MAORT Exercises and Solutions

May 17, 2018

Homework 1 (Due Mar 8). *Let $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R})$ with*

$$\phi_2(\theta) = \frac{\sin(\theta)}{\theta}\phi_1(\theta) \quad \textit{for all } \theta \in \mathbb{R}.$$

Write $d\theta$ for Lebesgue measure on the real line. For $k \in \mathbb{R}$, set $\widehat{\phi_j}(k) = \int_\theta e^{ik\theta}\phi_j(\theta)\, d\theta$. Let ν_j denote the compactly supported measure on $\mathfrak{g}$ given in standard coordinates by $\int_{\mathfrak{g}} f\, d\nu_j = \int_\theta f(0,0,\theta)\phi_j(\theta)\, d\theta$.

1. *Using Kirillov's formula and Archimedes's theorem, verify that*

$$\int_{x\in\mathfrak{g}} \chi_n(e^x)\sqrt{j(x)}\, d\nu_1(x) = \int_{k=-n-1}^{n+1} \widehat{\phi_1}(k)\, \frac{dk}{2},$$

where dk denotes Lebesgue measure.

2. *Define $T \in \mathrm{End}(V_n)$ by $T := \int_{x\in\mathfrak{g}} \pi_n(e^x)\, d\nu_2(x)$. Verify that T is diagonalized by the basis $X^{\frac{n+k}{2}}Y^{\frac{n-k}{2}}$ of π_n, where $k = -n, -n+2, \ldots, n$, with eigenvalues $\widehat{\phi_2}(k)$, hence that*

$$\int_{x\in\mathfrak{g}} \chi_n(e^x)\, d\nu_2(x) = \mathrm{trace}(T) = \sum_{k=-n,-n+2,\ldots,n} \widehat{\phi_2}(k).$$

3. *Verify that $d\nu_2 = \sqrt{j}\, d\nu_1$. Conclude that*

$$\int_{k=-n-1}^{n+1} \widehat{\phi_1}(k)\, \frac{dk}{2} = \sum_{k=-n,-n+2,\ldots,n} \widehat{\phi_2}(k).$$

[This may be deduced more directly by noting that the Fourier transform of $\sin(x)/x$ is a multiple of the characteristic function of the interval $[-1,1]$ and that the Fourier transform transports multiplication to convolution. The point is to observe the relationship between this fact and the $\mathrm{SU}(2)$ case of the Kirillov formula.]

Solution

1. From the Kirillov's formula in SU(2) and definition of the measure ν_1 we get that

$$\begin{aligned}\int_{\mathfrak{g}} \chi_n(e^x)\sqrt{j(x)}d\nu_1(x) &= \int_{\mathbb{R}} \chi_n(e^{\theta J_3})\phi_1(\theta)d\theta \\ &= \int_{\mathbb{R}}\int_{S_{n+1}} e^{i\langle \theta J_3,\xi\rangle}\phi_1(\theta)\frac{d\mu(\xi)}{4\pi(n+1)}d\theta \\ &= \int_{-(n+1)}^{(n+1)}\int_{\mathbb{R}} \phi_1(\theta)e^{i\theta\xi_3}d\theta\frac{d\xi_3}{2} \\ &= \int_{-n-1}^{n+1}\hat{\phi}_1(k)\frac{dk}{2},\end{aligned}$$

   where in the third equality we used the Archimedes's theorem and justified the interchange of the integrations as $\phi_1 \in C_c^\infty(\mathbb{R})$.

2. We need to check that $Te_k = \hat{\phi}_2(k)e_k$ where $e_k := X^{\frac{n+k}{2}}Y^{\frac{n-k}{2}}$. We note that

$$\pi_n(e^{\theta J_3})e_k = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta}\end{pmatrix} X^{\frac{n+k}{2}}Y^{\frac{n-k}{2}} = e^{ik\theta}e_k.$$

   Thus,

$$Te_k = \int_{\mathbb{R}} \pi_n(e^{\theta J_3})e_k\phi_2(\theta)d\theta = \left(\int_{\mathbb{R}} e^{ik\theta}\phi_2(\theta)d\theta\right)e_k = \hat{\phi}_2(k)e_k.$$

   Hence we calculate that,

$$\sum_{k=-n(2)}^{n}\hat{\phi}_2(k) = \mathrm{trace}(T) = \int_{\mathfrak{g}}\mathrm{trace}(\pi_n(e^x))d\nu_2(x) = \int_{\mathfrak{g}}\chi_n(e^x)d\nu_2(x)dx.$$

3. This is clear from the fact that $\sqrt{j(x)} = \frac{\sin(x)}{x}$. Hence we conclude the final statement by (1) and (2).

Homework 2 (Due Mar 8)**.** *Following the steps outlined below, establish the following:* there is a constant $C > 0$ so that for all $n \in \mathbb{Z}_{\geq 0}$ and each $\xi \in \mathcal{O}_{\pi_n}$, there is a unit vector $v \in \pi_n$ so that for all $x \in \mathfrak{g}$,

$$\|\pi_n(\exp(x))v - e^{i\langle x,\xi\rangle}v\| \leq C(\sqrt{n}|x| + |x|^{1/2}). \tag{1}$$

The moral is that the vector v is an approximate eigenvector, with eigenvalue corresponding to ξ, of group elements $g = \exp(x)$ satisfying $|x| < \epsilon/\sqrt{n}$ for some small $\epsilon > 0$.

1. *Using that* $\mathrm{Ad}^*(G)$ *acts transitively on* $\mathcal{O}_{\pi_n}$*, reduce to the case*

$$\xi = (0, 0, n+1).$$

[Hint: if (1) *holds for a given pair* (ξ, v)*, then a modified form of* (1) *holds for* $(\mathrm{Ad}^*(g)\xi, \pi_n(g)v)$*.]*

2. *Choose* $v \in \pi_n$ *to be a unit vector that is a multiple of the monomial* X^n*. Using the* G*-invariance of the inner product* $\langle , \rangle$*, verify that* v *is orthogonal to the vectors* $X^{n-1}Y, \ldots, Y^n$ *and that*

$$\langle \pi_n(g)v, v\rangle = a^n \text{ for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \tag{2}$$

3. *Using the triangle inequality, reduce to the case*

$$|x| \leq 1/\sqrt{n}. \tag{3}$$

For $x = \begin{pmatrix} ix_3 & ix_2 + x_1 \\ ix_2 - x_1 & -ix_3 \end{pmatrix} \in \mathfrak{g}$ *satisfying* (3)*, write* $\exp(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*, and verify that*

$$a = 1 + ix_3 + \mathcal{O}(|x|^2) = e^{ix_3}(1 + \mathcal{O}(|x|^2)) \tag{4}$$

and

$$a^n = e^{inx_3}(1 + \mathcal{O}(n|x|^2)) = e^{i(n+1)x_3} + \mathcal{O}(n|x|^2 + |x|), \tag{5}$$

where $\mathcal{O}(A)$ *denotes a quantity bounded in magnitude by* $C|A|$ *for some constant* $C > 0$*.*

4. *Square both sides of* (1)*, expand the norm squared as a sum of four inner products, and apply* (2) *and* (5) *to conclude.*

Solution

1. We follow the hint. As Ad^* acts transitively on $\mathcal{O}_{\pi_n}$ we may assume that $\mathrm{Ad}^*(g)\xi_0 = \xi$ for some $g \in G$, where $\xi_0 = (0, 0, n+1)$ satisfies (1) for some unit vector $v \in \pi_n$. Thus we see that

$$\begin{aligned} \|\pi_n(e^x)\pi_n(g)v - e^{i\langle x, \xi\rangle}\pi_n(g)v\| &= \|\pi_n(g)\pi_n(\mathrm{Ad}(g^{-1})e^x)v - e^{i\langle \mathrm{Ad}(g^{-1})x, \xi\rangle}\pi_n(g)v\| \\ &= \|\pi_n(e^{\mathrm{Ad}(g^{-1}x)}))v - e^{i\langle \mathrm{Ad}(g^{-1})x, \xi\rangle}\pi_n(g)v\| \\ &\ll \sqrt{n}|\mathrm{Ad}(g^{-1})x| + |\mathrm{Ad}(g^{-1})x|^{1/2}, \end{aligned}$$

where in the last inequality comes from (1) with an absolute implied constant. But we note that, as $G := \mathrm{SU}(2)$ is compact, there is an absolute constant $C > 0$ such that $|\mathrm{Ad}(g)x| \leq C|x|$ for all $x \in \mathfrak{g}$ and $g \in G$. This concludes the proof.

2. Let $g_\theta := \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$ as usual. Note that,

$$\pi_n(g_\theta)X^jY^k = e^{i(j-k)\theta}X^jY^k.$$

As the the inner product is G invariant we see that for $j+k=n$

$$\langle X^n, X^jY^k\rangle = \langle g_\theta.X^n, g_\theta.X^jY^k\rangle = \langle e^{in\theta}X^n, e^{i(j-k)\theta)}X^jY^k\rangle = e^{2ik\theta}\langle X^n, X^jY^k\rangle.$$

If $k \neq 0$ then $e^{ik\theta} \neq 1$, which implies that $\langle X^n, X^jY^k\rangle = 0$.
Recall the action of G on V by $\pi_n(g)f(X,Y) = f(g(X,Y))$. This implies that

$$\pi_n(g)X^n = (aX+cY)^n = a^nX^n + \sum_{j=1}^{n}\alpha_j X^{n-j}Y^j,$$

for some $\alpha_j \in \mathbb{C}$ implicitly depending on a, c. Thus

$$\begin{aligned}\langle \pi_n(g)v, v\rangle &= \frac{1}{\|X^n\|^2}\langle \pi_n(g)X^n, X^n\rangle \\ &= \frac{1}{\|X^n\|^2}\left(a^n\|X^n\|^2 + \sum_{j=1}^{n}\alpha_j\langle X^{n-j}Y^j, X^n\rangle\right) \\ &= a^n,\end{aligned}$$

as the each term in the sum vanishes because of orthogonality from the first part.

3. Note that if $|x| > \frac{1}{\sqrt{n}}$ then the RHS of (1) is $\gg 1$. While using a triangle inequality we obtain that the LHS

$$\leq \|\pi_n(e^x)v\| + \|e^{i\langle x,\xi\rangle}v\| \leq 2,$$

as π_n is unitary and v is unit vector. Thus from now on we assume that $|x| \leq \frac{1}{\sqrt{n}}$.
Recall that

$$\exp(x) = 1 + x + E(x),$$

where $E(x) = \sum_{n=2}^{\infty} x^n/n!$. Note that each summand matrix in the sum has entries which are homogeneous polynomial of degree 2 or higher. Thus $E(x) \ll |x|^2$ as $|x| < 1$. So we obtain that,

$$\exp(x) = 1 + x + O(|x|^2).$$

In particular, $a = 1 + ix_3 + O(|x|^2)$. On the other hand,

$$e^{ix_3} = 1 + ix_3 + O(|x_3|^2) = 1 + ix_3 + O(|x|^2).$$

Thus,

$$a = e^{ix_3} + O(|x|^2) = e^{ix_3}(1 + O(|x|^2)).$$

Now note that if $y \ll \frac{1}{n}$ then

$$(1+y)^n = 1 + O(ny).$$

Thus,

$$\begin{aligned} a^n &= e^{inx_3}(1 + O(|x|^2)^n = e^{inx_3}(1 + O(n|x|^2)) \\ &= e^{i(n+1)x_3}(e^{-ix_3} + O(n|x|^2)) = e^{i(n+1)x_3}(1 + O(|x_3|) + O(n|x|^2)) \\ &= e^{i(n+1)x_3} + O(n|x|^2 + |x|), \end{aligned}$$

Concluding the proof.

4. We let $\xi = (0, 0, n+1)$, $v = \frac{X^n}{\|X^n\|}$, and $|x| \leq \frac{1}{\sqrt{n}}$. We calculate the following inner product:

$$\begin{aligned} &\langle \pi_n(e^x)v, e^{i\langle x,\xi\rangle}v\rangle \\ &= e^{-i(n+1)x_3}\langle \pi_n(e^x)v, v\rangle \\ &= e^{-i(n+1)x_3}(e^{i(n+1)x_3} + O(n|x|^2 + |x|)) \\ &= 1 + O(n|x|^2 + |x|), \end{aligned}$$

where in the last two lines we have used (2) and (5) respectively. Thus squaring the LHS of (1) we get that

$$2\|v\|^2 - 2\Re\left(\langle \pi_n(e^x)v, e^{i\langle x,\xi\rangle}v\rangle\right) \ll n|x|^2 + |x|.$$

Taking square roots we conclude.

Homework 3 (Due Mar 15)**.** *Prove the Stone-von Neumann theorem when $V = (\mathbb{Z}/p\mathbb{Z})^n$, for p an odd prime.*

Hint: *Calculate the characters χ_h of π_h defined by*

$$\pi_h \begin{pmatrix} 1 & a & c \\ & 1 & b^t \\ & & 1 \end{pmatrix} f(x) := \psi(bx + hc) f(x + ha), \quad f \in \ell^2(V),$$

where ψ is an additive character of $\mathbb{Z}/p\mathbb{Z}$. Use the fact from representation theory of finite groups that, qny irreducible representation π of a finite group G appears in the regular representation $\dim(\pi)$ *times.*

Solution

Let $F = \mathbb{Z}/p\mathbb{Z}$ and $V = F^n$. Also the cardinality of $H(W)$ is p^{2n+1}. We write an element of $H(W)$ by

$$M(a, b, c) = \begin{pmatrix} 1 & a & c \\ & 1 & b^t \\ & & 1 \end{pmatrix} \text{ for } a, b \in V \text{ and } c \in F.$$

For a non-zero $h \in F$ we define the representation π_h on the finite dimensional inner-product space $\ell^2(V)$ (that is, V with usual 2-norm) by

$$\pi_h(M(a, b, c) f(x) := \psi(bx + hc) f(x + ha).$$

We will prove that these are only possible irreducible inequivalent representation of $H(W)$. In particular, once we fix the action the action of F the representation is unique, which proves the Stone-von Neumann theorem in this case.

We will calculate the characters of π_h's. Note that $\{1_v\}_{v \in V}$ form a basis of $\ell^2(V)$. Therefore,

$$\begin{aligned} \chi_h(M(a, b, c)) = \text{trace}(\pi_h(M(a, b, c)) &= \sum_{v \in V} \langle \pi_h(M(a, b, c)) 1_v, 1_v \rangle_{\ell^2(V)} \\ &= \sum_{v \in V} \sum_{x \in V} \psi(bx + hc) 1_{v-ha}(x) 1_v(x) \\ &= \delta_{a=0} \sum_{v \in V} \psi(hc) \sum_{x \in V} \psi(bx) \\ &= \delta_{a=0} \delta_{b=0} \psi(hc) p^n. \end{aligned}$$

From this we conclude that,

$$\sum_{g \in H(W)} |\chi_h(g)|^2 = p^{2n}.p = |H(W)|. \tag{6}$$

We also check that,

$$\sum_{g\in H(W)} \chi_{h_1}(g)\overline{\chi_{h_2}(g)} = p^{2n}\sum_{c\in F}\psi((h_1-h_2)c) = p^{2n+1}\delta_{h_1=h_2},$$

which (orthogonality of characters of inequivalent representations of finite groups) shows that π_h's are pairwise non-isomorphic and irreducible. Using the following well-known fact from the representation of finite groups:

Using the fact in the hint we get that

$$|H(W)| = \sum_{\pi \text{ inequivalent irreducible representation of H(W)}} |\dim(\pi)|^2.$$

However, the fact that $\dim(\pi) = \|\chi_\pi\|_{\ell^2(G)}$ along with equation (1), show that $\{\pi_h\}_{h\in F}$ are only possible irreducible in-equivalent representations of $H(W)$, proving the main theorem.

Homework 4 (Due Mar 22). *For $\lambda > 0$, let $\mathcal{O}_\lambda := \{\xi \in \mathfrak{g}^* : |\xi|= \lambda\}$ denote the corresponding coadjoint orbit and ω_λ its canonical symplectic volume measure, of total volume λ. Let $p : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ denote the addition map. For $\lambda_1, \lambda_2 > 0$, verify the measure disintegration*

$$p_*(\omega_{\lambda_1} \otimes \omega_{\lambda_2}) = \int_{|\lambda_1-\lambda_2|}^{\lambda_1+\lambda_2} \omega_\lambda \, d\lambda,$$

i.e. that for $f \in C_c^\infty(\mathfrak{g}^*)$,

$$\int_{x\in\mathcal{O}_{\lambda_1}} \int_{y\in\mathcal{O}_{\lambda_2}} f(x+y)\, d\omega_{\lambda_1}(x)\, d\omega_{\lambda_2}(y) = \int_{|\lambda_1-\lambda_2|}^{\lambda_1+\lambda_2} \int_{z\in\mathcal{O}_\lambda} f(z)\, dz\omega_\lambda \, d\lambda,$$

where $d\lambda$ denotes the Lebesgue measure. Compare with the decomposition

$$\pi_m \otimes \pi_n \cong \oplus_{k=|m-n|}^{m+n} \pi_k.$$

Homework 5 (Due Apr 12). *Prove the Kirillov formula for principal series of* $\mathrm{SL}_2(\mathbb{R})$.

Solution

Recall that the principal series of $G := \mathrm{SL}_2(\mathbb{R}) = BN^-$, B the subgroup of the upper triangular matrices i.e. Borel and N^- is the subgroup of lower unipotents, are of the form

$$\pi := I(\chi_1, \chi_2) := \mathrm{Ind}_B^G \chi_1 \otimes \chi_2, \quad \chi_1 \otimes \chi_2 \left(\begin{pmatrix} y & x \\ & z \end{pmatrix} \right) := \chi_1(y)\chi_2(z)|y/z|^{1/2},$$

for some unitary characters χ of $\mathbb{R}^\times$. We may assume that $\chi_j := |.|^{i\xi_j}$ for some $\xi_j \in \mathbb{R}$. To prove the Kirillov formula for π we follow similar process as in the nilpotent case. Let $f \in C_c^\infty(G)$. We will calculate $\mathrm{trace}(\pi(f))$. Note that for $v \in \pi$,

$$\begin{aligned} \pi(f)v(g) &= \int_G f(x)\pi(x)v(g)dx = \int_G f(x)v(gx)dx \\ &= \int_G f(g^{-1}x)v(x)dx = \int_{N^-}\int_B f(g^{-1}bn)v(bn)d_Lbdn \\ &= \int_{N^-} v(n) \int_B \chi_1 \otimes \chi_2(b) f(g^{-1}bn)d_Lbdn. \end{aligned}$$

Renaming $K_f(g,n) := \int_B \chi_1 \otimes \chi_2(b) f(g^{-1}bn)d_Lb$, we can write

$$\pi(f)v(g) = \int_{N^-} K_f(g,n)v(n)dn.$$

Thus,

$$\mathrm{trace}(\pi(f)) = \int_{N^-} K_f(n,n)dk = \int_{N^-}\int_B \chi_1 \otimes \chi_2(b) f(n^{-1}bn)d_Lbdn.$$

Note that if $\mathfrak{b} := \mathrm{Lie}(B)$ and $B = \begin{pmatrix} t & s \\ & -t \end{pmatrix} \in \mathfrak{b}$ then $\exp(B) = \begin{pmatrix} e^t & s\frac{\sinh t}{t} \\ & e^{-t} \end{pmatrix}$. Thus changing variable $b \mapsto \exp(B)$ in the last expression of the trace we get that

$$\mathrm{trace}(\pi(f)) = \int_{N^-}\int_{\mathfrak{b}} f(\exp(\mathrm{Ad}(n^{-1})B))e^{it(\xi_1-\xi_2)}j(t)dtdsdn,$$

where $j(t) := \frac{\sinh t}{t}$ is the jacobian due to change of variable. We also can identify $\mathfrak{g} \cong \mathbb{R}^3$ by $\begin{pmatrix} t & s_1 \\ s_2 & -t \end{pmatrix} \mapsto (t, s_1, s_2)$ and hence $\mathfrak{g}^* \cong \mathbb{R}^3$. Then letting $\phi := f \circ \exp$, and $\xi := (\xi_1 - \xi_2, 0, 0)$ we can rewrite the the last expression of the trace as

$$\begin{aligned} \mathrm{trace}(\pi(f)) &= \int_{N^-}\int_{\mathfrak{b}} \phi(\mathrm{Ad}(n^{-1})B)e^{i\langle B,\xi\rangle}j(t)dtdsdn \\ &= \int_{N^-}\int_{\mathrm{Ad}^*(n)(\mathfrak{b}^\perp+\xi)} \widehat{\phi j}(\eta)d\eta dn. \end{aligned}$$

Thus it is enough to prove that $K \times \mathrm{Ad}^*(n)(\mathfrak{b}^\perp + \xi) = \mathcal{O}_\xi$ where $\mathcal{O}_\xi$ is the orbit of ξ up to a measure zero set and to check that the symplectic volume form agrees with measure we have in the last expression.

First note that

$$\mathfrak{b}^\perp \cong \mathfrak{n}^{-*} \cong \mathfrak{n}.$$

Thus

$$\mathfrak{b}^\perp + \xi \cong \begin{pmatrix} \xi & \\ & -\xi \end{pmatrix} + \mathfrak{n} \cong \mathrm{Ad}^*(B)\xi.$$

We consider

$$N^- \times (\mathfrak{b}^\perp + \xi) \to \mathcal{O}_\xi : (n, \eta) \mapsto \mathrm{Ad}^*(n)\eta.$$

Note that if $\alpha : \mathrm{Ad}^*(n)\eta \in (\mathfrak{b}^\perp + \xi)$ then $n = 1$ for almost every η, which proves that the above map is injective (up to a measure zero set). On the other hand, for any $g = nb \in N^- B = G$ one has $\mathrm{Ad}^*(nb)\xi \in \mathrm{Ad}^*(n)(\mathfrak{b}^\perp + \xi)$, which proves the surjectivity.

Now as we have (almost) bijection α we just need to show that $\alpha_*(dnd\eta) = d\omega_\xi$. Differentiating we can get maps between the tangent spaces $d\alpha : \mathfrak{n}^- \oplus \mathfrak{b}^\perp \to T_\xi(\mathcal{O}_\xi) : (X, \eta) \mapsto \mathrm{ad}^*(X)\eta$. One can also check, using the above correspondence, that the symplectic measure in the Kirillov formula coincides with the measure in the last integral.

Homework 6 (Due Apr 19). *Let ϕ be a radial function in $\mathcal{S}(\mathbb{R}^3)$. We denote, as in the lecture,*

$$P : \mathcal{S}(\mathbb{R}^3) \to \mathcal{S}(\mathbb{R}), \quad \phi \mapsto \{x \mapsto x\phi(x,0,0)\}.$$

Prove that $P(\Delta_3\phi) = \Delta_1 P(\phi)$, where Δ_n is the Laplacian on $\mathbb{R}^n$.

Solution

Expanding the claimed identity we can see that it is enough to prove

$$x(\partial_2^2 + \partial_3^2)\phi = 2\partial_1\phi,$$

evaluated at $(x,0,0)$. Note that if $x = 0$ the identity holds as $\partial_1\phi$ is an odd function. So we may assume that $x \neq 0$, and prove that

$$x\partial_j^2\phi = \partial_1\phi, \quad j = 2,3.$$

Note that, as ϕ is radial, we have

$$\begin{aligned} \phi(x,y,0) &= \phi(\sqrt{x^2+y^2},0,0) \\ &= \phi(x\sqrt{1+y^2+x^2},0,0) \\ &= \phi(x + y^2/2x + O(y^4),0,0). \end{aligned}$$

Now if y is small enough we can expand both sides of the above in a Taylor series. Thus,

$$\phi(x,y,0) = \phi(x,0,0) + y\partial_2\phi(x,0,0) + \frac{y^2}{2}\partial_2^2\phi(x,0,0) + O(y^3).$$

On the other hand,

$$\begin{aligned} \phi(x + y^2/2x + O(y^4),0,0) &= \phi(x,0,0) + \left(\frac{y^2}{2x} + O(y^4)\right)\partial_1\phi(x,0,0) + O(y^4) \\ &= \phi(x,0,0) + \frac{y^2}{2x}\partial_1\phi(x,0,0) + O(y^4). \end{aligned}$$

Hence, matching coefficients of y^2 in the last two expressions we can conclude that

$$x\partial_2^2\phi(x,0,0) = \partial_1(x,0,0).$$

By a similar calculation for $j = 3$ we conclude the final result.

Homework 7 (Due Apr 26)**.** *Let $G := \mathrm{U}(n)$ and $T < G$ be the diagonal maximal torus in it, with Lie algebra $\mathfrak{g} := \mathfrak{u}(n)$ and $\mathfrak{t}$ respectively, and W be the the Weyl group. We identify*

$$\mathfrak{t}^*/W \to \mathbb{R}^n, \quad i\mathrm{diag}(\theta_1, \dots, \theta_n) =: i\theta \mapsto (e_1(\theta), \dots, e_n(\theta)),$$

where e_j is the j'th elementary symmetric polynomial in θ. Let $\lambda \in \mathfrak{t}^*$ be a regular element and $f \in C_c^\infty(\mathfrak{g}^*)$. Show that

$$\int_{\mathfrak{g}^*} f(x)dx = \int_{\mathfrak{t}^*/W} \int_{\mathcal{O}_\lambda} f(\xi)d\omega(\xi)d\mu(\lambda),$$

where $d\mu$ is the Lebesgue measure on $\mathbb{R}^n$ by the above identification of $\mathfrak{t}^*/W$ with $\mathbb{R}^n$.

Solution

Let us consider the map $\varphi : \mathfrak{t}^*/W \to \mathbb{R}^n$ as given above. This map is injective. To see that let $P_\theta(x) := \prod_{i=1}^n (x + \theta_i)$. Then $e_j(\theta)$ is the coefficient of x^j in P_θ. Thus if $\varphi(i\theta) = \varphi(i\theta')$ then $P_\theta = P_{\theta'}$ which implies that all the roots of these two polynomials are same, i.e. $\theta = \theta'$. We identify the image of $\varphi : \mathfrak{t}^*/W \hookrightarrow \mathbb{R}^n$ by same $\mathfrak{t}^*/W$, so that $\varphi_*\mu =$ Lebesgue.

Now, for an $h \in C_c^\infty(\mathfrak{t}^*/W)$, we consider the integral $\int_{\mathfrak{t}^*/W} f(\lambda)d\lambda$, $d\lambda$ is the Lebesgue measure on $\mathfrak{t}^*/W$ under the standard embedding in $\mathbb{R}^n$. We claim that

$$\int_{\mathfrak{t}^*/W} h(\lambda)|D(\lambda)|d\lambda = \int_{\varphi(\mathfrak{t}^*/W)} h(x)d\mu(x),$$

where, as defined in the class, $D(\lambda) := \prod_{j<k}(\lambda_j - \lambda_k)$. First assuming this claim we complete the proof of the main identity.

First using the Weyl integration formula on the Lie algebra $\mathfrak{g}$ we write the LHS as

$$\int_{\mathfrak{g}*} f(x)dx = \int_G \int_{\mathfrak{t}^*/W} f(\mathrm{Ad}^*(g)\lambda)|D(\lambda)|^2 d\lambda dg.$$

On the other hand recall that one can rewrite the symplectic integral on the coadjoint orbit $\mathcal{O}_\lambda := \mathrm{Ad}^*(G)\lambda$ as

$$\int_{\mathcal{O}_\lambda} f(\xi)d\omega(\xi) = |D(\lambda)| \int_G f(\mathrm{Ad}^*(g)\lambda)dg.$$

Hence we can rewrite the RHS as

$$\int_{\mathfrak{t}^*/W} \int_{\mathcal{O}_\lambda} f(\xi)d\omega(\xi)d\mu(\lambda) = \int_{\mathfrak{t}^*/W} |D(\lambda)| \int_G f(\mathrm{Ad}^*(g)\lambda)dgd\mu(\lambda).$$

Now letting $h(\lambda) := |D(\lambda) \int_G f(\mathrm{Ad}^*(g)\lambda)dg$, and using the claim we conclude the proof.

Now to prove the claim we need to compute the jacobian under the change of variable φ which maps

$$(\theta) = (\theta_1, \ldots, \theta_n) \mapsto (e_1(\theta), \ldots, e_n(\theta)),$$

and check that the jacobian is $|D(\theta)|$. We see that

$$\begin{aligned}\partial_{\theta_i} e_j(\theta) &= \partial_{\theta_i} \left[\theta_i(e_{j-1}(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n) + \text{independent of } \theta_i\right] \\ &= e_{j-1}(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n).\end{aligned}$$

Thus the jacobian is

$$j(\theta) = |\det((\partial_{\theta_i} e_j(\theta))_{i,j})| = |\det((e_{j-1}(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n))_{i,j})|.$$

It is easy to see that if $\theta_j = \theta_k$ for some $j \neq k$ then two rows of the matrix of $j(\theta)$ become identical, thus $j(\theta) = 0$. So $j(\theta)$ must be divisible by $D(\theta)$. Now checking the highest degrees and leading coefficients of $j(\theta)$ and $D(\theta)$ one can conclude that $j(\theta) = |D(\theta)|$, which ends the proof.

Homework 8 (Due 17 May). *Let* $H = \mathrm{U}(n-1) < G = \mathrm{U}(n)$ *with lie algebra* $\mathfrak{h}$ *and* $\mathfrak{g}$, *respectively. Fix* $f \in C_c^\infty(\mathfrak{h}^*)$. *Recall the restriction map* $\mathrm{res} : \mathfrak{g}^* \to \mathfrak{h}^*$. *Show that, for* $\lambda \in \mathfrak{g}^*$ *dominant regular,*

$$\int_{\mathcal{O}_\lambda} f(\mathrm{res}(\xi)) d\omega_\lambda(\xi) = \int_{\mu \prec \lambda} \left(\frac{1}{\mathrm{vol}(\mathcal{O}_\mu)} \int_{\mathcal{O}_\mu} f(\eta) d\omega_\mu(\eta) \right) d_a(\mu).$$

here $d_a(\mu)$ *is the measure on* $\mathbb{R}^{n-1}_{\mathrm{dom}}$ *s.t.*

$$\int_{\mathfrak{h}^*} f(x) dx \propto \int_{\mathbb{R}^{n-1}_{\mathrm{dom}}} \int_{\mathcal{O}_\mu} f d\omega_\mu d_a(\mu).$$

Solution

From the solution of Homework 7 we get that

$$\mathrm{vol}(\mathcal{O}_\mu) = \int_{\mathcal{O}_\mu} d\omega_\mu = |D(\mu)|.$$

We note that $\mathbb{R}^{n-1}_{\mathrm{dom}} \cong \mathfrak{t}^*/W$ where $\mathfrak{t}$ is the Lie algebra of the diagonal torus in H and W is the Weyl group of H. In the solution of Homework 7 we also checked that, for $h \in C_c^\infty(\mathfrak{t}^*/W)$,

$$\int_{\mathfrak{t}^*/W} h(\mu)|D(\mu)| d\mu = \int_{\mathbb{R}^{n-1}_{\mathrm{dom}}} h(x) d_a x,$$

where $d_a x$ was described in terms of the symmetric polynomials. Now from the class we recall that

$$\int_{\mathcal{O}_\lambda} f(\mathrm{res}(\xi)) d\omega_\lambda(\xi) = \int_{\mu \prec \lambda} \int_{\mathcal{O}_\mu} f(\eta) d\omega_\mu(\eta) d\mu.$$

Thus letting $h(\mu) := \int_{\mathcal{O}_\mu} f(\eta) d\omega_\mu(\eta)$ it is enough to show that

$$\int_{\mu \prec \lambda} h(\mu) d\mu = \int_{\mu \prec \lambda} h(\mu) \frac{d_a \mu}{\mathrm{vol}(\mathcal{O}_\mu)}.$$

But identifying $d_a\mu$ with $|D(\mu)| d\mu$ on $\mathbb{R}^{n-1}_{\mathrm{dom}}$ we can conclude.