Notes on microlocal aspects of representation theory

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Contents

1 Overview

These notes are recorded to complement the lectures. Material here has been rearranged from how it was presented in lecture to serve as a reference rather than an introduction.

A principal aim of the lectures is that after attending them, you should be able to do the homework problems given below their summaries.

2 Lecture summaries and homework

2.1 2/21 and 2/22: Kirillov formula for $\mathrm{SU}(2)$

Objectives. You should be able to state the Kirillov formula for compact Lie groups and to prove it for SU(2).

Summary. We proved the Kirillov formula for SU(2) by explicitly computing the Fourier transforms of the characters of the irreducible representations. We philosophized about the possibility of a "combinatorial proof" and indicated the aims of the course. We stated the general Kirillov formula in the nilpotent and compact cases. By computing the canonical symplectic forms on the coadjoint orbits of SU(2), we verified that the general formula specialized to what we had shown directly.

Homework 1 (Due Mar 8)**.**

1. Skim at least the first few pages of Chapter 1 of Kirillov's book (which should be available via mathscinet from the ETH network).

   The remaining exercises concern $G := \mathrm{SU}(2)$ and the representation $\pi_n : G \to \mathrm{GL}(V_n)$, as in the lecture, with standard basis $X^n, X^{n-1}Y, \dots, Y^n$ and character χ_n. We fix coordinates $\mathfrak{g} \cong \mathbb{R}^3 \cong \mathfrak{g}^*$ as in lecture. We denote by $\mathcal{O}_{\pi_n} \subseteq \mathfrak{g}^*$ the coadjoint orbit of π_n (i.e., the sphere of radius $n+1$) and fix a G-invariant inner product $\langle , \rangle$ on V_n, with associated norm $\|v\| = \langle v, v\rangle^{1/2}$.

2. Let $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R})$ with

$$\phi_2(\theta) = \frac{\sin(\theta)}{\theta}\phi_1(\theta) \quad \text{for all } \theta \in \mathbb{R}.$$

   Write $d\theta$ for Lebesgue measure on the real line. For $k \in \mathbb{R}$, set $\widehat{\phi_j}(k) = \int_\theta e^{ik\theta}\phi_j(\theta)\,d\theta$. Let ν_j denote the compactly supported measure on $\mathfrak{g}$ given in standard coordinates by $\int_{\mathfrak{g}} f\,d\nu_j = \int_\theta f(0,0,\theta)\phi_j(\theta)\,d\theta$.

   (a) Using Kirillov's formula and Archimedes's theorem, verify that

$$\int_{x\in\mathfrak{g}} \chi_n(e^x)\sqrt{j(x)}\,d\nu_1(x) = \int_{k=-n-1}^{n+1} \widehat{\phi_1}(k)\,\frac{dk}{2},$$

   where dk denotes Lebesgue measure.

(b) Define $T \in \operatorname{End}(V_n)$ by $T := \int_{x \in \mathfrak{g}} \pi_n(e^x)\, d\nu_2(x)$. Verify that T is diagonalized by the basis $X^{\frac{n+k}{2}} Y^{\frac{n-k}{2}}$ of π_n, where $k = -n, -n+2, \dots, n$, with eigenvalues $\widehat{\phi_2}(k)$, hence that

$$\int_{x \in \mathfrak{g}} \chi_n(e^x)\, d\nu_2(x) = \operatorname{trace}(T) = \sum_{k=-n,-n+2,\dots,n} \widehat{\phi_2}(k).$$

(c) Verify that $d\nu_2 = \sqrt{j}\, d\nu_1$. Conclude that

$$\int_{k=-n-1}^{n+1} \widehat{\phi_1}(k)\, \frac{dk}{2} = \sum_{k=-n,-n+2,\dots,n} \widehat{\phi_2}(k).$$

[This may be deduced more directly by noting that the Fourier transform of $\sin(x)/x$ is a multiple of the characteristic function of the interval $[-1, 1]$ and that the Fourier transform transports multiplication to convolution. The point is to observe the relationship between this fact and the SU(2) case of the Kirillov formula.]

3. Following the steps outlined below, establish the following: *there is a constant $C > 0$ so that for all $n \in \mathbb{Z}_{\geq 0}$ and each $\xi \in \mathcal{O}_{\pi_n}$, there is a unit vector $v \in \pi_n$ so that for all $x \in \mathfrak{g}$,*

$$\|\pi_n(\exp(x))v - e^{i\langle x, \xi\rangle} v\| \leq C(\sqrt{n}|x| + |x|^{1/2}). \tag{1}$$

The moral is that the vector v is an approximate eigenvector, with eigenvalue corresponding to ξ, of group elements $g = \exp(x)$ satisfying $|x| < \varepsilon/\sqrt{n}$ for some small $\varepsilon > 0$.

(a) Using that $\operatorname{Ad}^*(G)$ acts transitively on $\mathcal{O}_{\pi_n}$, reduce to the case

$$\xi = (0, 0, n+1).$$

[Hint: if (1) holds for a given pair (ξ, v), then a modified form of (1) holds for $(\operatorname{Ad}^*(g)\xi, \pi_n(g)v)$.]

(b) Choose $v \in \pi_n$ to be a unit vector that is a multiple of the monomial X^n. Using the G-invariance of the inner product $\langle , \rangle$, verify that v is orthogonal to the vectors $X^{n-1}Y, \dots, Y^n$ and that

$$\langle \pi_n(g)v, v\rangle = a^n \text{ for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \tag{2}$$

(c) Using the triangle inequality, reduce to the case

$$|x| \leq 1/\sqrt{n}. \tag{3}$$

For $x = \begin{pmatrix} ix_3 & ix_2 + x_1 \\ ix_2 - x_1 & -ix_3 \end{pmatrix} \in \mathfrak{g}$ satisfying (3), write $\exp(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and verify that

$$a = 1 + ix_3 + \mathrm{O}(|x|^2) = e^{ix_3}(1 + \mathrm{O}(|x|^2)) \tag{4}$$

and

$$a^n = e^{inx_3}(1 + \mathrm{O}(n|x|^2)) = e^{i(n+1)x_3} + \mathrm{O}(n|x|^2 + |x|), \qquad (5)$$

where $\mathrm{O}(A)$ denotes a quantity bounded in magnitude by $C|A|$ for some constant $C > 0$.

(d) Square both sides of (1), expand the norm squared as a sum of four inner products, and apply (2) and (5) to conclude.

4. For $\lambda > 0$, let $\mathcal{O}_\lambda := \{\xi \in \mathfrak{g}^* : |\xi| = \lambda\}$ denote the corresponding coadjoint orbit and ω_λ its canonical symplectic volume measure, of total volume λ. Let $p : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ denote the addition map. For $\lambda_1, \lambda_2 > 0$, verify the measure disintegration

$$p_*(\omega_{\lambda_1} \otimes \omega_{\lambda_2}) = \int_{|\lambda_1 - \lambda_2|}^{\lambda_1 + \lambda_2} \omega_\lambda \, d\lambda,$$

i.e., that for $f \in C_c(\mathfrak{g}^*)$,

$$\int_{x \in \mathcal{O}_{\lambda_1}} \int_{y \in \mathcal{O}_{\lambda_2}} f(x+y) \, d\omega_{\lambda_1}(x) \, d\omega_{\lambda_2}(y) = \int_{|\lambda_1 - \lambda_2|}^{\lambda_1 + \lambda_2} (\int_{z \in \mathcal{O}_\lambda} f(z) \, d\omega_\lambda(z)) \, d\lambda,$$

where $d\lambda$ denotes Lebesgue measure. Compare with the decomposition

$$\pi_m \otimes \pi_n \cong \oplus_{k=|m-n|}^{m+n} \pi_k.$$

3 Prerequisites

- basics on Lie groups, e.g., definitions of Lie algebra, exponential map, adjoint action
- basic language of representation theory (e.g., of finite groups)
- definition of Haar measure; reading the Wikipedia article should suffice

4 Some background

4.1 Peter–Weyl theorem

Let G be a group. Recall that $f : G \to \mathbb{C}$ is a *class function* if it is constant on conjugacy classes: $f(ghg^{-1}) = f(h)$.

Assume henceforth that G is compact. Then it has a Haar measure dg, which we may normalize to have total volume one. The square-integrable class functions on G form a Hilbert space with inner product

$$\langle f_1, f_2 \rangle := \int_{g \in G} f_1(g) \overline{f_2(g)} \, dg.$$

Let $\pi : G \to \mathrm{GL}(V)$ be a finite-dimensional representation. Its *character* $\chi = \chi_\pi = \chi_V : G \to \mathbb{C}$ is the class function given by $\chi(g) := \mathrm{trace}(\pi(g))$.

Theorem 1 (Peter–Weyl theorem)**.** *Every finite-dimensional representation of G decomposes as a finite direct sum of irreducible subrepresentations. The characters of the irreducible finite-dimensional representations of G, taken up to isomorphism, form an orthonormal basis for the space of class functions.*

In the special case that G is finite, this is proved in a first course on representation theory. The case of compact G may be proved similarly assuming the existence of Haar measure. We note also that for a finite-dimensional representation V,

$$V \text{ is irreducible iff } \langle \chi_V, \chi_V \rangle = 1,$$

as follows from (the easy part of) Theorem 1.

If $\pi : G \to \mathrm{GL}(V)$ is any finite-dimensional representation, then Maschke's theorem (proved as in the case of finite groups by averaging with respect to Haar measure) shows that there is a G-invariant inner product on V, which is moreover unique if V is irreducible.

4.2 Symplectic manifolds

4.2.1

Let W be a vector space of finite-dimension over the reals. A *symplectic form* on W is a map $\sigma : W \times W \to \mathbb{R}$ that is

1. bilinear,
2. alternating ($\sigma(v,v)=0$, hence $\sigma(u,v)=-\sigma(v,u)$), and
3. non-degenerate: the map $W \to W^*$ given by $v \mapsto \sigma(v,\cdot)$ is an isomorphism.

The first two properties say that

$$\sigma \in \Lambda^2(W)^* \cong \Lambda^2(W^*).$$

Lemma 2. *If W admits a symplectic form σ, then W is even-dimensional, say $\dim(W)=2n$, and there is a basis $e_1,\dots,e_n,f_1,\dots,f_n$ of W so that $\sigma(e_i,e_j)=0=\sigma(f_i,f_j)$ and $\sigma(e_i,f_j)=\delta_{ij}=-\sigma(f_i,e_j)$.*

Proof sketch. Choose any nonzero e_1, choose f_1 with $\langle e_1,f_1\rangle=1$, set $W':=\{v\in W:\langle v,e_1\rangle=\langle v,f_1\rangle=0\}$, and continue. ☐

Denoting by $e_1^*,\dots,f_n^*$ the basis of W^* dual to that furnished by the lemma, we have

$$\sigma = e_1^*\wedge f_1^* + \dots + e_n^*\wedge f_n^*.$$

We check readily that

$$\sigma^n := \underbrace{\sigma\wedge\dots\wedge\sigma}_{n \text{ times}} = n! e_1^*\wedge f_1^*\wedge\dots\wedge e_n^*\wedge f_n^* \in \Lambda^{2n}(W^*).$$

In particular, $\sigma^n \neq 0$.

4.2.2

Let M be a manifold. Recall that a 2-form σ on M is the smooth assignment $M \ni \xi \mapsto \sigma_\xi \in \Lambda^2(T_\xi^* M)$, thus σ_ξ is a bilinear alternating pairing on the tangent space $T_\xi M$. Its exterior derivative $d\sigma$ is a 3-form; recall that σ is *closed* if $d\sigma=0$. We call σ *non-degenerate* if each σ_ξ is non-degenerate in the sense of §4.2.1. A *symplectic form* σ on M is a closed, non-degenerate 2-form. A symplectic form exists only if M has even dimension, say $2n$. In that case the n-fold wedge product ω^n defines a volume form (i.e., a nonvanishing top degree form), hence an orientation and a volume measure on M.

4.3 Unitary representations

Let V be a Hilbert space. Recall that a sequence $v_j \in V$ converges to $v \in V$

- *strongly* if $\|v_j - v\| \to 0$, and
- *weakly* if $\langle v_j, w\rangle \to \langle v, w\rangle$ for each $w \in V$.

Let $\mathcal{U}(V)$ denote the space of unitary operators on V. The topologies on this space include

- the *norm* topology: $T_j \to T$ if $\|T_j - T\| \to 0$, where $\|.\|$ denotes the operator norm.
- the *strong* topology: $T_j \to T$ if $T_j v \to Tv$ strongly for each $v \in V$.
- the *weak* topology: $T_j \to T$ if $T_j v \to Tv$ weakly for each $v \in V$.

Exercise 1. Verify that if $v_j \to v$ weakly and $\|v_j\| \leq C$, then $v_j \to v$ strongly.

Let G be a topological group.

Exercise 2. Let $\pi : G \to \mathcal{U}(V)$ be a homomorphism. Verify that if π is continuous for the weak topology on the target, then it is continuous for the strong topology on the target.

Exercise 3. Verify that the map $\pi : \mathbb{R} \to \mathcal{U}(L^2(\mathbb{R}))$ given by $\pi(x)f(y) = f(y+x)$ is not continuous for the norm topology on the target.

A *unitary representation* of a group G is a pair (π, V), where V is a Hilbert space and $\pi : G \to \mathcal{U}(V)$ is a homomorphism. If G is a topological group, then we require π to be continuous for the weak (and hence the strong) operator topology on the target; in other words, the map $G \times V \to V$ is continuous. It is *irreducible* if it is nonzero and if $\{0\}$ and V are the only closed invariant subspaces.

Schur's lemma: if (π, V) is a unitary representation of a group G, then the following are equivalent:

1. π is irreducible.
2. Each bounded operator T on V that commutes with π is a scalar.

5 Kirillov formula: the case of $\mathrm{SU}(2)$

Set

$$G := \mathrm{SU}(2) := \{g \in \mathrm{SL}_2(\mathbb{C}) : gg^* = 1\},$$

where g^* denotes conjugate transpose. In this section we will compute explicitly the characters of the irreducible representations of G and show that, after pulling back to the Lie algebra and "twisting" a bit, they are given by Fourier transforms of surface measures on spheres. We then philosophize a bit.

5.1

Every element of G is conjugate to a diagonal element

$$g_\theta := \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix},$$

for some unique $\theta \in [0, \pi]$, so class functions $f : G \to \mathbb{C}$ are determined by the values $f(g_\theta)$.

Theorem 3 (Weyl integral formula)**.** *Let f be an integrable class function on G. Then*

$$\int_{g\in G} f(g)\,dg = \int_0^\pi f(g_\theta)|e^{i\theta} - e^{-i\theta}|^2 \frac{d\theta}{2\pi}. \tag{6}$$

Proof sketch. Let T denote the diagonal subgroup of G, thus $T = \{g_\theta : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$. The map

$$G/T \times T \to G$$

$$(hT, g_\theta) \mapsto h g_\theta h^{-1}$$

is generically 2-to-1. Computing its Jacobian leads to the required formula. □

Exercise 4. Verify directly that (6) holds in the special case $f = 1$.

5.2

The group G acts on the space $\mathbb{C}[X, Y]$ of polynomials in two variables:

$$g \cdot f(X, Y) := f((X, Y)g) := f(aX + cY, bX + dY)$$

$$\text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \quad f \in \mathbb{C}[X, Y].$$

For each $n \geq 0$, the space V_n of homogeneous $f \in \mathbb{C}[X, Y]$ of degree n is a G-invariant subspace of dimension $n + 1$, with basis $X^n, X^{n-1}Y, \ldots, Y^n$. Let $\pi_n : G \to \mathrm{GL}(V_n)$ denote the corresponding representation and $\chi_n : G \to \mathbb{C}$ its character. We compute χ_n. Since it is a class function, we need only compute $\chi_n(g_\theta)$. We have

$$g_\theta \cdot X^j Y^k = e^{(j-k)\theta} X^j Y^k,$$

so $\pi_n(g_\theta)$ has the basis of eigenvectors $X^n, X^{n-1}Y, \ldots, Y^n$ with eigenvalues $e^{in\theta}, e^{i(n-2)\theta}, \ldots, e^{-in\theta}$; thus

$$\chi_n(g_\theta) = e^{in\theta} + e^{i(n-2)\theta} + e^{i(n-4)\theta} + \cdots + e^{-in\theta}. \tag{7}$$

Summing the geometric series, we obtain

$$\chi_n(g_\theta) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}}. \tag{8}$$

Exercise 5.

1. Using Theorem 3 and the formula (8), verify that $\langle \chi_m, \chi_n \rangle = \delta_{mn}$. Deduce in particular that the V_n are irreducible.

2. Verify that the χ_n span a dense subspace of the space of class functions.

3. Using Theorem 1, deduce that the V_n give a complete collection of finite-dimensional irreducible representations of G, up to isomorphism.

5.3

We record some preliminaries concerning the Lie algebra

$$\mathfrak{g} = \mathrm{Lie}(G) = \{x : 1 + \varepsilon x \in G \pmod{\varepsilon^2}\}.$$

We have $1 + \varepsilon x \in G = \mathrm{SU}(2)$ iff $(1+\varepsilon x)(1+\varepsilon x)^* \equiv 1 \pmod{\varepsilon^2}$ iff $x + x^* = 0$, so

$$\mathfrak{g} = \mathfrak{su}(2) = \{x \in \mathrm{Mat}_2(\mathbb{C}) : x + x^* = 0\}.$$

Explicitly,

$$\mathfrak{g} = \left\{ x = \begin{pmatrix} ix_3 & ix_2 + x_1 \\ ix_2 - x_1 & -ix_3 \end{pmatrix} : (x_1, x_2, x_3) \in \mathbb{R}^3 \right\}.$$

Using these coordinates, we identify $\mathfrak{g}$ with $\mathbb{R}^3$. We may write $x = x_1 J_1 + x_2 J_2 + x_3 J_3$ with respect to the basis elements

$$J_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \quad J_2 = \begin{pmatrix} & i \\ i & \end{pmatrix}, \quad J_3 = \begin{pmatrix} i & \\ & -i \end{pmatrix}.$$

whose Lie brackets are given by $[J_i, J_j] = 2J_k$ for cyclic permutations $\{i, j, k\}$ of $\{1, 2, 3\}$.

The exponential map $\exp : \mathfrak{g} \to G$ is the matrix exponential function

$$\exp(x) = e^x = \sum_{n \geq 0} \frac{x^n}{n!}.$$

We note that $g_\theta = \exp(\theta J_3)$.

G acts on its Lie algebra $\mathfrak{g}$ via the adjoint action,

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}),$$

$$\mathrm{Ad}(g)x = gxg^{-1}.$$

For example, we may compute that with respect to the above coordinates, $\mathrm{Ad}(g_\theta)$ fixes x_3 and sends $x_1 + ix_2$ to $e^{2i\theta}(x_1 + ix_2)$, thus

$$\mathrm{Ad}(g_\theta) = (\text{rotation by } 2\theta \text{ radians clockwise around the } x_3\text{-axis}). \tag{9}$$

A similar description applies to $\mathrm{Ad}(\exp(\theta J_1)), \mathrm{Ad}(\exp(\theta J_2))$.

Among other general properties, one has

$$\exp(\mathrm{Ad}(g)x) = g \exp(x) g^{-1}. \tag{10}$$

The kernel of Ad is always the center of G, which in the present case of $\mathrm{SU}(2)$ is $\{\pm 1\}$.

Using the above coordinates, we may identify $\mathrm{GL}(\mathfrak{g})$ with $\mathrm{GL}_3(\mathbb{R})$. We equip $\mathfrak{g} \cong \mathbb{R}^3$ with the standard quadratic form $|x|^2 := x_1^2 + x_2^2 + x_3^2$. The following is proved in a first course on Lie groups:

Lemma 4. *The image* $\mathrm{Ad}(G) \subseteq \mathrm{GL}_3(\mathbb{R})$ *is the special orthogonal group* $\mathrm{SO}(3)$ *corresponding to the standard quadratic form on* $\mathfrak{g} \cong \mathbb{R}^3$. *The map* $\mathrm{Ad} : G \to \mathrm{SO}(3)$ *is thus surjective, with kernel* $\{\pm 1\}$.

Proof sketch. Since SO(3) is connected, it suffices to verify that $\mathrm{ad}(\mathfrak{g}) = \mathfrak{so}(3)$. Indeed, it follows from (9) and its analogues that $\mathrm{ad}(J_1)/2, \mathrm{ad}(J_2)/2, \mathrm{ad}(J_3)/2$ give the standard basis of $\mathfrak{so}(3)$. □

In particular, $\mathrm{Ad}(G)$ acts transitively on each sphere

$$\{x \in \mathfrak{g} : |x| = r\}, \tag{11}$$

so an $\mathrm{Ad}(G)$-invariant function $f : \mathfrak{g} \to \mathbb{C}$ is constant on such spheres, or equivalently, depends only upon $|x|$.

5.4

We now study the pullback of χ_n to $\mathfrak{g}$. Since χ_n is a class function on G, it follows from (10) that its pullback $\mathfrak{g} \ni x \mapsto \chi_n(e^x)$ is $\mathrm{Ad}(G)$-invariant; noting that each $x \in \mathfrak{g}$ is in the $\mathrm{Ad}(G)$-orbit of $(0, 0, |x|) = |x| J_3$, we see that the formula (8) translates to

$$\chi_n(e^x) = \frac{e^{i(n+1)|x|} - e^{-i(n+1)|x|}}{e^{i|x|} - e^{-i|x|}}. \tag{12}$$

It turns out that the best object to study now is not $\chi_n(e^x)$ but instead its twist $\chi_n(e^x)\sqrt{j(x)}$ by the Jacobian j of the exponential map, defined by writing

$$dg = j(x)\, dx \quad \text{for } g = e^x, \quad dx = \text{Haar on } \mathfrak{g}, \tag{13}$$

normalized by $j(0) = 1$. In the present case, one may compute that

$$j(x) = \left(\frac{e^{i|x|} - e^{-i|x|}}{2i|x|} \right)^2. \tag{14}$$

Remark 5. Recall that the χ_n give an orthonormal basis for class functions in $L^2(G, dg)$. To motivate the factor $\sqrt{j(x)}$, note that the exponential map is not a bijection, but "if it were," then it would follow from (13) that the functions $\chi_n(e^x)\sqrt{j(x)}$ give an orthonormal basis for $L^2(\mathfrak{g}, dx)$.

Combining (12) with (14) gives (after some cancellation) that

$$\chi_n(e^x)\sqrt{j(x)} = \frac{e^{i(n+1)|x|} - e^{-i(n+1)|x|}}{2i|x|}. \tag{15}$$

5.5

For each $x \in \mathfrak{g} \cong \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$, let $\langle x, \xi \rangle := x_1\xi_1 + x_2\xi_2 + x_3\xi_3$ denote the standard pairing. (Note that to work in a coordinate-free manner, we would take ξ in the dual space $\mathfrak{g}^*$, which we implicitly identify here with $\mathbb{R}^3$.)

For $\lambda > 0$, let $S_\lambda := \{\xi \in \mathbb{R}^3 : |\xi| = \lambda\}$ denote the sphere of radius λ. Let μ denote the standard area measure on S_λ, of total volume $4\pi\lambda^2$. The Fourier transform of this measure is defined by

$$\hat{\mu}(x) := \int_{\xi \in S_\lambda} e^{i\langle x, \xi \rangle} \, d\mu(\xi).$$

Let's evaluate this explicitly. Since μ is rotation-invariant, we may replace x with $(0, 0, |x|)$, giving

$$\hat{\mu}(x) = \int_{\xi \in S_\lambda} e^{i\xi_3|x|} \, d\mu(\xi).$$

Lemma 6 ("Archimedes's theorem"). *Under the map*

$$S_\lambda \to [-\lambda, \lambda]$$

$$\xi \mapsto \xi_3,$$

the measure μ on S_λ projects to the measure $2\pi\lambda\, dt$ on $[-\lambda, \lambda]$.

Thus

$$\hat{\mu}(x) = 2\pi\lambda \int_{t=-\lambda}^{\lambda} e^{it|x|} \, dt = 2\pi\lambda \frac{e^{i\lambda|x|} - e^{-i\lambda|x|}}{i|x|}. \tag{16}$$

5.6

Combining (15) with the specialization of (16) to $\lambda = n + 1$, we arrive at the following:

Theorem 7 (Kirillov formula for SU(2)). *Let $n \in \mathbb{Z}_{\geq 0}$. Set $\mathcal{O} := S_{n+1}$ and $\omega = \frac{\mu}{4\pi(n+1)}$, with μ the standard surface measure on $\mathcal{O}$. Then*

$$\boxed{\chi_n(e^x)\sqrt{j(x)} = \int_{\xi \in \mathcal{O}} e^{i\langle x, \xi \rangle} \, d\omega(\xi).} \tag{17}$$

This completes the rigorous part of today's discussion. Some natural questions now arise:

1. Why should something like this be true?
2. Why care?

We address those briefly below.

5.7

We may specialize (17) to $x = 0$. We have $j(0) = 1$ and $\chi(e^0) = \chi(1) = \dim(V_n) = n + 1$ and $e^{i\langle 0,\xi\rangle} = 1$, so

$$\mathrm{vol}(\mathcal{O}, d\omega) = \dim(V_n) = n + 1.$$

Given any such identity of integers, it is natural to ask for a "combinatorial proof." A vague idea in this direction is the following: given a nice partition $\mathcal{O} = \bigsqcup_k \mathcal{P}_k$ with $\mathrm{vol}(\mathcal{P}_k, d\omega) = 1$, there should be a basis v_k of V_n so that $v_k \leftrightarrow \mathcal{P}_k$ in the sense that for each $x \in \mathfrak{g}$, the set $\{i\langle x, \xi\rangle : \xi \in \mathcal{P}_k\}$ (typically something like an interval) is roughly the set of eigenvalues that show up if we write v_k as a sum of $d\pi_n(x)$-eigenvectors. (Here $d\pi(x) := \partial_{t=0}\pi(e^{tx})$.) In particular, this says that the full distribution of eigenvalues of $d\pi_n(x)$ should be approximately $\{i\langle x, \xi\rangle : \xi \in \mathcal{O}\}$; in the special case $x = J_3$, these two sets may be readily verified to be respectively

$$\{i(-n), i(-n+2), \ldots, in\} \quad \text{and} \quad [-i(n+1), i(n+1)],$$

which look similar in the $n \to \infty$ limit (cf. problem 2 on Homework 1). For another example, if each $\mathcal{P}_k$ is concentrated near some $\xi_k \in \mathcal{O}$, then the v_k given a basis of approximate eigenvectors for $\pi_n(e^x)$ when $x \in \mathfrak{g}$ is small enough, with eigenvalues $e^{i\langle x,\xi_k\rangle}$ (cf. problem 3 on Homework 1), hence

$$\chi_n(e^x) \approx \sum_k e^{i\langle x,\xi_k\rangle} \approx \int_{\mathcal{O}} e^{i\langle x,\xi\rangle}\, d\omega,$$

recovering an approximate form of the (exact) Kirillov formula (ignoring the $\sqrt{j(x)}$ factor, which is ≈ 1 for small x). We want to explain to what extent this naive idea is true and how it can be useful for estimating certain quantities that arise in representation theory and the study of periods of automorphic forms. We also want to explain how it fits into the broader picture of the orbit method and how the latter addresses the main problems in representation theory (e.g., determining the unitary dual and its topological structure, explicit description of restriction/induction functors, character formulas, computing Plancherel measure, etc.).

6 Kirillov formula: the general statement

Last time, we proved that the Fourier transform of the characters of irreducible representations of SU(2) (pulled back to the Lie algebra and twisted by the square root of the Jacobian of the exopnential map) have a particularly nice shape. However, our discussion was completely ad hoc. In particular, it wasn't clear at all how we might have "predicted" the normalization of the volume forms on the spheres without just doing the calculation. We now explain how that normalization arises naturally, putting the formula proved last time in some context.

Let G be a Lie group. We denote by $\mathfrak{g}$ its Lie algebra and by $\mathfrak{g}^* := \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$ the linear dual of its Lie algebra. For $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, we denote by $\langle x, \xi \rangle \in \mathbb{R}$ their natural pairing. Let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation. By duality, we obtain the *coadjoint representation* $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$, characterized by

$$\langle x, \xi \rangle = \langle \mathrm{Ad}(g)x, \mathrm{Ad}^*(g)\xi \rangle.$$

We may differentiate these to obtain Lie algebra representations

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*).$$

A *coadjoint orbit* is an $\mathrm{Ad}^*(G)$-orbit $\mathcal{O} \subseteq \mathfrak{g}^*$. An important observation (of Kirillov–Kostant–Souriau–Lie–Poisson) is that the Lie bracket on $\mathfrak{g}$ gives rise to a canonical *symplectic form* (cf. §4.2.2) σ on each coadjoint orbit $\mathcal{O}$, characterized by the identity

$$\sigma(x^*, y^*) = [x, y],$$

where

- for $x \in \mathfrak{g}$, x^* denotes the vector field on $\mathcal{O}$ given by $x^*(\xi) := \partial_{t=0} \mathrm{Ad}^*(e^{tx})\xi$;
- $[x, y] \in \mathfrak{g} \cong \mathfrak{g}^{**}$ defines the function $\mathcal{O} \ni \xi \mapsto \langle [x, y], \xi \rangle$

More explicitly, σ should smoothly assign to each $\xi \in \mathcal{O}$ an alternating bilinear form σ_ξ on the tangent space $T_\xi \mathcal{O}$. We may describe σ_ξ as follows. Let $\xi \in \mathcal{O}$. Then $\mathcal{O}$ is the $\mathrm{Ad}^*(G)$-orbit of ξ, thus[1] we may identify

$$T_\xi \mathcal{O} = \mathrm{ad}^*(\mathfrak{g})\xi = \{[x, \xi] : x \in \mathfrak{g}\}, \quad [x, \xi] := \mathrm{ad}^*(x)\xi.$$

The 2-form σ is then defined by

$$\sigma_\xi([x, \xi], [y, \xi]) = \langle [x, y], \xi \rangle \quad \text{for all } x, y \in \mathfrak{g}.$$

A few things need to be checked here:

- that σ_ξ is well-defined, i.e., that $\langle [x, y], \xi \rangle = 0$ if either x or y belongs to $\mathfrak{g}_\xi$, as follows from the identity
$$\langle [x, y], \xi \rangle = \langle x, [y, \xi] \rangle = -\langle y, [x, \xi] \rangle; \qquad (18)$$
- that σ_ξ is non-degenerate, i.e., that if $y \in \mathfrak{g}$ has the property that $\sigma_\xi(x, y) = 0$ for all $x \in \mathfrak{g}$, then $y =\in \mathfrak{g}_\xi$, as follows again from (18);
- that σ is closed, which boils down to the Jacobi identity $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

[1] We're taking for granted here basic properties of Lie group actions, typically proved in a first course.

The nondegeneracy of σ_ξ implies in particular that $\mathcal{O}$ is has even dimension, say $2n$, and also that the n-fold wedge product σ^n is a volume form on $\mathcal{O}$.

Exercise 6. Determine the adjoint and coadjoint orbits for

$$G = \text{Heis} := \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ & 1 & \mathbb{R} \\ & & 1 \end{pmatrix}.$$

Verify in particular that the coadjoint orbits are all even-dimensional, while some of the adjoint orbits are odd-dimensional.

We will refer to

$$\omega := \frac{1}{n!}(\frac{\sigma}{2\pi})^n$$

as the *canonical symplectic volume form* on $\mathcal{O}$. (Kirillov would write this as $e^{\sigma/2\pi}$, and omit the 2π, which may be considered an artefact of our normalization of Fourier transforms.) We'll eventually prove some chunk of the following:

Theorem 8. *Let G be a connected Lie group that is either nilpotent or compact. For each irreducible unitary representation (π, V) of G there is a coadjoint orbit $\mathcal{O} = \mathcal{O}_\pi$ so that the character $\chi = \chi_\pi$ of π is given by*

$$\chi(e^x)\sqrt{j(x)} = \int_{\xi \in \mathcal{O}} e^{i\langle x, \xi\rangle}\, d\omega(\xi),$$

where ω is the canonical symplectic volume form on $\mathcal{O}$.

Exercise 7. Take $G = \mathrm{SU}(2)$. Choose coordinates $\mathfrak{g} \cong \mathbb{R}^3$ and $\mathfrak{g}^* \cong \mathbb{R}^3$ as in the first lecture, by writing

$$x = \begin{pmatrix} ix_3 & ix_2 + x_1 \\ ix_2 - x_1 & -ix_3 \end{pmatrix}, \quad \langle x, \xi \rangle = x_1\xi_1 + x_2\xi_2 + x_3\xi_3.$$

Verify that the coadjoint orbits are the spheres $\mathcal{O} = \{\xi \in \mathbb{R}^3 : |\xi| = \lambda\}$, $\lambda \geq 0$. Assume now that $\lambda > 0$. Set $\xi = (0, 0, \lambda) \in \mathcal{O} \subseteq \mathfrak{g}^* \cong \mathbb{R}^3$, and let J_1, J_2, J_3 be the basis of $\mathfrak{g}$ considered above. Verify that $[J_1, \xi] = (0, -2\lambda, 0)$, $[J_2, \xi] = (2\lambda, 0, 0)$, and $\langle [J_1, J_2], \xi \rangle = 2\lambda$, hence that $\sigma_\xi((1, 0, 0), (0, 1, 0)) = 1/2\lambda$. Deduce that the canonical symplectic measure ω is given by $\mu/4\pi\lambda$, where μ is the standard surface measure. [Hint: μ identifies with the rotation invariant 2-form for which $\mu_\xi((1, 0, 0), (0, 1, 0)) = 1$.] Conclude that Theorem 8 specializes to Theorem 7.

Some refinements:

1. If G is nilpotent and simply-connected, then the correspondence between representations π and coadjoint orbits $\mathcal{O}$ turns out to be bijective. In particular, every coadjoint orbit arises. By contrast, for SU(2), only the spheres with positive integral radius were relevant.

2. The formula holds when G is reductive (generalizing the compact case) under the additional assumption that π is tempered, and with the caveat that in "rare" cases, one must take for $\mathcal{O}$ a finite union of orbits.

7 The Heisenberg group

7.1

We now consider the Lie group

$$G := \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ & 1 & \mathbb{R} \\ & & 1 \end{pmatrix}.$$

As generators we can take the elements

$$g_1^x := \begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \quad g_2^y := \begin{pmatrix} 1 & 0 & 0 \\ & 1 & y \\ & & 1 \end{pmatrix}, \quad g_3^z := \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

for $x, y, z \in \mathbb{R}$. The maps $x \mapsto g_1^x, y \mapsto g_2^y, z \mapsto g_3^z$ are homomorphisms, the elements g_3^z are all central, and we have the relation

$$g_1^x g_2^y = g_3^{xy} g_2^y g_1^x. \tag{19}$$

These remarks give a presentation for G.

7.2

We aim to work out the irreducible unitary representations of G. Motivated by Theorem 8, we might first try working out the coadjoint orbits. The Lie algebra of G is given by

$$\mathfrak{g} = \begin{pmatrix} 0 & \mathbb{R} & \mathbb{R} \\ & 0 & \mathbb{R} \\ & & 0 \end{pmatrix} = \left\{ [x, y, z] := \begin{pmatrix} 0 & x & z \\ & 0 & y \\ & & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \cong \mathbb{R}^3.$$

The exponential map (given by the exponential series) is a diffeomorphism, given explicitly by

$$e^{[x,y,z]} = \begin{pmatrix} 1 & x & z + xy/2 \\ & 1 & y \\ & & 1 \end{pmatrix} = g_3^{z+xy/2} g_2^y g_1^x. \tag{20}$$

One checks readily that Haar measures both on G and on $\mathfrak{g}$ may be given by $dx\, dy\, dz$, and that the Jacobian of the exponential map is trivial:

$$j(x) = 1 \text{ for all } x \in \mathfrak{g}.$$

The adjoint action is given in the above optic by conjugation:

$$\mathrm{Ad}(g)[x, y, z] = g[x, y, z]g^{-1}.$$

The dual of the Lie algebra may be identified with the quotient space of matrices modulo upper-triangular matrices, with the pairing given by taking the trace of the product:

$$\mathfrak{g}^* = \left\{ [\alpha,\beta,\gamma] := \begin{pmatrix} * & * & * \\ \alpha & * & * \\ \gamma & \beta & * \end{pmatrix} \right\} \cong \mathbb{R}^3,$$

$$\begin{aligned} \langle [x,y,z],[\alpha,\beta,\gamma]\rangle &:= \operatorname{trace}([x,y,z][\alpha,\beta,\gamma]) \\ &= \operatorname{trace}(\begin{pmatrix} 0 & x & z \\ & 0 & y \\ & & 0 \end{pmatrix} \begin{pmatrix} * & * & * \\ \alpha & * & * \\ \gamma & \beta & * \end{pmatrix}) \\ &= \operatorname{trace}(\begin{pmatrix} x\alpha + z\gamma & * & * \\ * & y\beta & * \\ * & * & 0 \end{pmatrix}) \\ &= x\alpha + y\beta + z\gamma. \end{aligned}$$

Since $\operatorname{trace}(g[x,y,z]g^{-1}g[\alpha,\beta,\gamma]g^{-1}) = \operatorname{trace}([x,y,z][\alpha,\beta,\gamma])$, we see that the coadjoint action is also given in the above optic by conjugation:

$$\operatorname{Ad}^*(g)[\alpha,\beta,\gamma] = g[\alpha,\beta,\gamma]g^{-1}.$$

Using this we compute readily that $\operatorname{Ad}^*(g_3^z)$ is trivial, while

$$\operatorname{Ad}^*(g_1^x)[\alpha,\beta,\gamma] = [\alpha,\beta - x\gamma,\gamma], \tag{21}$$

$$\operatorname{Ad}^*(g_2^y)[\alpha,\beta,\gamma] = [\alpha + x\gamma,\beta,\gamma]. \tag{22}$$

It follows that the coadjoint orbits for G are described as follows:

- For each $(\alpha,\beta) \in \mathbb{R}^2$, we have a zero-dimensional orbit

$$\mathcal{O}_{\alpha,\beta} := \{[\alpha,\beta,0]\}.$$

- For each $\gamma \in \mathbb{R}^\times$, we have a two-dimensional orbit

$$\mathcal{O}_\gamma := \left\{[\alpha,\beta,\gamma] : (\alpha,\beta) \in \mathbb{R}^2\right\} \cong \mathbb{R}^2.$$

7.3

It's easy to find representations $(\pi_{\alpha,\beta}, V_{\alpha,\beta})$ corresponding to the zero-dimensional orbits. The symplectic volumes of these orbits are 1 (by integrating the 0-form $\omega = (\sigma/2\pi)^0/0! = 1$ over a one-element set...), so we expect $\dim(V_{\alpha,\beta}) = 1$. The Kirillov formula tells us what the trace of $\pi_{\alpha,\beta}$ should be:

$$\operatorname{trace}(\pi_{\alpha,\beta}(\exp([x,y,z]))) = \int_{\mathcal{O}_{\alpha,\beta}} e^{i\langle [x,y,z],\cdot\rangle}\, d\omega = e^{i\langle [x,y,z],[\alpha,\beta,0]\rangle} = e^{i(x\alpha+y\beta)}.$$

Thus $\pi_{\alpha,\beta} : G \to \mathrm{GL}(V_{\alpha,\beta}) \cong \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ is the one-dimensional representation given by

$$\pi_{\alpha,\beta} = e^{i(x\alpha+y\beta)}.$$

7.4

The representations (π_γ, V_γ) corresponding to the two-dimensional orbits $\mathcal{O}_\gamma$ are more interesting. Let's start by computing the canonical symplectic forms σ and volume forms ω attached to these orbits. Let $J_1 = [1,0,0]$, $J_2 = [0,1,0]$, $J_3 = [0,0,1]$ be the obvious basis of $\mathfrak{g}$. Then J_3 is central, while $[J_1, J_2] = J_3$. For $\xi = [\alpha, \beta, \gamma] \in \mathcal{O}_\gamma$, we see by differentiating (21) and (22) that $[J_1, \xi] = [0, -\gamma, 0]$ and $[J_2, \xi] = [\gamma, 0, 0]$, so that $\sigma_\xi([J_1, \xi], [J_2, \xi]) = \langle [J_1, J_2], \xi \rangle = \langle J_3, \xi \rangle = \gamma$. It follows that $\sigma_\xi([1,0,0],[0,1,0]) = 1/\gamma$, hence that in the standard coordinates $(\alpha, \beta) \in \mathbb{R}^2$ for $\mathcal{O}_\gamma$,

$$\sigma = \frac{d\alpha \wedge d\beta}{\gamma}, \quad \omega = \frac{d\alpha \wedge d\beta}{2\pi\gamma}.$$

In particular, $\operatorname{vol}(\mathcal{O}_\gamma, d\omega) = \infty$, so we expect $\dim(V_\gamma) = \infty$. The Kirillov formula tells us what the character χ_γ should look like:

$$\chi_\gamma(e^{[x,y,z]}) = \int_{\alpha,\beta \in \mathbb{R}} e^{i(x\alpha + y\beta + z\gamma)} \frac{d\alpha \, d\beta}{2\pi|\gamma|}. \tag{23}$$

The integral doesn't converge, but we can interpret it distributionally by integrating against a smooth compactly-supported measure on $\mathfrak{g}$, which we may write as $\phi(x,y,z)\, dx\, dy\, dz$ for some $\phi \in C_c^\infty(\mathfrak{g}) \cong C_c^\infty(\mathbb{R}^3)$ with respect to Lebesgue measure $dx\, dy\, dz$. The precise way to interpret (23) is then that for each such ϕ, the integral operator

$$\pi_\gamma(\phi) := \int_{x,y,z \in \mathbb{R}} \phi(x,y,z) \pi_\gamma(e^{[x,y,z]}) \, dx \, dy \, dz$$

on V_γ is trace-class, with

$$\operatorname{trace}(\pi_\gamma(\phi)) = \int_{\alpha,\beta \in \mathbb{R}} \left(\int_{x,y,z \in \mathbb{R}} \phi(x,y,z) e^{i(x\alpha + y\beta + z\gamma)} \, dx \, dy \, dz \right) \frac{d\alpha \, d\beta}{2\pi|\gamma|}. \tag{24}$$

By Fourier inversion, we may rewrite the desired trace identity (24) in the form

$$\operatorname{trace}(\pi_\gamma(\phi)) = \frac{2\pi}{|\gamma|} \int_{z \in \mathbb{R}} \phi(0,0,z) e^{iz\gamma} \, dz. \tag{25}$$

How to construct the required representation π_γ? By noting that we may pull the factor $e^{iz\gamma}$ out of the integral in (23), we might guess that the element $e^{[0,0,z]} = g_3^z$ should act as multiplication by the scalar $e^{iz\gamma}$:

$$\pi_\gamma(g_3^z) := e^{iz\gamma} \quad \text{(scalar multiplication).}$$

We might then aim to find a Hilbert space V_γ on which the actions of g_1^x and g_2^y respect the relation (19). It turns out, as we'll explain later, that there is (up to isomorphism) only one way to do this. We'll give the answer first, and then explain how we could have arrived at it naturally.

7.5

We take

$$V_\gamma := L^2(\mathbb{R}),$$
$$\pi_\gamma(g_1^x)f(t) := f(t+x),$$
$$\pi_\gamma(g_2^y)f(t) := e^{i\gamma yt}f(t).$$

We compute readily that then

$$\pi_\gamma(g_1^x g_2^y)f(t) = e^{i\gamma y(t+x)}f(t+x) = e^{i\gamma xy}e^{i\gamma yt}f(t+x) = \pi_\gamma(g_3^{xy}g_2^y g_1^x)f(t),$$

so the relation (19) is respected, and we obtain a well-defined unitary representation of G.

Let's compute the traces of the integral operators $\pi_\gamma(\phi)$, $\phi \in C_c^\infty(\mathfrak{g})$. Using (20), we obtain

$$\pi_\gamma(\phi)f(t) = \int_{x,y,z\in\mathbb{R}} \phi(x,y,z)e^{i\gamma(z+xy/2+yt)}f(t+x)\,dx\,dy\,dz.$$

By the change of variables $x \mapsto x - t$, we may rewrite the above as $\pi_\gamma(\phi)f(t) = \int_{x\in\mathbb{R}} f(x)K_\phi(x,t)\,dx$, where

$$K_\phi(x,t) = \int_{y,z\in\mathbb{R}} \phi(x-t,y,z)e^{i\gamma(z+(x-t)y/2+yt)}\,dy\,dz.$$

Using that ϕ is smooth and compactly-supported and a bit of Fourier analysis, we see that the kernel K_ϕ belongs to the Schwartz space on $\mathbb{R}^2$. We'll show eventually (postponing for now the functional-analytic details) that $\pi_\gamma(\phi)$ is trace-class with

$$\mathrm{trace}(\pi_\gamma(\phi)) = \int_{t\in\mathbb{R}} K_\phi(t,t)\,dt = \int_{t,y,z\in\mathbb{R}} \phi(0,y,z)e^{i\gamma(z+ty)}\,dt\,dy\,dz.$$

Substituting $t = \gamma^{-1}\beta$, $dt = \gamma^{-1}d\beta$ and applying Fourier inversion to the y-integral, we readily derive the required formula (24).

In summary, we've been able to produce, for each coadjoint orbit, a unitary representation whose trace is as predicted by the Kirillov formula. Some natural questions now arise:

1. Is there some more direct way we could have arrived at the representation V_γ from the coadjoint orbit $\mathcal{O}_\gamma$, without just having to guess the answer and check that it works? (Yes.)
2. Are the representations V_γ irreducible? (We'll see that the answer is yes.)
3. Do we get *every* irreducible unitary representation of G in this way? (We'll see again that the answer is yes.)

We take these up next.