-1}) \\ \varphi \mapsto \tilde{\varphi} \end{gathered} \tag{6.16}$$

is surjective, then we can find φ with $\tilde{\varphi} = \Phi$, and $\int_K \Phi(k)dk = \int_G \varphi(g)dg$ is $G-$invariant.

To see that the map is surjective, we check the $H-$invariant part for any compact open subgroup H. For any $g \in P\backslash G/H$. The functions $\Phi \in C^\infty(G, \Delta_P^{-1})$ supported on PgH is at most $1-$dimensional. On the other hand let $\varphi \in C_c^\infty(G)$ be the constant function supported only on gH. Then $\tilde{\varphi}$ is supported only on PgH. It is nontrivial because

$$\tilde{\varphi}(g) = \int_P \varphi(pg)d_L p = \mathrm{Vol}(P \cap gHg^{-1}, d_L P), \tag{6.17}$$

which is nonzero as $P \cap gHg^{-1}$ is an open subgroup of P.

What we haven't checked is that the pairing defined above is non-degenerate. To do this, we need to check that the pairing between $H-$invariant parts are non-degenerate. We skip this step here. □

In the light of this result, from now on we refined the parabolic induction as follows

$$\mathrm{i}_{\mathrm{M,G}}\,\sigma := \mathrm{Ind}_P^G(\sigma \otimes \Delta_P^{-1/2}) \tag{6.18}$$

Then the contragredient representations of $\mathrm{i}_{\mathrm{M,G}}\,\sigma$ is $\mathrm{i}_{\mathrm{M,G}}\,\check{\sigma}$.

Similarly let $r_{G,M}$ be the normalized Jacquet functor

$$\mathrm{r}_{\mathrm{G,M}}\,\pi = \pi_N \otimes \Delta_P^{1/2}. \tag{6.19}$$

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6.3. **Frobenius reciprocity and primitive classification.**

Lemma 6.5 (Frobenius reciprocity 3)**.**

$$\mathrm{Hom}_G(\pi, \mathrm{i}_{\mathrm{M,G}}\,\sigma) = \mathrm{Hom}_M(\mathrm{r}_{\mathrm{G,M}}\,\pi, \sigma). \tag{6.20}$$

Proof. By Frobenius reciprocity 1 in Lemma 5.12,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma \otimes \Delta_P^{-1/2})) = \mathrm{Hom}_P(\pi|_P, \sigma \otimes \Delta_P^{-1/2}) \tag{6.21}$$

Note that Δ_P is trivial on N. We have a map

$$\mathrm{Hom}_M(\pi_N, \sigma \otimes \Delta_P^{-1/2}) \hookrightarrow \mathrm{Hom}_P(\pi|_P, \sigma \otimes \Delta_P^{-1/2})$$

by composing elements from $\mathrm{Hom}_M(\pi_N, \sigma \otimes \Delta_P^{-1/2})$ with the projection map $V \to V/V(N) = V_N$. On the other hand, as N acts trivially on $\sigma \otimes \Delta_P^{-1/2}$, any map in $\mathrm{Hom}_P(\pi|_P, \sigma \otimes \Delta_P^{-1/2})$ factor through π_N. Finally

$$\mathrm{Hom}_M(\pi_N, \sigma \otimes \Delta_P^{-1/2}) = \mathrm{Hom}_M(\pi_N \otimes \Delta_P^{1/2}, \sigma) = \mathrm{Hom}_M(\mathrm{r}_{\mathrm{G,M}}\,\pi, \sigma) \tag{6.22}$$

□

Definition 6.6. Let $\pi \in \mathrm{Irr}(G)$. If $\pi_N \neq 0$ for some nontrivial unipotent subgroup N, then by Lemma 6.5 we can find $\sigma \in \mathrm{Irr}(M)$ such that

$$\mathrm{Hom}_G(\pi, \mathrm{i}_{\mathrm{M,G}}\,\sigma) \neq 0, \tag{6.23}$$

i.e., π is a subrepresentation of $\mathrm{i}_{\mathrm{M,G}}\,\sigma$. Such π is called *non-supercuspidal*. On the other hand, if $\pi_N = 0$ for any non-trivial unipotent subgroup, it can't be constructed as a subrepresentation of $\mathrm{i}_{\mathrm{M,G}}\,\sigma$, and π is called *supercuspidal*.

By convention, all the characters χ of GL_1 will be considered as supercuspidal representations. Also note that if π is (non-)supercuspidal, so will be $\pi \otimes \chi$ for any character χ.

The goal now is to construct and classify the non-supercuspidal representations using supercuspidal ones. First of all, if $\pi_N \neq 0$ is not supercuspidal, then we can find another unipotent subgroup N' in M such that $(\pi_N)_{N'} \neq 0$. In particular we can assume WLOG that $\pi_N \neq 0$ is supercuspidal. Let $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$ in that case. Then there exists $\sigma = \otimes \sigma_i$ with $\sigma_i \in \mathrm{Irr}(\mathrm{GL}_{n_i})$ supercuspidal such that $\pi \hookrightarrow \mathrm{i}_{\mathrm{M,G}}\,\sigma$. The main questions are

(1) Can the set of representations $\{\sigma_i\}$ be used to parametrise π?
(2) Can we say something about when $\mathrm{i}_{\mathrm{M,G}}\,\sigma$ is irreducible?

6.4. **Variant of Mackey theorem and supercuspidal support.** Here we need a variant of Mackey theorem in the context of parabolic induction. Two main modifications are required. The first is to use parabolic induction and Jacquet functor to replace the usual induction and restriction. The second is related to the fact that parabolically induced representations are in general not semisimple for parabolic subgroups.

First of all we set up notations carefully. Let $\alpha = (\alpha_1, \cdots, \alpha_k)$ be a partition of n, and $M_\alpha \simeq \mathrm{GL}_{\alpha_1} \times \cdots \mathrm{GL}_{\alpha_k}$ be the corresponding Levi subgroup. G can be thought of M_α where $\alpha = (n)$. Let $\alpha\, \beta$ be two partitions of n. Let $\alpha \cap \beta$ be the partition corresponding to $M_\alpha \cap M_\beta$. β will be called a sub-partition of α if $\beta \cap \alpha = \beta$ (or equivalently $M_\beta < M_\alpha$).

When $M_\beta < M_\alpha < G$, let $i_{M_\beta, M_\alpha} : \mathrm{Rep}(M_\beta) \to \mathrm{Rep}(M_\alpha)$ be the normalised parabolic induction, where the parabolic subgroup is chosen to be $P_\beta \cap M_\alpha$ which is upper triangular. Similarly let $r_{M_\alpha, M_\beta} : \mathrm{Rep}(M_\alpha) \to \mathrm{Rep}(M_\beta)$ be the normalised Jacquet functor associated to $P_\beta \cap M_\alpha$.

As in Mackey theorem, we will be interested in $r_{G,M_\beta} i_{M_\alpha,G}\sigma$. It is expected that it can be 'decomposed' according to the double coset decomposition $P_\beta \backslash G/P_\alpha$. Using Bruhat decomposition and obvious inclusion of Weyl groups into corresponding parabolic subgroups, one can see that there exists an injection from $P_\beta \backslash G/P_\alpha$ to $W_{M_\beta} \backslash W/W_{M_\alpha}$, where $W_{M_\beta} < P_\beta$ is the Weyl group of M_β. (Actually one can prove that $P_\beta \backslash G/P_\alpha = W_{M_\beta} \backslash W/W_{M_\alpha}$. But we shall not do it here as we only need an injection)

Note that M_α^w is usually not a block wise diagonal matrix. But we can pick a set of special representatives.

Lemma 6.7. *A set of representatives of $W_{M_\beta} \backslash W/W_{M_\alpha}$ can be chosen as follows.*

$$W_{\beta,\alpha} = \{w \in W, w(i) < w(j) \text{ if } i < j \text{ and } i, j \text{ comes from same block of } M_\alpha; \tag{6.24}$$

$$w^{-1}(i) < w^{-1}(j) \text{ if } i < j \text{ and } i, j \text{ comes from same block of } M_\beta\} \tag{6.25}$$

Then for $w \in W_{\beta,\alpha}$, $M_\alpha \cap M_\beta^{w^{-1}}$ is block-wisely diagonal, and $M_\alpha^w \cap M_\beta$ is also block-wisely diagonal.

Proof.

$$W_{\beta,\alpha} = W_{M_\beta} \backslash W/W_{M_\alpha} \tag{6.26}$$

is clear as $W_{M_\beta}, W_{M_\alpha}$ can change ordering in the preimage and image. For the second part of the lemma, we shall only prove that $M_\alpha \cap M_\beta^{w^{-1}}$ is block-wisely diagonal. It suffice to look at a single block M_{α_i} from M_α. (6.24) would guarantee that the preimage of M_{β_j} in M_{α_i}, if nontrivial, will be a block along the diagonal. Thus $M_{\alpha_i} \cap M_\beta^{w^{-1}}$ is block-wisely diagonal. ◻

Theorem 6.8 (Geometric lemma in [1])**.** *Let α, β be any two partitions of n. Then*

$$(r_{G,M_\beta} i_{M_\alpha,G}\sigma)_{ss} = \bigoplus_{w \in W_{\beta,\alpha}} i_{M_\alpha^w \cap M_\beta, M_\beta}(r_{M_\alpha, M_\alpha \cap M_\beta^{w^{-1}}}\sigma)^w \tag{6.27}$$

Remark 6.9. It would seem more consistent with Mackey theorem if we write $r_{M_\alpha^w, M_\alpha^w \cap M_\beta}\sigma^w$ instead of $(r_{M_\alpha, M_\alpha \cap M_\beta^{w^{-1}}}\sigma)^w$, but M_α^w is not block-wisely diagonal in general.

Example 6.10. Let $n = 2$ and $\alpha = \beta = (1, 1)$, $\chi_1 \otimes \chi_2$ be a character on $M = M_\alpha$. Then $W_{\beta,\alpha} = \{1, \omega\}$ for $\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that $(\chi_1 \otimes \chi_2)^\omega = (\chi_2 \otimes \chi_1)$ By the theorem above we have

$$(r_{G,M}\, \mathrm{i}_{\mathrm{M,G}}(\chi_1 \otimes \chi_2))_{ss} = (\chi_1 \otimes \chi_2) \oplus (\chi_2 \otimes \chi_1). \tag{6.28}$$

The unnormalised version says

$$(\mathrm{Ind}(\chi_1|\cdot|^{1/2}\otimes\chi_2|\cdot|^{-1/2}))_N = (\chi_1|\cdot|^{1/2}\otimes\chi_2|\cdot|^{-1/2})\oplus(\chi_2|\cdot|^{1/2}\otimes\chi_1|\cdot|^{-1/2}). \tag{6.29}$$

Theorem 6.11. *Let σ_i be supercuspidal representations of M_i and $\pi_i = i_{M_i,G}\sigma_i$. If π_1 and π_2 have common composition factor, (i.e., $(\pi_i)_{ss}$ have common irreducible subrepresentation) then there exist $w \in W$ such that $M_2 = M_1^w$ and $\sigma_2 = \sigma_1^w$.*

Proof. Let π_0 be such a common composition factor. Let M_0 be a Levi subgroup such that $r_{G,M}\pi_0$ is supercuspidal. Then $r_{G,M_0}\pi_0$ is a common composition factor of $r_{G,M_0}\pi_i$ as r_{G,M_0} sends s.e.s to s.e.s. By Theorem 6.8, we have

$$(r_{G,M_0}\pi_i)_{ss} = \bigoplus_{w\in W_{M_0,M_i}} i_{M_i^w\cap M_0,M_0}(r_{M_i,M_i\cap M_0^{w^{-1}}}\sigma_i)^w. \tag{6.30}$$

But since σ_i is supercuspidal, we have that $r_{M_i,M_i\cap M_0^{w^{-1}}}\sigma_i = 0$ unless $M_i = M_i \cap M_0^{w^{-1}}$. As a result we can write

$$(r_{G,M_0}\pi_i)_{ss} = \bigoplus_{w\in W_{M_0,M_i}, M_i=M_i\cap M_0^{w^{-1}}} i_{M_i^w\cap M_0,M_0}\sigma_i^w, \tag{6.31}$$

Let ρ be a composition factor of $r_{G,M_0}\pi_0$, which is automatically supercuspidal. Then ρ is also a composition factor for one of $i_{M_i^w\cap M_0,M_0}\sigma_i^w$. We need a lemma for which we will postpone the proof.

Lemma 6.12. *Let σ an irreducible supercuspidal representation of M. Then $\mathrm{i}_{\mathrm{M,G}}\,\sigma$ doesn't have any supercuspidal composition factor/subquotient.*

By this Lemma, we must have $M_i^w \cap M_0 = M_0$. So we can further write

$$(r_{G,M_0}\pi_i)_{ss} = \bigoplus_{w\in W_{M_0,M_i}, M_i^w=M_0} \sigma_i^w. \tag{6.32}$$

Then we have that $(M_0,\rho) = (M_i^w, \sigma_i^w)$, and the claim in the theorem follows immediately. □

Corollary 6.13. *For $\pi \in Irr(G)$, suppose that π is a composition factor of $\mathrm{i}_{\mathrm{M,G}}\,\sigma$ for a supercuspidal representation $\sigma \simeq \otimes\sigma_{\alpha_i}$ of block-wisely diagonal $M_\alpha \simeq GL_{\alpha_1}\times\cdots\times GL_{\alpha_k}$. Then the set $\{(GL_{\alpha_i}, \sigma_{\alpha_i})\}$ are uniquely determined by π up to a permutation of blocks.*

Definition 6.14. The set $\{\sigma_{\alpha_i}\}$ is called the cuspidal support of π. This parametrises all $\pi \in \mathrm{Irr}(G)$ in terms of supercuspidal ones.

6.5. **Criterion for irreducibility.**

Theorem 6.15. *Let $\sigma \simeq \otimes\sigma_{\alpha_i}$ be supercuspidal representation of the block-wisely diagonal subgroup $M_\alpha \simeq GL_{\alpha_1}\times\cdots\times GL_{\alpha_k}$. Then $\pi = \mathrm{i}_{\mathrm{M,G}}\,\sigma$ is irreducible if and only if $\sigma_{\alpha_i} \not\simeq \sigma_{\alpha_j}\otimes|\cdot|_{\mathbb{F}}$ for any i, j.*

We shall give the proof in the case of GL_2. In this case, $\sigma = \chi_1\otimes\chi_2$, and the theorem claims that π is irreducible if and only if $\chi_i = \chi_j|\cdot|$. Let $P_1 = \begin{pmatrix} * & * \\ 0 & 1\end{pmatrix}$, $G_1 = \begin{pmatrix} * & 0 \\ 0 & 1\end{pmatrix} \simeq \mathrm{GL}_1$, $P_0 = \{I\}$. Let θ be a nontrivial character of N. To avoid confusion, we shall not do any normalisation in this proof.

We shall study the following functor: $\Psi^- : \mathrm{Rep}(P_1) \to \mathrm{Rep}(G_1)$ is the functor which takes V to $V_N = V/V(N)$, just like Jacquet module. $\Psi^+ : \mathrm{Rep}(G_1) \to \mathrm{Rep}(P_1)$ is the extension by the trivial character on N as in (6.2). $\Phi^-: \mathrm{Rep}(P_1) \to \mathrm{Rep}(P_0)$ be the functor which takes V to $V_{N,\theta} = V/V(N,\theta)$. Let $\Phi^+: \mathrm{Rep}(P_0) \to \mathrm{Rep}(P_1)$ taking any vector space V to $c-\mathrm{Ind}_N^{P_1} V \otimes \theta$, where we let N acts on V by θ. Then we have the following results.

Lemma 6.16. *Let ρ be a representation of P_1. If $\Psi^-\rho = \Phi^-\rho = 0$, then $\rho = 0$.*

Proof. When restricting to the action of N which is abelian, ρ is a direct sum of characters of N. Then we have the following direct sum of P_1−representations.

$$\rho = \rho_1 \oplus (\oplus_{\theta'}\rho_{\theta'}), \tag{6.33}$$

where N acts trivially on ρ_1 and by nontrivial characters on $(\oplus_{\theta'}\rho_{\theta'})$. $\Psi^-\rho = 0$ implies that $\rho_1 = 0$. On the other hand, $\Phi^-\rho = 0$ implies that $\rho_\theta = 0$. Suppose that $\rho_{\theta'} \neq 0$. Then there exists a nontrivial $v \in \rho_{\theta'}$ such that N acts on v by θ'. By Proposition 2.16, $\theta = \theta'_a$ for some $a \in \mathbb{F}^*$. Then

$$\rho(n)\rho(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix})v = \rho(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix})\rho(an)v = \rho(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix})\theta'_a(n)v = \theta(n)\rho(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix})v. \tag{6.34}$$

Thus $\rho_\theta \neq 0$, contradiction. □

Proposition 6.17. (1) *$\Psi^\pm, \Phi^\pm$ are exact.*
(2) *Ψ^+ is adjoint to Ψ^-. Φ^+ is left adjoint to Φ^-.*
(3) *$\Phi^-\Psi^+ = 0$, $\Psi^-\Phi^+ = 0$.*
(4) *$\Psi^-\Psi^+, \Phi^-\Phi^+$ are isomorphic to identity maps.*
(5) *For any representation π of P_1, we have the natural s.e.s*

$$0 \to \Phi^+\Phi^-\pi \to \pi \to \Psi^+\Psi^-\pi \to 0. \tag{6.35}$$

We first show a result of this proposition

Corollary 6.18. *Φ^+ and Ψ^+ send irreducible representations to irreducible representations of P_1.*

Proof. Let $\pi = \Phi^+\sigma$ be reducible. So there exists s.e.s

$$0 \to \pi_1 \to \pi \to \pi_2 \to 0. \tag{6.36}$$

$\Psi^-\pi_i = 0$ as $\Psi^-\pi = 0$. So $\Phi^-\pi_i$ are nontrivial by Lemma 6.16, and we get s.e.s

$$0 \to \Phi^-\pi_1 \to \Phi^-\pi \to \Phi^-\pi_2 \to 0. \tag{6.37}$$

Using that $\Phi^-\pi = \Phi^-\Phi^+\sigma \simeq \sigma$, we conclude that σ should also be reducible. □

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Proof of Proposition 6.17. We will not check all the details. For example (1) is always expected. We also remark that Ψ^+ is really not induction, so the first part in (2)-(4) are direct. Second part of (2) is not directly Lemma 5.14, as H is assumed to be open in G in Lemma 5.14. But we shall assume it's true without justification here.

Let σ be any 1−dimensional vector space. Then

$$\rho = \Phi^+\sigma = \{f \text{ locally constant and compactly supported}, f(nx) = \theta(n)f(x)\}.$$

They can be identify with functions $f': \mathbb{F}^* \to \mathbb{C}$ by restricting on the diagonal

$$f'(x) = f(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}). \tag{6.38}$$

Then the action of ρ can be described as follows

$$\rho(\begin{pmatrix} b & u \\ 0 & 1 \end{pmatrix})f'(x) = f(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & u \\ 0 & 1 \end{pmatrix}) = f(\begin{pmatrix} 1 & ux \\ 0 & 1 \end{pmatrix}\begin{pmatrix} bx & 0 \\ 0 & 1 \end{pmatrix}) = \theta(ux)f'(bx). \tag{6.39}$$

Note that we can choose a basis for ρ to be $f'_{a,m} = \mathrm{char}(a(1 + \varpi^m O_{\mathbb{F}}))$, the characteristic function of a neighbourhood of $a \in \mathbb{F}$. WLOG, we assume that θ is unramified.

Now we show that $\rho = \rho(N)$ and thus $\Psi^-\Phi^+\sigma = 0$. For this we choose the basis in such a way that $m > 1$. We claim that then every $f'_{a,m} \in V(N)$. This is because we can choose $u \in N$ such that $\theta(ua\varpi^m O_{\mathbb{F}}) = 1$, but $\theta(ua) \neq 1$. Then $\rho(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})f'_{a,m}(x) - f'_{a,m} = (\theta(ua) - 1)f'_{a,m}$ is a nonzero multiple of $f'_{a,m}$. Done.

Now we further use this description to study $\Phi^-\Phi^+\sigma$. This time we look at the elements such that

$$m \geq \max\{v_{\mathbb{F}}(a - 1) + 1 - v_{\mathbb{F}}(a), 0\}. \tag{6.40}$$

Then $\rho(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})f'_{a,m}(x) = \theta(ux)f'_{a,m}$ is constant on the support when $v_{\mathbb{F}}(u) = -v_{\mathbb{F}}(a) - m$.

$$\rho(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})f'_{a,m}(x) - \theta(u)f'_{a,m} = (\theta(ua) - \theta(u))f'_{a,m} \tag{6.41}$$

is a nonzero multiple of $f'_{a,m}$ as $v_{\mathbb{F}}(u(a - 1)) = -v_{\mathbb{F}}(a) - m + v_{\mathbb{F}}(a - 1) \leq -1$ and $\theta(u(a - 1)) \neq 1$. This implies that $f'_{a,m} \in V(N, \theta)$ as long as $a \not\equiv 1$ and $m \geq \max\{v_{\mathbb{F}}(a - 1) + 1 - v_{\mathbb{F}}(a), 0\}$.

Now if $a = 1$, any $f'_{1,m}$ will give a nontrivial image in $V_{N,\theta} = V/V(N, \theta)$, and they give the same image in $V_{N,\theta}$ as $f'_{1,m}$'s differ by characteristic functions near $a \not\equiv 1$, which are in $V(N, \theta)$. So $\dim V_{N,\theta} = 1$ and $\Phi^-\Phi^+\sigma \simeq \sigma$.

Now we prove part (5). While it's possible to prove it using the the second part of (2), we shall circumvent the using of it. From the proof of Lemma 6.16, we have a direct sum decomposition for P_1 representations $\pi = \pi_1 \oplus (\oplus\pi_{\theta'})$. We actually want to show that the short exact sequence in (5) splits. Let $\pi_2 = (\oplus\pi_{\theta'})$. It's obvious to see that $\pi_1 = \Psi^+\Psi^-\pi$ and $\pi_2 = \pi(N)$. We need to identify π_2 with $\Phi^+\Phi^-\pi_2$. This identification could follow from Frobenius reciprocity for compact inductions, but as we haven't verified it, we do as follows. Let $\hat{\Phi}^+ : \mathrm{Rep}(P_0) \to \mathrm{Rep}(P_1)$ taking any vector space V to $\mathrm{Ind}_N^{P_1} V \otimes \theta$. Then we have a standard Frobenius reciprocity for smooth induction

$$\mathrm{Hom}_{P_1}(\pi_2, \hat{\Phi}^+\Phi^-\pi_2) = \mathrm{Hom}_{P_0}(\Phi^-\pi_2, \Phi^-\pi_2),$$

and a map $\varphi : \pi_2 \to \hat{\Phi}^+\Phi^-\pi_2$ corresponding to the identity map on the right hand side. Note that $\pi_2 = \pi_2(N)$ so the image of φ is in $\Phi^+\Phi^-\pi_2(N)$.

First of all, we show that $\hat{\Phi}^+\sigma(N) = \Phi^+\sigma$. By the proof for $\Psi^-\Phi^+ = 0$ above, we already have that $\Phi^+\sigma \subset \hat{\Phi}^+\sigma(N)$. On the other hand for any $f \in \hat{\Phi}^+\sigma$, we can use the model above, identifying f with f' on $\mathbb{F}^*$, which is now no longer compactly supported. But as $\rho(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix})f'(x) = \theta(nx)f'(x)$ while f' has to be smooth and thus fixed by some compact subgroup of N, we have that $f'(x) = 0$ when $v_{\mathbb{F}}(x) \to -\infty$. $\rho(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix})f'(x) - f'(x) = 0$ as $v_{\mathbb{F}}(x) \to \infty$ because $\theta(nx) \to 1$. So we get that $\hat{\Phi}^+\sigma(N) \subset \Phi^+\sigma$. Done.

Thus we actually have $\varphi : \pi_2 \to \Phi^+\Phi^-\pi_2$.

Exercise 6.19. Show that φ is a bijection. Hint: apply Φ^- and Ψ^- to $\ker\varphi$ and cokerφ and use Lemma 6.16.

$\square$

Lemma 6.20. *There exists a non-degenerate pairing between* $1-$*dimensional representations* V_1 *and* V_2 *of* G_1*, if and only if there exists a* P_1-*invariant pairing between* $\Psi^+ V_1$ *and* $\Psi^+ V_2$.

Exercise 6.21. Prove this lemma.

Lemma 6.22. *For* $\pi = \mathrm{Ind}_P^G(\chi_1|\cdot|^{1/2} \otimes \chi_2|\cdot|^{-1/2})$*, we have that* $\Phi^-(\pi|_{P_1})$ *is* $1-$*dimensional.*

Proof. The map

$$\alpha : f \in \pi \mapsto f(1) \tag{6.42}$$

gives a homomorphism of P_1 representations, where N acts trivially on the image. Thus $\Phi^-(\mathrm{Image}\alpha) = 0$. For any $f \in \ker\alpha$, $f(1) = 0$ and $f = 0$ in a neighbourhood of form BN' where N' is a compact open subset of lower diagonal matrices. Using that

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} -x^{-1} & 1 \\ 0 & x \end{pmatrix} \omega \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \in B\omega N, \tag{6.43}$$

we get that f is compactly supported functions on N when restricting to ωN. So we have identified $\ker\alpha$ with

$$W = \{\tilde{f} = f|_{\omega N},\ \text{compactly supported smooth functions on } N\}.$$

By Example 6.10, we have that

$$\Psi^-(\ker\alpha) \subset \Psi^-\pi = \chi_1|\cdot|^{1/2} \oplus \chi_2|\cdot|^{1/2}, \tag{6.44}$$

and $\Psi^-\mathrm{Image}\alpha$ is $1-$dimensional. So $\Psi^- W = W_N$ is also $1-$dimensional. On the other hand we have the following twisting

$$W \to W \tag{6.45}$$

$$\tilde{f} \mapsto \tilde{f}\theta(x) \tag{6.46}$$

It maps $W(N, 1)$ to $W(N, \theta)$ as $\pi(n)\tilde{f}(x) - \tilde{f}(x)$ is mapped to

$$\begin{aligned} \theta(x)\pi(n)\tilde{f}(x) - \theta(x)\tilde{f}(x) &= \theta(x)\tilde{f}(x+n) - \theta(x)\tilde{f}(x) \\ &= \theta(n^{-1})\pi(n)\theta\tilde{f}(x) - \theta(x)\tilde{f}(x) \\ &= \theta(n^{-1})(\pi(n)\theta\tilde{f}(x) - \theta(n)\theta(x)\tilde{f}(x)) \end{aligned} \tag{6.47}$$

Thus $W/W(N,\theta) \simeq W/W(N)$ should also be $1-$dimensional. $\square$

Proof of Theorem 6.15 in case of GL_2. Now let $\pi = \mathrm{Ind}_P^G(\chi_1|\cdot|^{1/2} \otimes \chi_2|\cdot|^{-1/2})$, and $\check{\pi} = \mathrm{Ind}_P^G(\chi_1^{-1}|\cdot|^{1/2} \otimes \chi_2^{-1}|\cdot|^{-1/2})$. (To avoid confusion, we will use unnormalised version here.)

If π is not irreducible, then it has at least two composition factors. When restricting to P_1, only one of them will have nontrivial image under Φ^-. Let ρ be the composition factor of π with $\Phi^-\rho = 0$. Let $\check{\rho}$ be the contragredient representation of ρ. It is a composition factor of $\check{\pi}$, and the $G-$invariant pairing directly gives a $G-$invariant pairing between ρ and $\check{\rho}$ which is non-degenerate.

But by assumption, $\rho = \Psi^+\Psi^-\rho$, so the pairing above gives a nontrivial P_1-invariant pairing between ρ and $\Psi^+\Psi^-\check{\rho}$, and thus by Lemma 6.20 a nontrivial G_1-invariant pairing between $\Psi^-\rho$ and $\Psi^-\check{\rho}$.

By Example 6.10, we have that

$$\Psi^-\rho \subset \Psi^-\pi = \chi_1|\cdot|^{1/2} \oplus \chi_2|\cdot|^{1/2}, \tag{6.48}$$

$$\Psi^-\check{\rho} \subset \Psi^-\check{\pi} = \chi_1^{-1}|\cdot|^{1/2} \oplus \chi_2^{-1}|\cdot|^{1/2}. \tag{6.49}$$

So there is a nontrivial G_1−invariant pairing between $\Psi^-\rho$ and $\Psi^-\check{\rho}$ if and only if

$$\chi_1|\cdot|^{1/2} = (\chi_2^{-1}|\cdot|^{1/2})^{-1}, \text{ or } \chi_2|\cdot|^{1/2} = (\chi_1^{-1}|\cdot|^{1/2})^{-1}. \tag{6.50}$$

Note that the other two pairs are impossible. In summary, if π is not irreducible, then

$$\frac{\chi_1}{\chi_2} = |\cdot|^{\pm 1}. \tag{6.51}$$

Now we show that when $\frac{\chi_1}{\chi_2} = |\cdot|^{\pm 1}$, π is indeed reducible. We only have to look at the case $\frac{\chi_1}{\chi_2} = |\cdot|^{-1}$, as we can get the other case by taking a contragredient. Then let $\chi_1 = \chi|\cdot|^{-1/2}$ and $\chi_2 = \chi|\cdot|^{1/2}$, we have by definition $\pi = \chi \otimes \mathrm{Ind}_P^G(1 \otimes 1)$. There exist a 1−dimensional subrepresentation spanned by $f \in \pi$, $f(g) = \chi(\det g)$. It apparently satisfies

$$f(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} g) = \chi(ab)\chi(\det g) = \chi(ab)f(g).$$

□

Remark 6.23. (1) For $\pi = \mathrm{Ind}_P^G(\chi_1|\cdot|^{1/2} \otimes \chi_2|\cdot|^{-1/2})$, we have that $\Psi^-\pi$ is 2−dimensional and $\Phi^-\pi$ is 1−dimensional. Thus $\pi|_{P_1}$ has length at most 3 by Proposition 6.17 (5) and Corollary 6.18. In particular π is at most length 3 when it's not irreducible. It's possible to reduce the length down to 2, but we shall not pursue it here.
(2) The proof for irreducibility can be generalised to GL_n with some more works. But the proof for reducibility is limited to GL_2.
(3) Note that the proof above for contradiction doesn't work if π is irreducible, as $\pi|_{P_1}$ is in general not contragredient to $\check{\pi}|_{P_1}$. It is related to that $\Phi^+\sigma$ should be contragredient to $\mathrm{Ind}_N^G\,\theta$ instead of $c - \mathrm{Ind}_N^G\,\theta$.

7. Compact induction theory

7.1. **Alternative description of supercuspidal representations.**

Definition 6.6 is often not easy to use. The main result of this subsection is the following.

Proposition 7.1. *Let $\pi \in Irr(G)$ and $\check{\pi}$ be its contragredient representation and $\Phi_{v,\check{v}}(g) =< \pi(g)v, \check{v} >$ be the matrix coefficient associated to $v \in \pi, \check{v} \in \check{\pi}$. TFAE*

(1) *π is supercuspidal.*
(2) *$\Phi_{v,\check{v}}$ is compactly supported mod center for some pair $(v, \check{v})$.*
(3) *$\Phi_{v,\check{v}}$ is compactly supported mod center for any pair $(v, \check{v})$.*

Proof. We shall give the proof in the GL_2 case. The general case can be proven similarly.
(3) ⇒ (2) is obvious.
(2) ⇒ (3): recall from Lemma 5.24 that π and $\check{\pi}$ are all irreducible. Thus any other $v' \in \pi, \check{v}' \in \check{\pi}$ are of form

$$v' = \sum a_i\pi(g_i)v, \check{v}' = \sum b_j\check{\pi}(g_j)\check{v} \tag{7.1}$$

Thus

$$\Phi_{v',\check{v}'} = \sum a_i b_j \Phi_{v,\check{v}}(g_j^{-1} g g_i) \tag{7.2}$$

is still compactly supported mod center.

(1) $\Rightarrow$ (3): recall the Cartan decomposition $\mathrm{GL}_2 = Z \coprod K \begin{pmatrix} \varpi^i & 0 \\ 0 & 1 \end{pmatrix} K$. $\check{v}$ smooth implies that there exists N_1 compact open subgroup of N which fixes $\check{\pi}(K)\check{v}$. π supercuspidal implies that there exist N_2 compact open such that

$$\int_{N'} \pi(n) v' dn = 0 \tag{7.3}$$

whenever $N_2 < N'$ and $v' \in \pi(K)v$, according to Lemma 6.1. Then for any $g \in \mathrm{GL}_2$ whose Cartan decomposition corresponds to $t = \begin{pmatrix} \varpi^i & 0 \\ 0 & 1 \end{pmatrix}$ with i large enough such that $N_2 < t^{-1} N_1 t$, we have for some nonzero constants c_1,

$$\begin{aligned} < \pi(g)v, \check{v} >=< \pi(t)v', \check{v}' > &= c_1 \int_{N_1} < \pi(t)v, \check{\pi}(n^{-1})v' > dn \\ &= c_1 \int_{N_1} < \pi(t^{-1}nt)v, \check{\pi}(t^{-1})v' > dn \\ &= 0. \end{aligned} \tag{7.4}$$

(3) $\Rightarrow$ (1): Let $K_I(n) = I + \varpi^n M_{2\times 2}(O_{\mathbb{F}})$. For any $v \in \pi$, v is fixed by some $K_I(n)$. Then for $t = \begin{pmatrix} \varpi^i & 0 \\ 0 & 1 \end{pmatrix}$ when i is large enough, $< \pi(t)v, \check{v} >= 0$ for any $\check{v} \in \check{\pi}^{K_I(n)}$. This implies that

$$\int_{g \in K_I(n)} \pi(g)\pi(t) v dg \in \pi^{K_I(n)} \tag{7.5}$$

has to be zero. But on the other hand,

$$\int_{g \in K_I(n)} \pi(g)\pi(t) v dg = \int_{g \in K_I(n)} \pi(t)\pi(t^{-1} g t) v dg \tag{7.6}$$

is equivalently an integral over

$$t^{-1} K_I(n) t / K_I(n) \cap t^{-1} K_I(n) t = \begin{pmatrix} 1+\varpi^n & \varpi^{n-i} \\ \varpi^{n+i} & 1+\varpi^n \end{pmatrix} / \begin{pmatrix} 1+\varpi^n & \varpi^n \\ \varpi^{n+i} & 1+\varpi^n \end{pmatrix}$$

which is actually the same as an integral on some compact open $N_0 < N$. Then (i) follows by Lemma 6.1. □

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We show a sequence of results following from the proposition above.

Lemma 7.2. *Let π be smooth, irreducible and supercuspidal. Then π is admissible.*

Proof. Suppose that π is not admissible. Then there exists a compact subgroup K, such that π^K is infinite dimensional but countable by Lemma 5.21, and $\check{\pi}^K \simeq \check{\pi^K}$ is uncountable. The resulting matrix coefficients formed by π^K and $\check{\pi}^K$ would be uncountable. But on the other hand, such matrix coefficients are bi$-K-$invariant and also compactly supported, they must be countable. Contradiction. □

Lemma 7.3. *Let $\pi \in Irr(G)$ be supercuspidal, with unitary central character. Then π has a $G-$invariant unitary pairing (called unitary) given by*

$$(v_1, v_2) = \int_{Z\backslash G} < \pi(g)v_1, \check{v} > \overline{< \pi(g)v_2, \check{v} >} dg \tag{7.7}$$

for any nonzero $\check{v} \in \check{\pi}$.

Proof. The integral is convergent by Proposition 7.1. It is unitary by the symmetry. It is $G-$invariant by a change of variables. It is nontrivial when we take $v_1 = v_2$ to be some element such that $< v_1, \check{v} > \neq 0$. □

Lemma 7.4. *For any supercuspidal representation $\pi \in Irr(G)$, there exist a character χ such that $\pi \otimes \chi$ has unitary central character.*

Proof. Recall that a multiplicative character χ is unitary if and only if $|\chi| = 1$. Also recall that $\mathbb{F}^* = \varpi^{\mathbb{Z}} \times O_{\mathbb{F}}^*$. For any character χ, its values on $O_{\mathbb{F}}^*$ are always roots of unity and in particular satisfies $|\chi| = 1$. Then one can easily choose a twist so that $|w_\pi(\varpi)| = 1$ too. □

Lemma 7.5 (Formal degree)**.** *Let $\pi \in Irr(G)$ be supercuspidal. Then there exists a nonzero constant c_π such that*

$$I(v_1, v_2, \check{v}_1, \check{v}_2) = \int_{Z\backslash G} < \pi(g^{-1})v_1, \check{v}_1 >< \pi(g)v_2, \check{v}_2 > dg = c_\pi < v_1, \check{v}_2 >< v_2, \check{v}_1 > \tag{7.8}$$

for any $v_i \in \pi$ and $\check{v}_i \in \check{\pi}$. $d_\pi = \frac{1}{c_\pi}$ is called the formal degree of π.

Proof. The integral $I(v_1, v_2, \check{v}_1, \check{v}_2)$ in (7.8) gives an element of $\mathrm{Hom}_G(\pi \otimes \check{\pi}, \mathbb{C})^2$ as

$$I(\pi(g)v_1, v_2, \check{v}_1, \pi(g)\check{v}_2) = I(v_1, \pi(g)v_2, \pi(g)\check{v}_1, \check{v}_2) = I(v_1, v_2, \check{v}_1, \check{v}_2) \tag{7.9}$$

by change of variables. But on the other hand $\mathrm{Hom}_G(\pi \otimes \check{\pi}, \mathbb{C}) = \mathrm{Hom}_G(\pi, \check{\check{\pi}}) = \mathbb{C}$ by Schur's lemma. Thus there exists a constant c_π such that

$$I(v_1, v_2, \check{v}_1, \check{v}_2) = c_\pi < v_1, \check{v}_2 >< v_2, \check{v}_1 > . \tag{7.10}$$

We need to show that this constant is nonzero. First of all, note that

$$\begin{aligned} I(v_1, v_2, \check{v}_1, \check{v}_2) &= \int_{Z\backslash G} < \pi(g^{-1})v_1, \check{v}_1 >< \pi(g)v_2, \check{v}_2 > dg \\ &= \int_{Z\backslash G} < \pi \otimes \chi(g^{-1})v_1, \check{v}_1 >< \pi \otimes \chi(g)v_2, \check{v}_2 > dg \end{aligned} \tag{7.11}$$

Applying same argument again we get that $c_\pi = c_{\pi\otimes\chi}$. By Lemma 7.4, we can assume WLOG that π is unitary after a proper twist. Being unitary allow us to identify π with $\check{\pi}$ via unitary pairing. In particular let v_3, v_4 be such that $(v, v_3) =< v, \check{v}_1 >$, $(v, v_4) =< v, \check{v}_2 >$. Then we have

$$c_\pi(v_1, v_4)(v_2, v_3) = \int_{Z\backslash G} (\pi(g^{-1})v_1, v_3)(\pi(g)v_2, v_4)dg \tag{7.12}$$

$$= \int_{Z\backslash G} \overline{(\pi(g)v_3, v_1)}(\pi(g)v_2, v_4)dg. \tag{7.13}$$

Now taking $v_2 = v_3$ and $v_1 = v_4$, then the right hand side of the equality above is nonzero. Thus $c_\pi \neq 0$. □

Remark 7.6. All the discussion above applies to representations whose matrix coefficient belongs to $L^2(Z\backslash G)$. Such representations which are not supercuspidal exist. They are called discrete series representations.

Lemma 7.7. *Let σ an irreducible supercuspidal representation of M. Then $\mathrm{i}_{\mathrm{M,G}}\,\sigma$ doesn't have any supercuspidal composition factor/subquotient.*

Proof. From Definition 6.6, we know that $\mathrm{i}_{\mathrm{M,G}}\,\sigma$ can't have supercuspidal subrepresentation, as

$$\mathrm{Hom}_G(\rho, \mathrm{i}_{\mathrm{M,G}}\,\sigma) = \mathrm{Hom}_M(\mathrm{r}_{\mathrm{G,M}}\,\rho, \sigma) = 0. \tag{7.14}$$

Suppose now that W is a subrepresentation of $\mathrm{i}_{\mathrm{M,G}}\,\sigma$, and $\varphi : W \to V$ is a surjection to a supercuspidal representation (ρ, V). We shall construct a $G-$map from V back to W as follows. Let $v_0 \in V$ be a nontrivial element, and $w_0 \in W$ be any preimage of v_0 under the map φ. Let $\Phi_{v,\check{v}_0}$ be the matrix coefficient associated to v and some fixed $\check{v}_0 \in \check{\rho}$ such that $< v_0, \check{v}_0 > \neq 0$. Then define the map

$$\tilde{\varphi}(v) = \int\limits_{Z\backslash G} \Phi_{v,\check{v}_0}(g^{-1})\pi(g)w_0 dg. \tag{7.15}$$

The integral is convergent as $\Phi_{v,\check{v}_0}$ is compactly supported mod center by Proposition 7.1. It is $G-$equivalent as

$$\tilde{\varphi}(\rho(h)v) = \int\limits_{Z\backslash G} < \rho(g^{-1}h)v, \check{v}_0 > \pi(g)w_0 dg = \int\limits_{Z\backslash G} < \rho(g^{-1})v, \check{v}_0 > \pi(hg)w_0 dg = \pi(h)\tilde{\varphi}(v). \tag{7.16}$$

It is nontrivial because

$$< \varphi(\tilde{\varphi}(v_0)), \check{v}_0 >= \int\limits_{Z\backslash G} \Phi_{v_0,\check{v}_0}(g^{-1}) < \varphi(\pi(g)w_0), \check{v}_0 > dg = c_\pi < v_0, \check{v}_0 >^2 \neq 0. \tag{7.17}$$

Then we get a contradiction as $\mathrm{i}_{\mathrm{M,G}}\,\sigma$ can't have supercuspidal subrepresentation. □

7.2. General results for compact induction.

Definition 7.8. Let $\pi \in \mathrm{Rep}(G)$, J be compact open subgroup of G, and $\rho \in \mathrm{Rep}(J)$. We say ρ occurs in π, or π contains ρ if $\mathrm{Hom}_J(\rho, \pi|_J) \neq 0$.

This directly implies that $\mathrm{Hom}_G(c - \mathrm{Ind}\,\rho, \pi) \neq 0$ by Frobenius reciprocity.

Definition 7.9. Let J_i be two compact open subgroups of G, $\rho_i \in \mathrm{Irr}(J_i)$. We say $g \in G$ intertwines ρ_1 with ρ_2 if $\mathrm{Hom}_{J_1^g \cap J_2}(\rho_1^g, \rho_2) \neq 0$. Here recall that $J_1^g = g^{-1}J_1 g$, and $\rho_1^g(g^{-1}jg) = \rho_1(j)$. Note that if g intertwines ρ_1 with ρ_2, then g^{-1} intertwines ρ_2 with ρ_1. We simply say g intertwines ρ (with itself) when $\rho_1 = \rho_2 = \rho$.

Let J be a compact open subgroup of G, $\Lambda \in \mathrm{Irr}(J)$ and $\pi = c - \mathrm{Ind}_J^G \Lambda$. Recall that we can explicitly give a basis for π as in (5.5)

$$f_{g_i,w_j}(g) = \begin{cases} \Lambda(j)w_j, & \text{if } g = jg_i \text{ for } j \in J, \\ 0, & \text{otherwise.} \end{cases} \tag{7.18}$$

Here g_i are representatives for $J\backslash G$ and w_j are basis for Λ. Let $< \cdot, \cdot >_J$ be a $J-$invariant pairing between Λ and $\check{\Lambda}$. Let $\pi' = c - \mathrm{Ind}_J^G \check{\Lambda}$ and f'_{g_i,w'_j} be a basis of π'.

Lemma 7.10. *The pairing given by*

$$< f_{g_i,w}, f'_{g_j,w'} >= \delta_{i,j} < w, w' >_J \tag{7.19}$$

extends to a G−invariant pairing between π and π′. In particular if π is irreducible smooth, then it's supercuspidal.

Proof. For any $g \in G$, multiplication on right by g permutes the cosets Jg_i. So let

$$g_i g = a_{g,i} g_{s(i)} \tag{7.20}$$

for $s(i)$ a permutation of i's and $a_{g,i} \in J$. Then

$$\begin{aligned} < \pi(g) f_{g_i,w}, \pi(g) f'_{g_j,w'} > &=< f_{g_{s(i)},\Lambda(a_{g,i})w}, f'_{g_{s(j)},\Lambda(a_{g,j})w'} >= \delta_{s(i),s(j)} < \Lambda(a_{g,i})w, \Lambda(a_{g,j})w' >_J \\ &= \begin{cases} < w, w' >_J =< f_{g_i,w}, f'_{g_j,w'} >, & \text{if } i = j, \\ 0 =< f_{g_i,w}, f'_{g_j,w'} >, & \text{if } i \neq j. \end{cases} \end{aligned} \tag{7.21}$$

So this pairing is $G-$invariant. In particular we can choose w, w' such that $< w, w' >_J \neq 0$ and $< f_{1,w}, f'_{1,w'} > \neq 0$. Then the matrix coefficient Φ associated to $f_{1,w}, f'_{1,w'}$ satisfies

$$\Phi(g) = \begin{cases} < \Lambda(j)w, w' >_J, & \text{if } g = j \in J, \\ 0, & \text{otherwise.} \end{cases} \tag{7.22}$$

Apparently Φ is compactly supported on J. By Proposition 7.1, π is supercuspidal when it is irreducible. □

Proposition 7.11. *For $\pi = c - \mathrm{Ind}_J^G \Lambda$, suppose that g intertwines Λ iff $g \in J$. Then π is irreducible and supercuspidal.*

Proof. By the lemma above, we just have to show that π is irreducible. This is the analogue of Corollary 5.17.

The main tool is again an analogue of Mackey's theory. For another open compact subgroup J',

$$c - \mathrm{Ind}_J^G \Lambda|_{J'} = \oplus_{g \in J \backslash G / J'} \mathrm{Ind}_{J' \cap J^g}^{J'} (\Lambda^g|_{J' \cap J^g}) \tag{7.23}$$

Exercise 7.12. Verify this version of Mackey's theory. Hint: as we can explicitly give basis for compact inductions when J is open, the proof for Mackey's theory for finite groups should carry through. Unlike parabolic induction, we don't have to take semi-simplification for restriction to compact open subgroups.

Taking $J' = J$, we have that

$$\mathrm{Hom}_J(\Lambda, \pi|_J) = \oplus_{g \in J \backslash G / J} \mathrm{Hom}_J(\Lambda, \mathrm{Ind}_{J \cap J^g}^J (\Lambda^g|_{J \cap J^g})) = \oplus \mathrm{Hom}_{J^g \cap J}(\Lambda, \Lambda^g). \tag{7.24}$$

We have used the Frobenius reciprocity in the last equality. Then $\mathrm{Hom}_{J^g \cap J}(\Lambda, \Lambda^g) \neq 0$ iff g intertwines Λ iff, by the condition, $g \in J$. Thus only the coset representative 1 occurs on the RHS and $\mathrm{Hom}_J(\Lambda, \pi|_J)$ is $1-$dimensional as Λ is irreducible. Note that at this point we can't yet claim that π is irreducible by using reciprocity and that $\dim \mathrm{Hom}_G(\pi, \pi) = 1$, as π in general is not semi-simple.

Suppose that π is not irreducible and σ is a subrepresentation of π. Then we have

$$0 \neq \mathrm{Hom}_G(\sigma, \pi) \subset \mathrm{Hom}_G(\sigma, \mathrm{Ind}_J^G \Lambda) = \mathrm{Hom}_J(\sigma|_J, \Lambda) = \mathrm{Hom}_J(\Lambda, \sigma|_J) \tag{7.25}$$

so $\sigma|_J$ contains Λ. By the previous argument, we have that Λ occurs in $\pi|_J$ with multiplicity 1, and $\pi^\Lambda := \{f_{1,w}, w \in \Lambda\}$ is such a copy, generating the whole representation π by G action. Thus $\sigma|_J$ contains π^Λ and $\sigma = \pi$. □

Lemma 7.13. *Let π be irreducible smooth. Assume that π contains representations ρ_i of K_i for $i = 1, 2$. Then there exists g which intertwines ρ_1 with ρ_2.*

Proof. If π contains ρ_1 of K_1, then by Frobenius reciprocity, we have that $\mathrm{Hom}_G(c\text{−}\mathrm{Ind}_{K_1}^G \rho_1, \pi) \neq 0$, and any nontrivial element φ_1 in it is surjective as π is irreducible. π contains ρ_2 of K_2 implies that $\mathrm{Hom}_{K_2}(\pi|_{K_2}, \rho_2) \neq 0$, and let φ_2 be a nontrivial element in it. Then $\varphi_2 \circ \varphi_1$ gives a non-trivial element of $\mathrm{Hom}_{K_2}(c - \mathrm{Ind}_{K_1}^G \rho_1|_{K_2}, \rho_2) \neq 0$. By Mackey's theory and Frobenius reciprocity again, we get that

$$0 \neq \oplus_{g \in K_1\backslash G/K_2} \mathrm{Hom}_{K_2}(\mathrm{Ind}_{K_2 \cap K_1^g}^{K_2}(\rho_1^g|_{K_2\cap K_1^g}), \rho_2) = \oplus_{g\in K_1\backslash G/K_2} \mathrm{Hom}_{K_2\cap K_1^g}(\rho_1^g, \rho_2) \tag{7.26}$$

Thus one of the terms on the RHS is nonzero, meaning that some g intertwines ρ_i. □

Start of lecture 9

7.3. **Lattice chain and filtration of compact subgroups.** The goal now is to construct various compact subgroups, and proper representations for them (preferably characters) so that the condition in Proposition 7.11 is satisfied. For simplicity we shall only work with GL_2 from now on. Similar theory exists for GL_n and classical groups.

Recall the relation between compact open groups and lattices, which motivates the considerations in this section.

Let $V = \mathbb{F} \oplus \mathbb{F}$, so $G = \mathrm{Aut}_{\mathbb{F}} V$. Let $A = \mathrm{End}_{\mathbb{F}} V$, the ring of endomorphisms of V. Recall we have considered the $O_{\mathbb{F}}$−lattice L in V in Definition 3.3. For simplicity we shall just say lattice L.

Definition 7.14. An $O_{\mathbb{F}}$ lattice chain $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$ is a collection of lattices such that $L_{i+1} \subsetneq L_i$ and $xL_i = L_j$ for any $x \in \mathbb{F}^*$.

Lemma 7.15. *There exists an integer $e = e_{\mathcal{L}}$, called the ramification index of the lattice chain $\mathcal{L}$, such that*

$$xL_i = L_{i+ev_{\mathbb{F}}(x)}. \tag{7.27}$$

Proof. Let $x = u\varpi^k$ where $k = v_{\mathbb{F}}(x)$. Then $uL_i = L_i$ as L_i is $O_{\mathbb{F}}$ lattice and $u \in O_{\mathbb{F}}^*$. On the other hand there exists a function $e(i)$ such that $\varpi L_i = L_{i+e(i)}$. We just have to show that $e(i)$ is independent of i. For $L_{i+1} \subsetneq L_i$, we have by multiplying with ϖ,

$$L_{i+1+e(i+1)} \subsetneq L_{i+e(i)}.$$

So $e(i + 1) \geq e(i)$. If $e(i + 1) > e(i)$, then we have a tower of lattices $L_{i+1+e(i+1)} \subsetneq L_{i+e(i+1)} \subsetneq L_{i+e(i)}$. Multiplying with ϖ^{-1} gives us another lattice between L_i and L_{i+1}, which is impossible. Thus $e(i) = e(i + 1)$ for any i. □

We can classify lattice chains by $e_{\mathcal{L}}$ up to a change of basis.

Lemma 7.16. $e_{\mathcal{L}} = 1$ *or* 2.

(1) *If $e_{\mathcal{L}} = 1$, then there exists $g \in G$, s.t. $gL_i = \varpi^i O_{\mathbb{F}} \oplus \varpi^i O_{\mathbb{F}}$ for $i \in \mathbb{Z}$.*
(2) *If $e_{\mathcal{L}} = 2$, then there exists $g \in G$, s.t. $gL_{2i} = \varpi^i O_{\mathbb{F}} \oplus \varpi^i O_{\mathbb{F}}$, $gL_{2i+1} = \varpi^i O_{\mathbb{F}} \oplus \varpi^{i+1} O_{\mathbb{F}}$.*

Proof. $L_0/\varpi L_0 = L_0/L_e$ is a vector space over the residue field k, which is always 2−dimensional. L_i/L_e for $0 \leq i \leq e$ form a flag variety of this vector space, so $1 \leq e \leq 2$. Choose a basis first so that $gL_0 = O_{\mathbb{F}} \oplus O_{\mathbb{F}}$. Then if $e = 1$, $gL_i = \varpi^i gL_0 = \varpi^i O_{\mathbb{F}} \oplus \varpi^i O_{\mathbb{F}}$. If $e = 2$, then $gL_{2i} = \varpi^i gL_0 = \varpi^i O_{\mathbb{F}} \oplus \varpi^i O_{\mathbb{F}}$. gL_1/gL_2 is a k−subspace of $gL_0/gL_2 \simeq k \oplus k$. Then there exists

$\overline{h} \in \mathrm{GL}_2(k)$ such that $\overline{h}g(L_1/L_2) \simeq k \oplus 0$. Pick any lift h of $\overline{h}$ in $\mathrm{GL}_2(O_{\mathbb{F}})$. Then h stabilise L_{2i} and $hgL_1 = O_{\mathbb{F}} \oplus \varpi O_{\mathbb{F}}$. □

Definition 7.17. Let $\mathcal{U}_{\mathcal{L}} = \bigcap_i \mathrm{End}_{O_{\mathbb{F}}} L_i = \{x \in A, xL_i \subset L_i, \forall i \in \mathbb{Z}\}$, called chain order associated to $\mathcal{L}$.

Then as a corollary for the classification of $\mathcal{L}$, we have the following result

Lemma 7.18. *There exists $g \in G$ s.t.*

$$g\mathcal{U}_{\mathcal{L}}g^{-1} = \begin{cases} \begin{pmatrix} O_{\mathbb{F}} & O_{\mathbb{F}} \\ O_{\mathbb{F}} & O_{\mathbb{F}} \end{pmatrix}, & \text{if } e_{\mathcal{L}} = 1, \\ \begin{pmatrix} O_{\mathbb{F}} & O_{\mathbb{F}} \\ \varpi O_{\mathbb{F}} & O_{\mathbb{F}} \end{pmatrix}, & \text{if } e_{\mathcal{L}} = 2. \end{cases} \tag{7.28}$$

Proof. By a change of basis, we can assume that $\mathcal{L}$ is as in the standard form. Consider for example the case (2) in Lemma 7.16. Let $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in g\mathcal{U}_{\mathcal{L}}g^{-1}$. Then $hL_0 \subset L_0$ implies that $a, b, c, d \in O_{\mathbb{F}}$, this step actually proved the $e_{\mathcal{L}} = 1$ case. On the other hand, $hL_1 \subset L_1$ implies that $a, d \in O_{\mathbb{F}}$, $b \in \varpi^{-1}O_{\mathbb{F}}$, and $c \in \varpi O_{\mathbb{F}}$. Thus we get $h \in \begin{pmatrix} O_{\mathbb{F}} & O_{\mathbb{F}} \\ \varpi O_{\mathbb{F}} & O_{\mathbb{F}} \end{pmatrix}$. □

Definition 7.19. A $\mathcal{U}_{\mathcal{L}}$ lattice is a $O_{\mathbb{F}}$ lattice which is also closed under the action of $\mathcal{U}_{\mathcal{L}}$.

Lemma 7.20. *If L is an $\mathcal{U}_{\mathcal{L}}$ lattice, then $L \in \mathcal{L}$.*

Proof. Let $e_{\mathcal{L}} = 2$. We shall work with the standard form of $\mathcal{U}_{\mathcal{L}}$ after change of basis. In particular $\mathcal{U}_{\mathcal{L}}$ contains the element $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then by the condition $g_1L + g_2L \subset L$, hence $L = g_1L + g_2L$ is of form $\varpi^a O_{\mathbb{F}} \oplus \varpi^b O_{\mathbb{F}}$. As L is invariant under the action of $\begin{pmatrix} 1 & O_{\mathbb{F}} \\ 0 & 1 \end{pmatrix}$ (as column vectors), we have $a \le b$. Its invariance under $\begin{pmatrix} 1 & 0 \\ \varpi O_{\mathbb{F}} & 1 \end{pmatrix}$ implies that $b \le a + 1$. Then either $a = b$ or $a + 1 = b$. Either way, we have $L \in \mathcal{L}$. The case $e_{\mathcal{L}} = 1$ is similar and easier. □

By this lemma, we can recover the lattice chain $\mathcal{L}$ from chain order $\mathcal{U}$. We shall denote $e_{\mathcal{U}} = e_{\mathcal{L}}$. We also denote $\mathcal{U}_i$ to be the standard chain orders in (7.28) with $e_{\mathcal{U}} = i$ for $i = 1, 2$.

Definition 7.21. With the standard form for $\mathcal{U}$, let

$$\Pi = \begin{cases} \varpi, & \text{if } e_{\mathcal{U}} = 1, \\ \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}, & \text{if } e_{\mathcal{U}} = 2. \end{cases}$$

Let $\mathcal{B} = \Pi\mathcal{U} = \mathcal{U}\Pi$, and $\mathcal{B}^n = \Pi^n\mathcal{U} = \mathcal{U}\Pi^n, \forall n \in \mathbb{N}$.

Let $U_{\mathcal{U}} = U^0_{\mathcal{U}} = \mathcal{U}^*$, $U^n_{\mathcal{U}} = 1 + \mathcal{B}^n, \forall n \ge 1$.

Note that we have $\Pi^n L_i = L_{i+n}$.

Alternatively, $\mathcal{B}^n = \bigcap_{i \in \mathbb{Z}} \mathrm{Hom}_{O_{\mathbb{F}}}(L_i, L_{i+n})$.

Remark 7.22. One way to understand these objects is that they are analogue of various objects for $p-$adic field. More specifically, A is like p-adic field, $\mathcal{U}$ ring of integers, Π uniformizer, $\mathcal{B}^n$ ideals, $U^n_{\mathcal{U}}$ neighbourhood of identity.

Definition 7.23. Define the group

$$\mathcal{K}_{\mathcal{U}} = \{g \in G, g\mathcal{U}g^{-1} = \mathcal{U}\}. \tag{7.29}$$

When $\mathcal{U} = \mathcal{U}_{\mathcal{L}}$ for a lattice chain $\mathcal{L}$, $\mathcal{K}_{\mathcal{U}}$ can be alternatively defined as

$$\mathcal{K}_{\mathcal{U}} = \mathrm{Aut}_{O_{\mathbb{F}}}(\mathcal{L}) = \{g \in G, gL \in \mathcal{L}, \forall L \in \mathcal{L}\}.$$

Exercise 7.24. Show that these two definitions coincide. Hint: make use of the fact that $\mathcal{U}$ and $\mathcal{L}$ can determine each other.

Lemma 7.25. *There exists a $\mathbb{Z}-$valued function $k(g)$ on $\mathcal{K}_{\mathcal{U}}$ such that $gL_i = L_{i+k(g)}$ for any $L_i \in \mathcal{L}$.*

Proof. For fixed $g \in \mathcal{K}_{\mathcal{U}}$, by the second description there exists $\mathbb{Z}-$valued function $k_g(i)$, such that $gL_i = L_{i+k_g(i)}$. One can show as in the proof of Lemma 7.15 that $k_g(i)$ will be a constant function for any i. □

Lemma 7.26. *$\mathcal{K}_{\mathcal{U}}$ normalises all $U^i_{\mathcal{U}}$.*

Proof. For $i > 0$, $U^i_{\mathcal{U}} = 1 + \mathcal{B}^i$. $\mathcal{K}_{\mathcal{U}}$ normalise $\mathcal{B}$ because of the following. Let $u \in \mathcal{B} = \cap\,\mathrm{Hom}_{O_{\mathbb{F}}}(L_i, L_{i+1})$ and $g \in \mathcal{K}_{\mathcal{U}}$. By the above lemma, $gL_i = L_{i+k(g)}$ for any $L_i \in \mathcal{L}$, $ugL_i \in L_{i+k(g)+1}$, and $g^{-1}ugL_i \in L_{i+1}$. Thus $g^{-1}ug \in \mathcal{B}$.

One can similarly prove the case $i = 0$. □

It is convenient to understand $\mathcal{U}$, $\mathcal{B}$ and $\mathcal{K}_{\mathcal{U}}$ from quadratic extensions.

Lemma 7.27. *Let $\mathbb{E}$ be a quadratic field extension over $\mathbb{F}$, embedded into A. Then V can be viewed as $1-$dimensional $\mathbb{E}-$space. The collection of all $O_{\mathbb{E}}-$lattices in V form an $O_{\mathbb{F}}-$lattice chain $\mathcal{L}$. Then*

1. *$e_{\mathcal{L}} = e(\mathbb{E}/\mathbb{F})$ (so every $\mathcal{L}$ arises in this way), and $\mathcal{L}$ is the unique lattice chain in V which is stable under the action of $\mathbb{E}^*$.*
2. *$\mathcal{U} = \mathcal{U}_{\mathcal{L}}$ is the unique chain order such that $\mathbb{E}^* \subset \mathcal{K}_{\mathcal{U}}$.*
3. *For $\mathcal{B} = \mathcal{B}_{\mathcal{L}}$, $x\mathcal{U} = \mathcal{B}^{v_{\mathbb{E}}(x)}, \forall x \in \mathbb{E}^*$, and $\mathcal{K}_{\mathcal{U}} = \mathbb{E}^* U_{\mathcal{U}}$.*

Proof. For any $v \in V, v \neq 0$, $\mathcal{L} = \{\varpi^i_{\mathbb{E}} O_{\mathbb{E}} v, i \in \mathbb{Z}\}$ is the set of $O_{\mathbb{E}}-$lattices in V (i.e., the ideals). Then $\mathcal{L} = \mathbb{E}^* L$ for any $L \in \mathcal{L}$. $\varpi L_i = L_{i+e_{\mathcal{L}}} = \varpi_{\mathbb{E}}^{e(\mathbb{E}/\mathbb{F})+i} O_{\mathbb{E}} v$ and thus $e_{\mathcal{L}} = e(\mathbb{E}/\mathbb{F})$. As $\mathbb{E}$ can be unramified or ramified, all $\mathcal{L}$ arise in this way.

If $\mathcal{L}'$ is also stable under $\mathbb{E}^*$, then $L \in \mathcal{L}'$ must be stable under $O^*_{\mathbb{E}}$, which makes L an $O_{\mathbb{E}}-$lattice (as any element in $O_{\mathbb{E}}$ can be written as, for example, a difference of two elements in $O^*_{\mathbb{E}}$). Then as $\mathcal{L} = \mathbb{E}^* L$, we must have $\mathcal{L}' = \mathcal{L}$.

(2) is immediate by the relation between $\mathcal{U}$ and $\mathcal{L}$.

For the first part of (3), one can pick $\Pi = \varpi_{\mathbb{E}}$. For the second part of (3), it's obvious that $\mathbb{E}^* U_{\mathcal{U}} \subset \mathcal{K}_{\mathcal{U}}$. For the inverse direction, note that when we take $\Pi = \varpi_{\mathbb{E}}$. By definition and lemma above, $gL_i = L_{i+k(g)}$ for any i. Thus $\Pi^{-k(g)} gL_i = L_i$ implies that $\Pi^{-k} g \in U_{\mathcal{U}}$ and $g \in \mathbb{E}^* U_{\mathcal{U}}$.

□

Exercise 7.28. Fill in the details for this proof.

Start of lecture 10

7.4. **Type theory.**

7.4.1. *Characters and stratum.*

Lemma 7.29. *For $1 \leq m < n \leq 2m$, we have the identification*

$$\begin{aligned} \mathcal{B}^m/\mathcal{B}^n &\xrightarrow{\simeq} U^m_{\mathcal{U}}/U^n_{\mathcal{U}} \\ x &\mapsto 1 + x \end{aligned} \tag{7.30}$$

Now we give the analogue of Lemma 2.19.

For an additive character ψ of $\mathbb{F}$ and $a \in A$, define the character ψ_a on A by

$$\psi_a(x) = \psi(\mathrm{Tr}(ax)), \forall x \in A. \tag{7.31}$$

Take from now on ψ to be level 1, i.e., ψ is constant 1 on $\varpi O_{\mathbb{F}}$, but not on $O_{\mathbb{F}}$. The purpose for this unconventional assumption is to make the formulae uniform for $e_{\mathcal{L}} = 1$ or 2.

For an order Q in A, let $Q^* = \{x \in A, \psi_x(y) = 1, \forall y \in Q\}$.

Proposition 7.30. *Let $\mathcal{B} = \mathcal{B}_{\mathcal{L}}$ and $\mathcal{U} = \mathcal{U}_{\mathcal{L}}$.*

(1) $(\mathcal{B}^n)^* = \mathcal{B}^{1-n}$.

(2) *For $0 < m < n \leq 2m$, the following map*

$$\begin{aligned} \mathcal{B}^{1-n}/\mathcal{B}^{1-m} &\rightarrow \widehat{U^m_{\mathcal{U}}/U^n_{\mathcal{U}}} \\ a + \mathcal{B}^{1-m} &\mapsto (x \mapsto \psi_a(x-1)). \end{aligned} \tag{7.32}$$

Proof. We shall give the proof only in the case $e_{\mathcal{U}} = 2$. The case $e_{\mathcal{U}} = 1$ is similar and easier. For (1), let $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$, and we have

$$\psi(\mathrm{Tr}(xy)) = \psi(x_1y_1 + x_4y_4 + x_2y_3 + x_3y_2). \tag{7.33}$$

By assumption on the level of ψ, one can easily see that for $\mathcal{B}^0 = \begin{pmatrix} O_{\mathbb{F}} & O_{\mathbb{F}} \\ \varpi O_{\mathbb{F}} & O_{\mathbb{F}} \end{pmatrix}$, $(\mathcal{B}^0)^* = \begin{pmatrix} \varpi O_{\mathbb{F}} & O_{\mathbb{F}} \\ \varpi O_{\mathbb{F}} & \varpi O_{\mathbb{F}} \end{pmatrix} = \mathcal{B}^1$, and in general

$$(\mathcal{B}^n)^* = (\Pi^n\mathcal{B}^0)^* = \Pi^{-n}\mathcal{B}^1 = \mathcal{B}^{1-n}.$$

For (2), note that by the previous lemma, we have

$$\begin{aligned} \mathcal{B}^m/\mathcal{B}^n &\xrightarrow{\simeq} U^m_{\mathcal{U}}/U^n_{\mathcal{U}} \\ x &\mapsto 1 + x. \end{aligned} \tag{7.34}$$

So $\widehat{U^m_{\mathcal{U}}/U^n_{\mathcal{U}}} \simeq \widehat{\mathcal{B}^m/\mathcal{B}^n}$. Note that $A \simeq \mathbb{F}^4$ and $\hat{A} \simeq \hat{\mathbb{F}}^4 \simeq \mathbb{F}^4$, and by (7.33), all characters in $\hat{A}$ is of form ψ_a. Then (2) follows from (1) directly. □

In particular we shall care about characters on $U^n_{\mathcal{U}}/U^{n+1}_{\mathcal{U}}$.

Definition 7.31. The depth of π is

$$l(\pi) = \min\{n/e_{\mathcal{U}}, \pi \text{ contains the trivial character of } U^{n+1}_{\mathcal{U}}\}. \tag{7.35}$$

Definition 7.32. A stratum in A is a triple $(\mathcal{U}, n, a)$ where $\mathcal{U}$ is a chain order in A, n is an integer and $a \in \mathcal{B}^{-n}$. The strata $(\mathcal{U}, n, a_1)$ and $(\mathcal{U}, n, a_2)$ are equivalent if $a_1 \equiv a_2 \mod \mathcal{B}^{1-n}$.

One can easily see by Proposition 7.30 that when $n \geq 1$, a stratum defines a non-trivial character $\psi_a = \psi_a(x-1)$ on $U^n_{\mathcal{U}}/U^{n+1}_{\mathcal{U}}$, and two strata are equivalent if they correspond to the same character. We shall mainly focus on such cases.

The following lemma is easy, but will be used multiple times.

Lemma 7.33. *Suppose that π contains a character $\psi_a \in \widehat{U^{m_1}_{\mathcal{U}}/U^n_{\mathcal{U}}}$ for $a \in \mathcal{B}^{1-n}/\mathcal{B}^{1-m_1}$ and $m_1 \leq n \leq 2m_1$. Let m_2 be such that $n/2 \leq m_2 < m_1$. Then ψ_a can be extended to $U^{m_2}_{\mathcal{U}}/U^n_{\mathcal{U}}$. More precisely, there exists an element $v \in \pi^{\psi_a}$ and $a' \in \mathcal{B}^{1-n}/\mathcal{B}^{1-m_2}$ such that $a' \equiv a \mod \mathcal{B}^{1-m_1}$ and $U^{m_2}_{\mathcal{U}}/U^n_{\mathcal{U}}$ acts on v by $\psi_{a'}$.*

Proof. By condition, let

$$W = \pi^{\psi_a} = \{v' \in \pi, \pi(x)v' = \psi_a(x-1) \text{ for } x \in U^{m_1}_{\mathcal{U}_1}\} \neq \varnothing. \tag{7.36}$$

In particular $U^n_{\mathcal{U}}$ acts trivially on W. For any $v' \in W$, $1+x \in U^{m_2}_{\mathcal{U}}$ and $1+y \in U^{m_1}_{\mathcal{U}}$, we have

$$\pi((1+y)(1+x))v' = \pi((1+y)(1+x)(1+y)^{-1}(1+y))v' = \psi_a(y)\pi((1+y)(1+x)(1+y)^{-1})v'. \tag{7.37}$$

As $m_2 \geq n/2$, we have by Taylor expansion that

$$(1+y)(1+x)(1+y)^{-1} \equiv 1+x \mod \mathcal{B}^n \tag{7.38}$$

and $\pi((1+y)(1+x)(1+y)^{-1})v' = \pi(1+x)v'$. Thus $\pi(1+x)v' \in W$ and the action of $U^{m_2}_{\mathcal{U}}$ on W is closed.

Further as $U^{m_2}_{\mathcal{U}}/U^n_{\mathcal{U}}$ is abelian, W decomposes into characters $\psi_{a'}$ for $U^{m_2}_{\mathcal{U}}/U^n_{\mathcal{U}}$. As the restriction of $\psi_{a'}$ to $U^{m_1}_{\mathcal{U}}/U^n_{\mathcal{U}}$ must be ψ_a, we get the congruence $a' \equiv a \mod \mathcal{B}^{1-m_1}$. □

Lemma 7.34. *Let $(\mathcal{U}_i, n_i, a_i)$ be two strata. An element $g \in G$ intertwines ψ_{a_1} of $U^{n_1}_{\mathcal{U}_1}$ with ψ_{a_2} of $U^{n_2}_{\mathcal{U}_2}$ iff the intersection $g^{-1}(a_1 + \mathcal{B}_1^{1-n_1})g \cap (a_2 + \mathcal{B}_2^{1-n_2})$ is non-empty.*

Proof. By taking conjugation by g for a stratum, we can assume WLOG that $g = 1$. For $\Leftarrow$, if $a \in (a_1 + \mathcal{B}_1^{1-n_1}) \cap (a_2 + \mathcal{B}_2^{1-n_2})$, then $\psi_a = \psi_{a_i}$ on $U^{n_i}_{\mathcal{U}_i}$, and $\psi_{a_1} = \psi_{a_2}$ on the common support. For $\Rightarrow$, if $\psi(\mathrm{Tr}(a_1 x)) = \psi(\mathrm{Tr}(a_2 x))$ for $x \in \mathcal{B}_1^{n_1} \cap \mathcal{B}_2^{n_2}$, then

$$a_1 - a_2 \in (\mathcal{B}_1^{n_1} \cap \mathcal{B}_2^{n_2})^* = \mathcal{B}_1^{1-n_1} + \mathcal{B}_2^{1-n_2} \tag{7.39}$$

by Proposition 7.30 (1). This implies that $(a_1 + \mathcal{B}_1^{1-n_1}) \cap (a_2 + \mathcal{B}_2^{1-n_2}) \neq \varnothing$. □

7.4.2. *Fundamental stratum and depth.*

Lemma 7.35 (Lemma-Definition)**.** *A stratum is called fundamental if $a + \mathcal{B}^{1-n}$ contains no nilpotent element of A (eg. conjugates of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$). Equivalently, there exists $r \geq 1$ s.t. $a^r \in \mathcal{B}^{1-rn}$.*

Proof. We first note that the second definition is actually a property for any element in $a + \mathcal{B}^{1-n}$, as it also implies that $(a+b)^r = a^r + ra^{r-1}b + \cdots \in \mathcal{B}^{1-rn}$ for $b \in \mathcal{B}^{1-n}$. Further more, the statements remain true after conjugation. For $\Rightarrow$, we just pick $a = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}$ by condition (as any nilpotent element is conjugate to such form), and $a^r \in \mathcal{B}^{1-rn}$ is obvious. For $\Leftarrow$, we note that multiplying by a power of p also preserve the equivalence, while changing the stratum $(\mathcal{U}, n, a)$ to $(\mathcal{U}, n - e_{\mathcal{U}}, pa)$. So we reduce the problem into three cases: $(\mathcal{U}_1, 0, a)$, $(\mathcal{U}_2, 0, a)$, $(\mathcal{U}_2, 1, a)$. One can check case by case. For the case $(\mathcal{U}_2, 0, a)$, we have $a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in \mathcal{B}^0/\mathcal{B}_1$ for some $a_i \in O_{\mathbb{F}}/\varpi O_{\mathbb{F}}$. Then $a^r \in \mathcal{B}^{1-rn}$ iff $a_1 \equiv a_2 \equiv 0 \mod \varpi O_{\mathbb{F}}$, in which case $a + \mathcal{B}^1 = \mathcal{B}^1$ automatically contains nilpotent elements. For the case $(\mathcal{U}_2, 1, a)$, we have $a = \begin{pmatrix} 0 & \varpi^{-1}a_1 \\ a_2 & 0 \end{pmatrix} \in \mathcal{B}^{-1}/\mathcal{B}^0$ for some $a_i \in O_{\mathbb{F}}/\varpi O_{\mathbb{F}}$. Then $a^r \in \mathcal{B}^{1-rn}$ iff one of $a_i \equiv 0 \mod \varpi$, in which case the coset contains a nilpotent element. □

Exercise 7.36. Check the case $(\mathcal{U}_1, 0, a)$.

Note that we actually get a trivial stratum in the case $(\mathcal{U}_2, 0, a)$.

Corollary 7.37. *Any nontrivial, non-fundamental stratum is conjugate to one of the followings:* $(\mathcal{U}_1, n, p^{-n}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix})$*,* $(\mathcal{U}_2, 2n-1, p^{-n}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix})$*.*

The motivation of study of fundamental stratum is its relation with depth of a representation.

Proposition 7.38. *Let π be an irreducible smooth representation of G which contains a stratum $(\mathcal{U}, n, a)$ with $n \geq 1$. (Recall that it associates to a character when $n \geq 1$.) Then this stratum is fundamental iff $l(\pi) = n/e_{\mathcal{U}}$.*

For the proof, we need the following lemmas.

Lemma 7.39. *If $(\mathcal{U}, n, a)$ is a non-fundamental stratum contained in π with $n \geq 1$, then there exists another chain order $\mathcal{U}'$ and n' s.t. $a + \mathcal{B}^{1-n} \subset \mathcal{B}'^{-n'}$ and $n'/e_{\mathcal{U}'} < n/e_{\mathcal{U}}$. In particular π contains the trivial character of $U_{\mathcal{U}'}^{n'+1}$ and $l(\pi) < n/e_{\mathcal{U}}$.*

Proof. By the proof of Lemma 7.35 and Corollary 7.37, we can just check for standard cases.

For $(\mathcal{U}_1, 0, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix})$ we have $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \mathcal{B}_1 \subset \mathcal{B}_2$ and $-1/2 < 0$.

For $(\mathcal{U}_2, 1, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix})$ we have $\begin{pmatrix}0 & \varpi^{-1}\\ 0 & 0\end{pmatrix} + \mathcal{B}_2^0 \subset \begin{pmatrix}\varpi^{-1} & 0\\ 0 & 1\end{pmatrix}\mathcal{B}_1^0\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}$ and $0 < 1/2$.

Note that $a+\mathcal{B}^{1-n} \subset \mathcal{B}'^{-n'}$ in particular implies that $\mathcal{B}^{1-n} \subset \mathcal{B}'^{-n'}$ and $\mathcal{B}^{n} \supset \mathcal{B}'^{1+n'}$ by taking $*$ and Lemma 7.30(1), thus $U_{\mathcal{U}}^{n} \supset U_{\mathcal{U}'}^{n+1}$. Now $a + \mathcal{B}^{1-n}$ defines a character on $U_{\mathcal{U}}^{n}$, its restriction to $U_{\mathcal{U}'}^{n+1}$ is trivial as $a + \mathcal{B}^{1-n} \subset \mathcal{B}'^{-n'}$ and any element in $\mathcal{B}'^{-n'}$ gives trivial character on $U_{\mathcal{U}'}^{n+1}$ by Lemma 7.30(1). Then $l(\pi) < n/e_{\mathcal{U}}$. □

Lemma 7.40. *If $(\mathcal{U}, n, a)$ is fundamental contained in π and $(\mathcal{U}', n', a')$ is another stratum contained in π, then $n/e_{\mathcal{U}} \leq n'/e_{\mathcal{U}'}$.*

Proof. By Lemma 7.13 and 7.34, we get, after a conjugation,

$$a' \in a + \mathcal{B}^{1-n}. \tag{7.40}$$

Suppose that $n/e_{\mathcal{U}} > n'/e_{\mathcal{U}'}$. Then $-n'e_{\mathcal{U}} > -ne_{\mathcal{U}'}$ and there exists $r \geq 1$ such that

$$p^{-rn'e_{\mathcal{U}}}\mathcal{U}' \subset p^{1-rne_{\mathcal{U}'}}\mathcal{U}. \tag{7.41}$$

On the other hand for $x := (a')^{re_{\mathcal{U}}e_{\mathcal{U}'}}$, we have that

$$x \in (\mathcal{B}')^{-re_{\mathcal{U}}e_{\mathcal{U}'}n'} \subset p^{-rn'e_{\mathcal{U}}}\mathcal{U}' \subset p^{1-rne_{\mathcal{U}'}}\mathcal{U} = \mathcal{B}^{1-re_{\mathcal{U}}e_{\mathcal{U}'}n} \tag{7.42}$$

contradicting that $(\mathcal{U}, n, a)$ is fundamental.

□

Proof of Proposition 7.38. One just have to note that if π contains a trivial character for $U_{\mathcal{U}}^{n+1}$ with $n \geq 1$, then it automatically contains a stratum $(\mathcal{U}, n, a)$ for some $a \in \mathcal{B}^{-n}$ by Lemma 7.33. □

Remark 7.41. Whether $l(\pi) = 0$ divides the situation into two cases. When $l(\pi) > 0$, there exists a fundamental stratum which defines a character on the corresponding compact open subgroup, starting from which we can further construct π by compact induction.

When $l(\pi) = 0$, π can be constructed by compact induction from a representation of K which is inflated from a representation of $\mathrm{GL}_2(k)$.

We shall mainly focus on the case $l(\pi) > 0$ first, and discuss the case $l(\pi) = 0$ if time allows.

7.4.3. *Classifying fundamental stratum.* From definition, we have that $l(\pi) \in \frac{1}{2}\mathbb{Z}$. If $l(\pi) \notin \mathbb{Z}$, then $l(\pi) = \frac{n}{2}$ for an odd integer n, and $e_{\mathcal{U}} = 2$.

Lemma 7.42. (1) *Let $n = 2k+1$ be odd, then $(\mathcal{U}_2, n, a)$ is fundamental iff $a \in \mathcal{B}^{-n}/\mathcal{B}^{1-n}$ if of form $\begin{pmatrix} 0 & b\varpi^{-k-1} \\ c\varpi^{-k} & 0 \end{pmatrix} + \mathcal{B}^{1-n}$ for $b, c \in O_{\mathbb{F}}^*$. In that case, $(\mathcal{U}_2, n, a)$ is called a ramified simple stratum.*

(2) *If $l(\pi) = \frac{n}{2} \notin \mathbb{Z}$, then π contains a ramified simple stratum.*

Proof. For (1) when $n = 2k+1$, we have $\mathcal{B}^{-n} = \begin{pmatrix} \varpi^{-k}O_{\mathbb{F}} & \varpi^{-k-1}O_{\mathbb{F}} \\ \varpi^{-k}O_{\mathbb{F}} & \varpi^{-k}O_{\mathbb{F}} \end{pmatrix}$, $\mathcal{B}^{1-n} = \begin{pmatrix} \varpi^{-k}O_{\mathbb{F}} & \varpi^{-k}O_{\mathbb{F}} \\ \varpi^{-k+1}O_{\mathbb{F}} & \varpi^{-k}O_{\mathbb{F}} \end{pmatrix}$. Thus a is always of form $\begin{pmatrix} 0 & b\varpi^{-k-1} \\ c\varpi^{-k} & 0 \end{pmatrix} + \mathcal{B}^{1-n}$ for $b, c \in O_{\mathbb{F}}$. If any of b, c, say $c \notin O_{\mathbb{F}}^*$, then

$$\begin{pmatrix} 0 & b\varpi^{-k-1} \\ c\varpi^{-k} & 0 \end{pmatrix} + \mathcal{B}^{1-n} = \begin{pmatrix} 0 & b\varpi^{-k-1} \\ 0 & 0 \end{pmatrix} + \mathcal{B}^{1-n},$$

which is not fundamental as $\begin{pmatrix} 0 & b\varpi^{-k-1} \\ 0 & 0 \end{pmatrix}$ is nilpotent.

(2) Follows directly from definition and Proposition 7.38. □

Note that in this case a satisfies the quadratic equation $a^2 \equiv bc\varpi^{-2k-1}$ and $\mathbb{F}[a]$ is a ramified quadratic field extension.

The following lemma shows that we don't have to consider the case $e_{\mathcal{U}} = 2$ and n even.

Lemma 7.43. *If $l(\pi) = k \in \mathbb{Z}_{>0}$, then π contains a fundamental stratum $(\mathcal{U}_1, k, a)$.*

Proof. By Proposition 7.38, π contains a stratum as claimed, or a stratum of form $(\mathcal{U}_2, n, a)$ for $n = 2k$, $k \geq 1$. This implies that there exists $v \in \pi$ such that $\pi(x)v = \psi_a(x-1)v$ for any $x \in U_{\mathcal{U}_2}^{2k}$ while $U_{\mathcal{U}_2}^{2k}$ acts trivially. Note however

$$U_{\mathcal{U}_1}^{k} \supset U_{\mathcal{U}_2}^{2k} \supset U_{\mathcal{U}_2}^{2k+1} \supset U_{\mathcal{U}_1}^{k+1}$$

as

$$\begin{pmatrix} \varpi^k & \varpi^k \\ \varpi^k & \varpi^k \end{pmatrix} \supset \begin{pmatrix} \varpi^k & \varpi^k \\ \varpi^{k+1} & \varpi^k \end{pmatrix} \supset \begin{pmatrix} \varpi^{k+1} & \varpi^k \\ \varpi^{k+1} & \varpi^{k+1} \end{pmatrix} \supset \begin{pmatrix} \varpi^{k+1} & \varpi^{k+1} \\ \varpi^{k+1} & \varpi^{k+1} \end{pmatrix}.$$

This implies that $U_{\mathcal{U}_1}^{k+1}$ acts on v trivially. Let W be the subspace of elements of π which are $U_{\mathcal{U}_1}^{k+1}$−invariant. Note that $U_{\mathcal{U}_1}^{k}/U_{\mathcal{U}_1}^{k+1}$ is abelian. Thus $U_{\mathcal{U}_1}^{k}$ acts on some $v' \in W$ by some $\psi_{a'}$ according to Lemma 7.33, which is nontrivial because $U_{\mathcal{U}_2}^{2k}$ has to act nontrivially. □

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Let $e_{\mathcal{U}} = 1$ from now on. Then $a = \varpi^{-n}a_0$ for $a_0 \in \mathcal{U}_1^*$. Let $f_a(t) \in O_{\mathbb{F}}[t]$ be the characteristic polynomial of a_0, with $\overline{f}_a(t) \in k[t]$ being its reduction mod ϖ and characteristic polynomial of $\overline{a}_0 \in M_2(k)$. These polynomials are conjugacy-invariant.

One can easily check that a stratum is non-fundamental iff the associated $\overline{f}_a(t) = t^2$. Otherwise, it is called an unramified fundamental stratum, which can further classified into three cases:

(1) $\overline{f}_a(t)$ is irreducible of degree 2. The corresponding stratum is called unramified simple stratum.
(2) $\overline{f}_a(t)$ has distinct roots. The corresponding stratum is called split stratum.

(3) $\overline{f}_a(t)$ has repeated roots. The corresponding stratum is said to be essentially scalar.

Apparently in case (1), $\mathbb{F}[a]$ is an unramified quadratic field extension of $\mathbb{F}$. A simple stratum is either a ramified simple stratum or an unramified simple stratum.

Lemma 7.44. (1) *A ramified simple stratum can never intertwine with an unramified fundamental stratum.*

(2) *If $(\mathcal{U}_1, n, a)$ intertwines with $(\mathcal{U}_2, n, b)$, then $\overline{f}_a(t) = \overline{f}_b(t)$.*

Proof. For (1), as both stratums are fundamental, one can easily get a contradiction by Lemma 7.40.

For (2), Lemma 7.34 and the condition implies that there exists $g \in G$ such that $g^{-1}bg \in a + \mathcal{B}_1^{1-n}$, thus $\overline{f}_a(t) = \overline{f}_{g^{-1}bg}(t) = \overline{f}_b(t)$. □

Later on we shall construct supercuspidal representations from simple stratums. Right now we discuss essentially scalar stratum and split stratum.

Definition 7.45. π is called minimal if $l(\pi) \leq l(\pi \otimes \chi)$ for any multiplicative character χ.

Thus it is sufficient to classify all minimal representations and then get all representations by proper twisting.

Proposition 7.46. *Let $l(\pi) > 0$. Then π contains an essentially scalar stratum iff there exists a χ such that $l(\pi \otimes \chi) < l(\pi)$*

Proof. $\Rightarrow$: after conjugation, assume that π contains the essentially scalar stratum $(\mathcal{U}_1, n, a = \varpi^{-n}\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix})$ for $a \in O_{\mathbb{F}}^*$. So there exists $v \in \pi$ such that $\pi(1+x)v = \psi_a(x)v$ for any $x \in \mathcal{B}_1^n$.

Let χ be a character of level $n+1$ such that $\chi(1+u) = \psi(-\alpha\varpi^{-n}u)$ for any $u \in \varpi^{\lceil \frac{n+1}{2} \rceil}O_{\mathbb{F}}$. Then for $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathcal{B}_1^n$, $\chi \circ \det(1+x) = \chi(1 + x_1 + x_4 + x_1x_4 - x_2x_3) = \psi(-\alpha\varpi^{-n}(x_1 + x_4))$.

As a result, $\pi \otimes \chi(1+x)$ acts on v by the character associated to $\varpi^{-n}\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} - \varpi^{-n}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \mathcal{B}_1^{1-n}$ which now contains a non-fundamental stratum. Thus $l(\pi \otimes \chi) < n$.

$\Leftarrow$: if $l(\pi \otimes \chi) < n$, then $\pi \otimes \chi$ contains a trivial character of $\mathcal{U}_1^n$. This is obvious if π contains some stratum $(\mathcal{U}_1, i, a)$ for $i \leq n-1$. When $l(\pi \otimes \chi) = n - 1/2$, for example, then it contains a trivial character of $\mathcal{U}_2^{2n}$, which contains $\mathcal{U}_1^n$. When twisting back, the trivial character of $\mathcal{U}_1^n$ becomes the essentially scalar stratum $(\mathcal{U}_1, n, \varpi^{-n}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix})$. □

From this, we get that minimal representations will not contain essentially scalar stratum. Now we discuss split stratum.

Proposition 7.47. *Suppose that, after conjugation if necessary, π contains a split stratum of form $(\mathcal{U}_1, n, a)$ with $a \in T = \{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\}$, then the Jacquet module π_N contains the character $\psi_a|_{U_{\mathcal{U}_1}^n \cap T}$. In particular π is not supercuspidal.*

Proof. By assumption, we have that

(7.43)
$$\pi^{\psi_a} = \{v \in \pi, \pi(x)v = \psi_a(x-1) \text{ for } x \in U_{\mathcal{U}_1}^n\} \neq \emptyset$$

Suppose that all elements in π^{ψ_a} belongs to $\pi(N)$. Then by Lemma 6.1 we have for all $v \in \pi^{\psi_a}$

$$\int_{N_j} \pi(n) v dn = 0 \tag{7.44}$$

where $N_j = \begin{pmatrix} 1 & \varpi^j \mathcal{O}_{\mathbb{F}} \\ 0 & 1 \end{pmatrix}$. Choose j to be maximal for this property, s.t. there exists $v_1 \in \pi^{\psi_a}$ such that

$$\int_{N_{j+1}} \pi(n) v_1 dn \neq 0. \tag{7.45}$$

Take $t = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$ and $v_2 = \pi(t^{-1}) v_1$. On the one hand, we have

$$\int_{N_j} \pi(n) v_2 dn = \pi(t^{-1}) \int_{N_j} \pi(tnt^{-1}) v_1 dn \doteq \pi(t^{-1}) \int_{N_{j+1}} \pi(n') v_1 dn' \neq 0. \tag{7.46}$$

On the other hand, let $Y = U^n_{\mathcal{U}_1} \cap t^{-1} U^n_{\mathcal{U}_1} t = 1 + \begin{pmatrix} \varpi^n \mathcal{O}_{\mathbb{F}} & \varpi^n \mathcal{O}_{\mathbb{F}} \\ \varpi^{n+1} \mathcal{O}_{\mathbb{F}} & \varpi^n \mathcal{O}_{\mathbb{F}} \end{pmatrix}$. For any $y \in Y$, $y = t^{-1} x t$ for $x \in U^n_{\mathcal{U}_1}$, we have

$$\pi(y) v_2 = \pi(t^{-1} x) v_1 = \pi(t^{-1}) \psi_a(x-1) v_1 = \psi_a(x-1) v_2 \tag{7.47}$$

and $\psi_a(x-1) = \psi_a(y-1)$ as a is diagonal and y is conjugate to x by the diagonal matrix t.

Lemma 7.48. (1) *Any irreducible representation of $U^n_{\mathcal{U}_1}$ containing $\psi_a(y-1)$ of Y is 1−dimensional.*
(2) *Let ϕ be a character of $U^n_{\mathcal{U}_1}$ s.t. $\phi|_Y = \psi_a(y-1)$. Then there exists $u \in N$ s.t. $\phi^u(x) = \psi_a(x-1)$ for $x \in U^n_{\mathcal{U}_1}$.*

Proof. For (1), one can argue similarly as in the proof of Lemma 7.33, as $\psi_a(y-1)$ is trivial on $U^{n+1}_{\mathcal{U}_1}$ and $U^n_{\mathcal{U}_1}/U^{n+1}_{\mathcal{U}_1}$ is abelian.

For (2), $\phi|_Y = \psi_a(y-1)$ implies that $\phi = \psi_\delta$ for $\delta \in \varpi^{-n} \begin{pmatrix} a_1 & m \\ 0 & a_2 \end{pmatrix} + \mathcal{B}_1^{1-n}$. As $a_1 \not\equiv a_2$, there exists $n \in \mathcal{O}_{\mathbb{F}}$ s.t. $m + (a_2 - a_1) n \equiv 0 \mod \varpi$. Then

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & m \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & m + (a_2 - a_1) n \\ 0 & a_2 \end{pmatrix} \equiv \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \tag{7.48}$$

and $\phi^u(x) = \psi_a(x-1)$ for $u = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. □

Now we return to the proof of Proposition 7.47. Let $W \subset \pi$ be the subspace s.t. Y acts by $\psi_a(y-1)$.

By part (1) of the lemma we have $v_2 \in W = \oplus_\delta W^{\psi_\delta}$ so that there exists $v_\delta \in W^{\psi_\delta}$ s.t.

$$\int_{N_j} \pi(n) v_\delta dn \neq 0. \tag{7.49}$$

But by part (2) of the lemma we have $\psi_\delta = \psi_a^u$ for some $u \in N$, so $v_3 := \pi(u^{-1}) v_\delta$ is in V^{ψ_a} and

$$0 = \int_{N_j} \pi(n) v_3 dn = \pi(u^{-1}) \int_{N_j} \pi(u)\pi(n)\pi(u^{-1}) v_\delta dn \doteq \pi(u^{-1}) \int_{N_j} \pi(n) v_\delta dn \neq 0, \tag{7.50}$$

contradiction. □

Corollary 7.49. *Let π be an irreducible minimal supercuspidal smooth representation with $l(\pi) > 0$. Then π contains a simple stratum.*

7.5. **Extending to larger compact subgroup.** From the last section, we saw that for a simple stratum $(\mathcal{U}, n, a)$, we can associated a quadratic field extension $\mathbb{E} = \mathbb{F}[a]$ which satisfies $e_{\mathbb{E}/\mathbb{F}} = e_{\mathcal{U}}$ and $\mathbb{E}^* \in \mathcal{K}_{\mathcal{U}}$.

Lemma 7.50. *For any $a' \in a + \mathcal{B}^{1-n}$, and $\mathbb{E}' = \mathbb{F}[a']$, we have $e_{\mathbb{E}'/\mathbb{F}} = e_{\mathcal{U}}$ and $\mathbb{E}'^* \in \mathcal{K}_{\mathcal{U}}$.*

Proof. $e_{\mathbb{E}'/\mathbb{F}} = e_{\mathcal{U}}$ follows directly from that the ramification of $\mathbb{E}'$ only depends on whether the simple stratum is ramified or unramified. As $a' \in a + \mathcal{B}^{1-n}$ for $a \in \mathcal{B}^{-n} \backslash \mathcal{B}^{1-n}$, we can write $a' = au$ for some $u \in U^1_{\mathcal{U}}$, while $\mathcal{K}_{\mathcal{U}} = \mathbb{E}^* U^0_{\mathcal{U}}$. Thus $a' \in \mathcal{K}_{\mathcal{U}}$ and so does $\mathbb{E}'^*$. □

First of all by Lemma 7.33, if π contains a simple stratum $(\mathcal{U}, n, a)$ which gives a character in $U^n_{\mathcal{U}}/U^{n+1}_{\mathcal{U}}$, then π contains a character ψ_a of $U^{\lfloor n/2 \rfloor+1}_{\mathcal{U}}/U^{n+1}_{\mathcal{U}}$. (Note there that $\lfloor n/2 \rfloor + 1 = \lceil \frac{n+1}{2} \rceil$.) It is now a simple representation of a relatively large compact open subgroup, and we care about its intertwining property.

Proposition 7.51. *TFAE:*

(1) *g intertwines ψ_a of $U^{\lfloor n/2 \rfloor+1}_{\mathcal{U}}$,*
(2) *g normalise/stabilise ψ_a of $U^{\lfloor n/2 \rfloor+1}_{\mathcal{U}}$,*
(3) *$g \in \mathbb{E}^* U^{\lfloor \frac{n+1}{2} \rfloor}_{\mathcal{U}}$.*

Proof. (2)⇒ (1) is obvious. (3)⇒ (2): Let $g = u(1 + y)$ for $u \in \mathbb{E}^*$, $y \in \mathcal{B}^{\lfloor \frac{n+1}{2} \rfloor}$, $x \in \mathcal{B}^{\lfloor \frac{n}{2} \rfloor+1}$. Then

$$\begin{aligned} \psi^g_a(1 + x - 1) &= \psi_a(u(1 + y)x(1 + y)^{-1}u^{-1}) = \psi \circ \operatorname{Tr}(au(1 + y)x(1 + y)^{-1}u^{-1}) \\ &= \psi \circ \operatorname{Tr}(a(1 + y)x(1 + y)^{-1}) \\ &= \psi \circ \operatorname{Tr}(ax) = \psi_a(1 + x - 1). \end{aligned} \tag{7.51}$$

Here in the second line, we have used that $u, a \in \mathbb{E}^*$ commute with each other and trace is conjugation-invariant. In the last line we have used that

$$(1 + y)x(1 + y)^{-1} = x + yx - xy + \cdots \equiv x \mod \mathcal{B}^{n+1}. \tag{7.52}$$

(1)⇒ (3): If g intertwines ψ_a of $U^{\lfloor n/2 \rfloor+1}_{\mathcal{U}}$ with itself, it in particular intertwines the related stratum.

Lemma 7.52. *Suppose that $g \in G$ intertwines two simple stratum $(\mathcal{U}, n, a_i)$ for $i = 1, 2$. Then $g \in \mathcal{K}_{\mathcal{U}}$ and it conjugates the two stratums (i.e., the two stratums become equivalent after conjugation).*

Proof. By Lemma 7.34, there exists a non-trivial element $\gamma \in a_1 + \mathcal{B}^{1-n} \cap g^{-1}(a_2 + \mathcal{B}^{1-n})g$. From $\gamma \in a_1 + \mathcal{B}^{1-n}$, we have that by Lemma 7.50 $\mathbb{F}[\gamma]^* \subset \mathcal{K}_{\mathcal{U}}$, and $\mathcal{L} = \mathcal{L}_{\mathcal{U}}$ is an $O_{\mathbb{F}[\gamma]}$ lattice chain in V. On the other hand by the same reasoning, $\gamma \in g^{-1}(a_2 + \mathcal{B}^{1-n})g$ implies that $\mathcal{L}$ is also an $O_{\mathbb{F}[g\gamma g^{-1}]}$ lattice chain, which is equivalent to that $g^{-1}\mathcal{L}$ is an $O_{\mathbb{F}[\gamma]}$ lattice chain. But the $O_{\mathbb{F}[\gamma]}$ lattice chain is unique by Lemma , thus $g^{-1}\mathcal{L} = \mathcal{L}$ and $g \in \mathcal{K}_{\mathcal{U}} = \mathbb{E}^* U^0_{\mathcal{U}}$ by definition. Then by Lemma 7.26, g normalise $\mathcal{B}^i$. $a_1 + \mathcal{B}^{1-n} \cap g^{-1}(a_2 + \mathcal{B}^{1-n})g = a_1 + \mathcal{B}^{1-n}$ when non-empty and the two stratums become equivalent after conjugation. □

At this step we actually proved that (1) ⇒ (2). By Proposition 7.30, ψ_a of $U^{\lfloor n/2 \rfloor+1}_{\mathcal{U}}$ is given by $a \in \mathcal{B}^{-n}/\mathcal{B}^{-\lfloor n/2 \rfloor}$. Now g normalise ψ_a iff

$$g^{-1}ag \equiv a \mod \mathcal{B}^{-\lfloor \frac{n}{2} \rfloor}. \tag{7.53}$$

As a is conjugation-invariant by elements in $\mathbb{E}^*$, we can assume WLOG that $g \in U^0_{\mathcal{U}}$. As $a \in \mathcal{B}^{-n}$, the above equation is further equivalent to that

$$aga^{-1} \equiv g \mod \mathcal{B}^{\lfloor \frac{n+1}{2} \rfloor} \tag{7.54}$$

as $n - \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n+1}{2} \rfloor$. The required result follows form the following lemma. □

Lemma 7.53. *For $k \geq 1$, $g \in U^0_{\mathcal{U}}$, $aga^{-1} \equiv g \mod \mathcal{B}^k$ iff $g \in O_{\mathbb{E}} + \mathcal{B}^k$*

Exercise 7.54. Prove this lemma. Hint: $\Leftarrow$ is obvious as $a \in \mathbb{E}^* \subset \mathcal{K}_{\mathcal{U}}$ normalises $\mathcal{B}^k$. For the other direction, one can use induction and reduce the problem into one on quotient. Then the main point of the proof is that the centraliser of $\mathbb{E}$ in M_2 is $\mathbb{E}$ itself.

Remark 7.55. Using similar proof as for Lemma 7.52, one can show that if $g \in G$ intertwines two characters ψ_{a_i} of $U^{\lfloor n/2 \rfloor+1}_{\mathcal{U}}$, then $g \in \mathcal{K}_{\mathcal{U}}$ and g conjugates the two characters.

Let $J = \mathbb{E}^* U^{\lfloor \frac{n+1}{2} \rfloor}_{\mathcal{U}}$.

Proposition 7.56. *Let Λ be an irreducible representation of J containing ψ_a of $U^{\lfloor \frac{n}{2} \rfloor+1}_{\mathcal{U}}$. Then $\Lambda|_{U^{\lfloor \frac{n}{2} \rfloor+1}_{\mathcal{U}}}$ is a multiple of ψ_a and $\pi = c - \mathrm{Ind}^G_J \Lambda$ is irreducible supercuspidal.*

Proof. By Lemma 7.13, other possible component of $\Lambda|_{U^{\lfloor \frac{n}{2} \rfloor+1}_{\mathcal{U}}}$ would intertwine with ψ_a by some element $j \in J$, but by Proposition 7.51, any $j \in J$ normalise ψ_a. Hence the first part of proposition. Now let $g \in G$ intertwines Λ. It in particular intertwines ψ_a on $U^{\lfloor \frac{n}{2} \rfloor+1}_{\mathcal{U}}$ with itself, which implies $g \in J$ by Proposition 7.51. By Proposition 7.11 we get that π is irreducible and supercuspidal. □

Let

$$C(\psi_a, \mathcal{U}) = \{\Lambda \in \mathrm{Irr}(J),\ \Lambda \text{ contains } \psi_a \text{ of } U^{\lfloor \frac{n}{2} \rfloor+1}_{\mathcal{U}}\}.$$

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Proposition 7.57. *For $i = 1, 2$, let $(\mathcal{U}_i, n_i, a_i)$ be two simple stratum, $\Lambda_i \in C(\psi_{a_i}, \mathcal{U}_i)$, and $\pi_i = c - \mathrm{Ind}^G_{J_i} \Lambda_i$. If $\pi_1 \simeq \pi_2$, then $n_1 = n_2$ and there exists $g \in G$ such that*

$$\mathcal{U}_2 = g^{-1}\mathcal{U}_1 g, J_2 = g^{-1} J_1 g, \Lambda_2 = \Lambda_1^g. \tag{7.55}$$

If $\mathcal{U}_1 = \mathcal{U}_2$, we can pick $g \in U^0_{\mathcal{U}_1}$.

Proof. By $l(\pi_1) = l(\pi_2)$, we get $n_1 = n_2$ and $\mathcal{U}_i$ are conjugate. After proper conjugation, assume WLOG $\mathcal{U}_1 = \mathcal{U}_2$ now.

By Lemma 7.13, there exists $g \in \mathcal{K}_{\mathcal{U}}$ which intertwines Λ_i. In particular it intertwines ψ_{a_i} of $U^{\lfloor n/2 \rfloor+1}_{\mathcal{U}}$. By Remark 7.55, $g \in \mathcal{K}_{\mathcal{U}}$ conjugates the two characters.

At this step we can assume after proper conjugation that $\mathcal{U}_1 = \mathcal{U}_2$, $n_1 = n_2$ and $\psi_{a_1} = \psi_{a_2}$. Thus the intertwining groups $J_1 = J_2$. Going through the above intertwining argument again, we see that if g intertwines Λ_i, then g intertwines ψ_{a_1} itself and we get that $g \in J$ and g conjugates Λ_1. Thus follows the claims in the proposition. □

Thus it remains to extend ψ_a of $U^{\lfloor \frac{n}{2} \rfloor+1}_{\mathcal{U}}$ to an irreducible representation Λ of J.

7.5.1. *n odd.* This is a particular simple case for this task, as $U_{\mathcal{U}}^{\lfloor\frac{n}{2}\rfloor+1} = U_{\mathcal{U}}^{\lfloor\frac{n+1}{2}\rfloor}$, so we just have to specify how to extend to $\mathbb{E}^*$.

Lemma 7.58. *When n is odd and $p > 2$, elements in $C(\psi_a, \mathcal{U})$ extending ψ_a of $U_{\mathcal{U}}^{\lfloor\frac{n}{2}\rfloor+1}$ are parametrised by characters θ satisfying $c(\theta) = n + 1$ and*

$$\theta(1 + x) = \psi(ax) \tag{7.56}$$

for any $x \in \mathbb{E} = \mathbb{F}[a]$ with $v_{\mathbb{E}}(x) \geq \frac{n+1}{2}$. $\Lambda = \tilde{\theta} \in C(\psi_a, \mathcal{U})$ can be explicitly given by $\tilde{\theta}(eu) = \theta(e)\psi_a(u - 1)$ for $e \in \mathbb{E}^*$ and $u \in U_{\mathcal{U}}^{\lfloor\frac{n}{2}\rfloor+1}$

Proof. First of all we show that such $\tilde{\theta}$'s are well defined, thus giving elements in $C(\psi_a, \mathcal{U})$. Apparently by the assumption on θ, we have that $\theta(u) = \psi_a(u-1)$ for $u \in \mathbb{E}^* \cap U_{\mathcal{U}}^{\lfloor\frac{n}{2}\rfloor+1}$. So $\tilde{\theta}$ is well-defined as a function. To show that it's indeed a character, let $ue = eu'$ as $e \in \mathbb{E} \subset \mathcal{K}_{\mathcal{U}}$ which normalise these compact subgroups. Then

$$\begin{aligned} \tilde{\theta}(ue) &= \theta(e)\psi_a(u' - 1) = \theta(e)\psi \circ \mathrm{Tr}(ae^{-1}ue - a) \\ &= \theta(e)\psi \circ \mathrm{Tr}(e^{-1}aue - a) = \theta(e)\psi \circ \mathrm{Tr}(au - a) \\ &= \theta(e)\psi_a(u - 1) = \tilde{\theta}(eu). \end{aligned} \tag{7.57}$$

Here we have used that $a \in \mathbb{E}^*$ and thus $e^{-1}a = ae^{-1}$.

On the other hand, for any $\Lambda \in C(\psi_a, \mathcal{U})$, $\Lambda|_{\mathbb{E}^*}$ is a direct sum of characters of $\mathbb{E}^*$, whose restriction to $\mathbb{E}^* \cap U_{\mathcal{U}}^{\lfloor\frac{n}{2}\rfloor+1}$ must agree with ψ_a. But for each eigenvector for the action of $\mathbb{E}^*$, both $\mathbb{E}^*$ and $U_{\mathcal{U}}^{\lfloor\frac{n}{2}\rfloor+1}$ acts on it by a multiple, thus it is stable under the action of J. □

7.5.2. *n even.* This case is more complicated due to that $U_{\mathcal{U}}^{\lfloor\frac{n}{2}\rfloor+1} \neq U_{\mathcal{U}}^{\lfloor\frac{n+1}{2}\rfloor}$. By the classification of simple stratum, this case only occurs when $e_{\mathcal{U}} = 1$ and $\mathbb{E}$ is unramified. We shall sketch the construction and proofs.

Let $\mathbb{E}^1$ be the elements in $O_{\mathbb{E}}$ which are congruent to 1 $\mod \varpi_{\mathbb{E}}$, $H_1 = \mathbb{E}^1 U_{\mathcal{U}}^{\lfloor\frac{n}{2}\rfloor+1}$, $H = \mathbb{E}^1 U_{\mathcal{U}}^{\lfloor\frac{n+1}{2}\rfloor}$, $J_1 = \mathbb{E}^* U_{\mathcal{U}}^{\lfloor\frac{n}{2}\rfloor+1}$ and J as before. One can similarly define a character $\tilde{\theta}$ on H_1 and J_1 as the previous case, and show that any $\Lambda \in C(\psi_a, \mathcal{U})$ contains some $\tilde{\theta}$.

There are two steps to extend $\tilde{\theta}$ to be a representation Λ of J.

The first step is called Heisenberg extension, which gives a $q = |k|$ dimension representation of H. In particular H/H_1 is a 2−dimensional vector space over k, and there exists a polarisation B/H_1 for some intermediate group $H_1 \subset B \subset H$, in the sense that $\tilde{\theta}([B, B]) = 1$. Thus $\tilde{\theta}$ can be extended to be a character of B and then $\eta = \mathrm{Ind}_B^H \tilde{\theta}$.

Lemma 7.59. *η is an irreducible representation of H with dimension $[B : H] = q$. It's independent of polarisation and extension of $\tilde{\theta}$ to B, and $\eta|_{H_1}$ is a multiple of $\tilde{\theta}$. Further more $\mathrm{Ind}_{H_1}^H \tilde{\theta} = q\eta$.*

Proof. Essentially finite group version of Stone-Von Neumann theorem. □

The next step is to further extend the group action to J. One basically need to specify the action of $\mu_{\mathbb{E}} = (O_{\mathbb{E}}/\varpi_{\mathbb{E}} O_{\mathbb{E}})^*$, while $\mu_{\mathbb{F}}$ is already determined by the central character.

Lemma 7.60. *There exists a $q-$dimension irreducible representation Λ of J such that $\Lambda|_H = \eta$, and*

$$\Lambda|_{\mathbb{E}^*} = \bigoplus_{\theta'|_{Z\mathbb{E}^1} = \theta, \theta' \neq \theta} \theta'. \tag{7.58}$$

Proof. First of all, note that as $e_{\mathcal{U}} = 1$, $[ZH : J] = q + 1$, $[J_1 : J] = q^2$. The construction of Λ is give as

$$\Lambda = \mathrm{Ind}_{ZH}^{J}\,\eta - \mathrm{Ind}_{ZJ_1}^{J}\,\tilde{\theta}. \tag{7.59}$$

Here we used the representation theoretical subtraction in Grothendieck of representations, but we get an actual representation, which we shall not justify here. One can easily check that $\dim\Lambda = q(q+1) - q^2 = q$, and $\Lambda|_H = (q+1)\eta - q^{\eta} = \eta$ by using the last statement of Lemma 7.59. From this one can also see why Λ is irreducible.

On the other hand, we have $J = \mathbb{E}^* H$, By Mackey theory, we have

$$\mathrm{Ind}_{ZH}^{J}\,\eta|_{\mathbb{E}^*} = \mathrm{Ind}_{Z\mathbb{E}^1}^{\mathbb{E}^*}\,\theta|_{Z\mathbb{E}^1} = \bigoplus_{\theta'|_{Z\mathbb{E}^1}=\theta} q\theta'. \tag{7.60}$$

One can also check that the action of $\mathbb{E}^*$ on J/J_1 has to type of orbits: the orbit of 1 with stabiliser $\mathbb{E}^*$, or other $q-1$ orbits of $q+1$ elements with stabiliser $Z\mathbb{E}^1$. For $g \in \mathbb{E}^*\backslash J/J_1$,

$$J_1^g \cap \mathbb{E}^* = \begin{cases} \mathbb{E}^*, & \text{if } g = 1; \\ Z\mathbb{E}^1, & \text{otherwise.} \end{cases}$$

Thus

$$\mathrm{Ind}_{ZJ_1}^{J}\,\tilde{\theta}|_{\mathbb{E}^*} = \theta \oplus (q-1)\Big(\bigoplus_{\theta'|_{Z\mathbb{E}^1}=\theta} \theta'\Big), \tag{7.61}$$

and

$$\Lambda|_{\mathbb{E}^*} = \bigoplus_{\theta'|_{Z\mathbb{E}^1}=\theta,\,\theta'\neq\theta} \theta'. \tag{7.62}$$

□

Exercise 7.61. Verify the action of $\mathbb{E}^*$ on J/J_1, and the sets $J_1^g \cap \mathbb{E}^*$. Hint: the two properties are closely related, in the sense that the number of elements in the orbit of g is $[J_1^g \cap \mathbb{E}^* : \mathbb{E}^*]$. One can check that $Z\mathbb{E}^1 \subset J_1^g \cap \mathbb{E}^*$ is always true, and $J_1^g \cap \mathbb{E}^*/Z \subset O_{\mathbb{E}}^*$ must be either $\mathbb{E}^1$ or $O_{\mathbb{E}}^*$. Thus one has only to show that there is one fixed point of the action of $\mathbb{E}^*$ on J/J_1, which is reduced to that the normaliser of $\mathbb{E}$ in A is $\mathbb{E}$ itself.

It is not difficult to check that any $\Lambda \in C(\psi_a, \mathcal{U})$ arise in this way.

7.6. **Depth zero supercuspidal representations.** When $l(\pi) = 0$, π has a vector fixed v by $U_{\mathcal{U}}^1 = 1 + \varpi M_2(O_{\mathbb{F}})$, thus for $K = U_{\mathcal{U}}^0$ (maximal compact), the representation of K generated by v is contained in π and is essentially a representation of $K/U_{\mathcal{U}}^1 \simeq \mathrm{GL}_2(k)$. Thus we need to know the representation theory of GL_2 over finite field. Let $\overline{G} = \mathrm{GL}_2(k)$, and similarly other notations for groups be over residue field.

The story for $\mathrm{GL}_2(k)$ is quite parallel to the theories we have seen before: there are parabolic induced representations from the characters on the diagonal subgroup. Some parabolically induced representations are not irreducible, with conditions for irreducibility similar to Section 6.5. Such $K-$representations will not arise from supercuspidal representations, parallel to what happens for split stratum. These parabolically induced representations contains the trivial character of $\overline{N}$. One can show that the Jacquet module of π containing such representations of $\mathrm{GL}_2(k)$ will be nonzero.

The remaining representations does not contain the trivial character of $\overline{N}$ (thus must contain nontrivial characters of $\overline{N}$) and are called cuspidal. They are parametrised by level 1 characters over unramified extension $\mathbb{E}$ with similar property as in Lemma 7.60. More precisely, a character θ of $\overline{\mathbb{E}^*}$ is called regular iff $\theta^q \neq \theta$. Then

Lemma 7.62. *Let $\tilde{\psi} : zu \in \overline{ZN} \mapsto \theta(z)\psi(u)$ and θ be a regular character of $\overline{\mathbb{E}^*}$.*

$$\sigma_\theta = \mathrm{Ind}_{\overline{ZN}}^{\overline{G}} \tilde{\psi} - \mathrm{Ind}_{\overline{\mathbb{E}^*}}^{\overline{G}} \theta. \tag{7.63}$$

(1) *It is irreducible and $q-1$dimensional. $\sigma_\theta|_{\overline{\mathbb{E}^*}} = \bigoplus_{\theta' \neq \theta, \theta^q} \theta'$.*
(2) *$\sigma_{\theta_1} \simeq \sigma_{\theta_2}$ iff $\theta_1 = \theta_2$ or $\theta_2 = \theta_1^q$.*
(3) *Every cuspidal representation of $\overline{G}$ arises in this way.*

Then one can inflate σ_θ to be a representation of K and then define $\pi = c - \mathrm{Ind}_K^G \sigma_\theta$.

Lemma 7.63. *$g \in G$ intertwines σ_θ iff $g \in ZK$ when σ_θ is cuspidal.*

Proof. By Bruhat decomposition, we have $\mathrm{GL}_2 = \cup_{i \geq 0} ZK \begin{pmatrix} \varpi^i & 0 \\ 0 & 1 \end{pmatrix} K$. If $g \in G$ intertwines σ_θ, but $g \notin ZK$, then there exists $i \neq 0$ such that $t = \begin{pmatrix} \varpi^i & 0 \\ 0 & 1 \end{pmatrix}$ intertwines two (possibly different) conjugates of σ_θ of K. Let ρ be the common factor of σ_θ^t and σ_θ when restricting to $K^t \cap K = \begin{pmatrix} O_\mathbb{F}^* & O_\mathbb{F} \\ \varpi^i O_\mathbb{F} & O_\mathbb{F}^* \end{pmatrix}$. Then σ_θ^t is trivial on $(U_\mathcal{U}^1)^t = \begin{pmatrix} 1+\varpi O_\mathbb{F} & \varpi^{1-i} O_\mathbb{F} \\ \varpi^{1_i} O_\mathbb{F} & 1+\varpi O_\mathbb{F} \end{pmatrix}$, implying that ρ is trivial on $\begin{pmatrix} 1 & O_\mathbb{F} \\ 0 & 1 \end{pmatrix}$. But ρ is also a component of σ_θ on $K^t \cap K$. Thus σ_θ contains the trivial character of $\overline{N}$, contradicting to cuspidality. □

Thus π is irreducible and supercuspidal by Proposition 7.11. Every depth 0 supercuspidal representation arise in this way.

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8. New vectors and minimal vectors

In this section we introduce the classical topic on levels and newforms, and show how type theory can be used to approach some of the topics.

Take $G = \mathrm{GL}_2$. Let $K_1(\varpi^n) = \{k \in K, k \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \varpi^n\}$. Let π have trivial central character.

Definition 8.1. We say the representation is level $c(\pi) = c$ if c is the minimal integer such that there exists a $K_0(\varpi^c)-$invariant element in π.

Lemma 8.2. *For $c = c(\pi)$, the space of $K_0(\varpi^c)-$invariant elements in π is $1-$dimensional. Any nontrivial element in this space is called a newform in π (thus unique up to a constant).*

8.1. Minimal vector vs. newform for supercuspidal representations.

We shall show how these notions relate to the depth and compact induction model for supercuspidal representations.

For simplicity let consider the case when n is odd. We have shown that minimal supercuspidal representations (which is true when central character is trivial) with $l(\pi) = n/e_\mathcal{U}$ arise as $\pi =$

$c - \mathrm{Ind}_J^G \tilde{\theta}$. By explicit construction of basis in, for example, (5.5), we can in particular get an element

$$\varphi_\theta(g) = \begin{cases} \tilde{\theta}(g), & \text{if } g \in J, \\ 0, & \text{otherwise.} \end{cases} \tag{8.1}$$

Furthermore one can easily check that

$$\pi(j)\varphi_\theta = \tilde{\theta}(j)\varphi_\theta. \tag{8.2}$$

Lemma 8.3. *φ_θ is uniquely determined by* (8.2).

Sketch of proof. One can use Mackey theory and known intertwining property to show that $\tilde{\theta}$ of J (and even proper subgroup of J) occurs in $\pi|_J$ at most once. □

Note that $\pi(g)\varphi_\theta$ can be similarly identified with $\tilde{\theta}^g$ on J^g.

Definition 8.4 (Temporary)**.** Any element of form $\pi(g)\varphi_\theta$ is called a minimal vector/micro-local lift.

For now we use the standard embedding of $\mathbb{E}^*$ in GL_2.

Proposition 8.5. *Let π be supercuspidal minimal with $l(\pi) = n/e_{\mathcal{U}}$ for n odd. Then $c(\pi) = 2(l(\pi) +$ 1) is odd or divisible by 4, and a new form can be chosen as*

$$\varphi_0 = \sum_{a \in (O_{\mathbb{F}}/\varpi^{n+1}O_{\mathbb{F}})^*} \pi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})\varphi_\theta. \tag{8.3}$$

Proof. For simplicity, let's consider only the case $e_{\mathcal{U}} = 1$. Let $n = l(\pi)$. By definition, if φ_0 is a newform, then $\pi(\begin{pmatrix} \varpi^{-\lfloor \frac{c}{2} \rfloor} & 0 \\ 0 & 1 \end{pmatrix})\varphi_0$ is invariant under $\begin{pmatrix} * & \varpi^{\lfloor \frac{c}{2} \rfloor} \\ \varpi^{\lceil \frac{c}{2} \rceil} & * \end{pmatrix}$, which contains $U_{\mathcal{U}_1}^{\frac{c}{2}}$ if c is even and $U_{\mathcal{U}_2}^{c}$ if c is odd. Thus $n + 1 \leq \frac{c}{2}$ in either case. On the other hand, φ_θ is $U_{\mathcal{U}}^{n+1}-$invariant, thus $\varphi = \sum_{a \in (O_{\mathbb{F}}/\varpi^{n+1}O_{\mathbb{F}})^*} \pi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})\varphi_\theta$ is still $U_{\mathcal{U}}^{n+1}-$invariant as $U_{\mathcal{U}}^{n+1}$ is a normal subgroup of K, while φ is $\begin{pmatrix} * & \varpi^{n+1} \\ \varpi^{n+1} & * \end{pmatrix}-$invariant. The averaging is non-vanishing as different translates of φ_θ form a basis. So φ is nontrivial, then $c \leq 2(n + 1)$ and all claims are proved. □

Exercise 8.6. Prove the proposition when $e_{\mathcal{U}} = 2$.

Remark 8.7. When $e_{\mathcal{U}} = 1$ and $l(\pi) = n$ is even, this approach will become slightly more complicated, due to the fact that π is now induced from a $q-$dimensional representation Λ. Any way in that case $c(\pi) = 2(l(\pi) + 1) \equiv 2 \mod 4$ is still true.

Remark 8.8. A classical approach to the newform for supercuspidal representations is to use the Kirillov model. Here we give the newform without referring to the Kirillov model. It's possible to prove the uniqueness of newform using Mackey theory, but we shall skip it here.

8.2. **Comparing test vectors.** For many problems in number theory like period integrals, sup norm problem, QUE, etc, one has to specify a choice of test vectors. The most natural choice for unramified representations is the spherical element. When there is ramification for the representation, then classically people developed the theory of newforms which is in most of cases the first test vector people tried. However recent developments motivate us to look at the corresponding problems using the minimal vector. The major advantage of the minimal vector is that we understand its behaviour under a larger subgroup when comparing with newforms.

For example when $c(\pi) = 4n$ corresponding to $l(\pi) = 2n-1$ and $c(\theta) = 2n$, we have that φ_0 is $K_0(\varpi^c)-$invariant, with $[K_0(\varpi^c) : K] \asymp q^c$. On the other hand φ_θ behaves by a character under the action of $J \cap K = O_{\mathbb{E}}^* U_{\mathcal{U}_1}^n$, with $[J \cap K : K] \asymp q^{2n} = \sqrt{q^c}$. Thus $J \cap K$ is a much larger subgroup compared to $K_0(\varpi^c)$.

A more practical advantage of φ_θ is that its matrix coefficient is very simple to describe and easy to use. In particular for the cases considered above, we have

$$\Phi_{\varphi_\theta}(g) = \begin{cases} \tilde{\theta}(g), & \text{if } g \in J, \\ 0, & \text{otherwise.} \end{cases} \tag{8.4}$$

On the other hand, the matrix coefficient for the new form is most of time involving all kinds of epsilon factors and difficult to evaluate explicitly. There are some recent success in using minimal vectors for period integrals and other analytic number theory problems.

8.3. newform for parabolically induced representations.

Proposition 8.9. *Let $\pi = \pi(\chi_1, \chi_2)$ and π irreducible, then we have $c(\pi) = c(\chi_1) + c(\chi_2)$.*

We first need a lemma.

Lemma 8.10. *For every positive integer c,*

$$GL_2(F) = \coprod_{0 \le i \le c} B \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} K_1(\varpi^c)$$

Proof. First we show it's a disjoint union. For $0 \le i \ne j \le c$, suppose

$$\begin{pmatrix} a_1 & m \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varpi^j & 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$$

for $\begin{pmatrix} a_1 & m \\ 0 & a_2 \end{pmatrix} \in B$ and $\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in K_1(\varpi^c)$. Note $k_1, k_4 \in O_v^*$ and $v(k_3) \ge c$. By equating respective elements of the matrices, we get $a_1 + m\varpi^i = k_1$, $m = k_2$, $a_2\varpi^i = k_1\varpi^j + k_3$, $a_2 = k_2\varpi^j + k_4$. Then we can get a contradiction from the last two equation.

Next we show that every matrix of GL_2 belongs to $B \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} K_1(\varpi^c)$ for some i. Note that $\mathrm{GL}_2(F) = B\mathrm{GL}_2(O_v)$ by the standard Iwasawa decomposition. As a result of this, we only have to look at matrices of form $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{GL}_2(O_v)$. If $i = v(x_3) > 0$, then $x_4 \in O_v^*$. When $i \ge c$, we have

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & x_4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3/x_4 & 1 \end{pmatrix}.$$

When $0 < i < c$, we have

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} \frac{x_1x_4 - x_2x_3}{x_3}\varpi^i & x_2 \\ 0 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} \begin{pmatrix} \frac{x_3}{x_4\varpi^i} & 0 \\ 0 & 1 \end{pmatrix}.$$

When $i = 0$ and $x_4 \in O_v^*$, we can still decompose $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ like the case $0 < i < c$. If $x_4 \notin O_v^*$, then $x_2, x_3 \in O_v^*$, and

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \frac{\det x}{x_3} \begin{pmatrix} 1 & \frac{x_1 - x_3}{x_3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{x_3^2}{\det x} & -1 + \frac{x_3 x_4}{\det x} \\ 0 & 1 \end{pmatrix}.$$

□

Proof of Proposition 8.9. Let $c = c(\pi)$. Let $f \in \pi$ be the newform given in the parabolically induced model. Then its value is left $B-$equivalent and right $K_0(\varpi^c)-$invariant. By the Lemma above, f is uniquely determined by its values on the double coset representatives $\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}$. We shall just check for which i can f be supported. This will not only shows existence, but also the uniqueness and the dimension of old forms.

Suppose that

$$\begin{pmatrix} a_1 & m \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \tag{8.5}$$

for some $k = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in K_1(\varpi^c)$, or equivalently

$$\begin{pmatrix} a_1 + m\varpi^i & m \\ a_2\varpi^i & a_2 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ \varpi^i k_1 + k_3 & \varpi^i k_2 + k_4 \end{pmatrix}. \tag{8.6}$$

Then f is supported on $\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}$ iff $\chi_1(a_1)\chi_2(a_2) = 1$ for a_i satisfying the above condition. From $m = k_2$, we get that $m \in O_{\mathbb{F}}$. From $a_2\varpi^i = \varpi^i k_1 + k_3$, $a_2 = \varpi^i k_2 + k_4$, we get that

$$a_2 \equiv 1 \mod \varpi^i, \tag{8.7}$$

and

$$k_1 \equiv 1 + \varpi^i k_2 \mod \varpi^{c-i}. \tag{8.8}$$

From $a_1 + m\varpi^i = k_1$, we get that

$$a_1 \equiv 1 \mod \varpi^{c-i}. \tag{8.9}$$

Now $\chi_1(a_1)\chi_2(a_2) = 1$ iff $i \geq c(\chi_2)$ and $c - i \geq c(\chi_1)$, i.e., $c(\chi_2) \leq i \leq c - c(\chi_1)$. It has solution iff $c(\chi_1) + c(\chi_2) \leq c$, and resulting space of functions is $c - c(\chi_1) - c(\chi_2) + 1-$ dimensional. By definition of newform and level, we have

$$c(\pi) = c(\chi_1) + c(\chi_2). \tag{8.10}$$

□

Definition 8.11. When $c(\chi_1) = c(\chi_2) = 0$, we have $c(\pi) = 0$. π is called unramified in this case, and the newform is $K-$invariant, sometimes called spherical.

Remark 8.12. The relation between depth and level is no longer true for parabolically induced representations even when they are irreducible. For example for $\pi = \pi(\chi_1, \chi_2)$ with $c(\chi_1) = k$ and $c(\chi_2) = 0$, then the level of π is $c(\chi_1) + c(\chi_2) = k$ while the depth of π is $k - 1$. The relation $c(\pi) = 2(l(\pi) + 1)$ holds only when $c(\chi_1) = c(\chi_2)$ if π is irreducible.

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