Topics in Minimal Surface Theory

Introduction to the Almgren-Pitts Theory and Applications

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Contents

Introduction

We give in the first two chapters an introduction to the basic objects appearing in Geometric Measure Theory, with an emphasis on the tools most useful to understand the technical details of the series of papers of Fernando Codá-Marques and André Neves, partly in collaboration with Yevgeny Liokumovich and Kei Irie, culminating in the proof of the Yau's conjectures about denseness of minimal hypersurfaces in Riemannian manifolds of generic metrics of dimension less or equal than 7 ([MN17], [MN16], [LMN16], [IMN17]).

Notations

We fix for all subsequent chapters an integer $n \geq 3$, and a Riemmanian submanifold $M \subset \mathbb{R}^n$. This is not restrictive thanks of the Nash embedding theorem [Nas56]. We fix notations for the sets we will consider, as they differ slightly from the ones of Pitts. Let $x \in \mathbb{R}^n, 0 < r < s < \infty$.

$$\begin{aligned}
B(x,r) &= \mathbb{R}^n \cap \{y : |y-x| < r\} \\
\overline{B}(x,r) &= \mathbb{R}^n \cap \{y : |y-x| \leq r\} \\
A(x,r,s) &= \mathbb{R}^n \cap \{y : r < |y-x| < s\} \\
\overline{A}(x,r,s) &= \mathbb{R}^n \cap \{y : r \leq |y-x| \leq s\}
\end{aligned}$$

and we define likewise notations $B_M, \overline{B}_M, A_M, \overline{A}_M$ with $\mathbb{R}^n$ and replaced by M and the Euclidean distance replaced by the geodesic distance of M.

Definition. For $0 < s < \infty$, we note $\alpha(s) = \dfrac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$. Let $n \geq 2$. Referring to [Fed69] (2.10), $\mathscr{H}^s$ is the Hausdorff measure on $\mathbb{R}^n$ (equipped with its euclidean distance) associated in the Carathéodory construction with the function

$$\zeta(A) = \alpha(s)\left(\frac{\mathrm{diam} A}{2}\right)^s.$$

This is the standard normalisation, which shows that for a k-submanifold of $\mathbb{R}^n$, $\mathscr{H}^k$ coincides with the induced volume form. For all definitions on measure theory, we refer to Federer [Fed69]. We recall some of the most basic definitions.

Definition. Let (X, μ, d) a metric measured space space, $0 \leq s < \infty$ and $x \in X$. We define the s dimensional lower and upper densities of μ at x by

$$\begin{aligned}
\Theta^s_*(\mu, x) &= \limsup_{r \to 0^+} \frac{\mu\left(\overline{B}(x,r)\right)}{\alpha(s) r^s} \\
\Theta^{s*}(\mu, x) &= \liminf_{r \to 0^+} \frac{\mu\left(\overline{B}(x,r)\right)}{\alpha(s) r^s}.
\end{aligned}$$

If these two numbers coincide, we denote this common value $\Theta^s(\mu, x)$ and we call it the density of μ.

Our definition of k-rectifiability coincides with the definition of countably $(\mathscr{H}^k, k)$-rectifiability of Federer ([Fed69], 3.2.14), as we will not need the stronger notions of rectifiability.

Definition. If k is an integer such that $1 \leq k \leq n$ and $A \subset \mathbb{R}^n$ is $\mathscr{H}^k$ measurable, we say that A is k-rectifiable if A is $\mathscr{H}^k$ measurable and if there exists a sequence of Lipschitz function $f_j : \mathbb{R}^k \to \mathbb{R}^n$ $(j \geq 1)$ such that

$$\mathscr{H}^k\left(A \setminus \bigcup_{j=1}^{\infty} f_j(\mathbb{R}^k)\right) = 0.$$

Chapter 1

Introduction to currents

1.1 Preliminary definitions

Theorem 1.1.1 (Besicovitch covering theorem)**.** *There exists a positive integer $B(n)$ with the following property. Let $A \subset \mathbb{R}^n$, and $\mathscr{B}$ be a family of balls such that all point of A is the center of a ball of $\mathscr{B}$. Assume that A is bounded or the radii of the ball of $\mathscr{B}$ uniformly bounded. Then there exists disjoint families $\mathscr{B}_1, \cdots, \mathscr{B}_{B(n)}$ of $\mathscr{B}$ such that*

$$A \subset \bigcup_{i=1}^{B(n)} \bigcup \mathscr{B}_i.$$

Remark 1.1.2. Note that the hypothesis of radius boundedness of necessary in this theorem if A is unbounded, contrary to the version quoted by Allard ([All72]). However, the reference cited therein is correct ([Fed69], 2.8.14). We can easily show that $B(n) \leq 2^{3n}$ (see [Aus12], 2.1.4), and this had been proved that there is an exponential lower bound of C (see for example [FL94]). This theorem shows the advantages to work in $\mathbb{R}^n$, as Besicovitch theorem is false in general. Aside from the counter-example of the Heisenberg group, see also the work of Séverine Rigot ([Rig04]).

We first start by recalling the definition of Radon measures.

Definition 1.1.3 ([Fed69], 2.2.5)**.** A Radon measure on a locally compact topological space X is a Borel regular locally finite measure.

We shall need the following simple property of Radon measures.

Lemma 1.1.4. *Suppose that X has a topology with a countable basis, and let $\{\mu_r\}_{r\in\mathbb{R}}$ be a family of Radon measures such that*

$$\mu_r \leq \mu_s \quad \forall r \leq s.$$

Then for $\mathscr{L}^1$ a.e. $r \in \mathbb{R}$, there exists a Radon measure $\mu'(r)$ on X such that

$$\mu'(f) = \lim_{h\to 0} \frac{(\mu_{r+h} - \mu_r)}{h}(f) \quad \forall f \in C_c(X).$$

Proof. Let $\mathscr{C}$ a countable base of $C_c(X)$. For all $f \in C_c(X)$, there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathscr{C}$ and a function $f_\infty \in \mathscr{C}$ such that

$$|f(x) - f_j(x)| \leq 2^{-j} f_\infty(x), \quad \forall x \in X$$

As for all $f \in C_c(X)$, the real function $r \mapsto \mu_r(f)$ is increasing, it has in particular a locally bounded variation, the limit

$$\lim_{h\to 0} \frac{\mu_{r+h}(f) - \mu_r(f)}{h}$$

exists for $\mathscr{L}^1$ a.e. $r \in \mathbb{R}$. Defining for all $g \in \mathscr{C}$ the set

$$N_g = \mathbb{R} \cap \left\{ \lim_{h \to 0} \frac{\mu_{r+h}(f) - \mu_r(f)}{h} \text{ does not exist} \right\} \tag{1.1.1}$$

we obtain $\mathscr{L}^1(N_g) = 0$, and as $\mathscr{C}$ is countable, we obtain $\mathscr{L}^1(\cup_{g \in \mathscr{C}} N_g) = 0$. As

$$|\mu_r(f) - \mu_r(f_j)| \leq 2^{-j} \mu_r(f_\infty) \quad \forall j \in \mathbb{N},$$

we deduce that for all $r \in \mathbb{R} \setminus N$

$$\lim_{h \to 0} \frac{\mu_{r+h}(f) - \mu_r(f)}{h} = \lim_{j \to \infty} \lim_{h \to 0} \frac{\mu_{r+h}(f_j) - \mu_r(f_j)}{h} \in \mathbb{R}$$

and we conclude the proof thanks of the Riesz representation theorem. $\square$

1.2 First definitions and theorems

The currents were first introduced by De Rham and their application to Geometric Measure Theory was made possible thanks of Federer and Fleming in their seminal paper *Normal and Integral Currents* ([FF]).

Let (M^m, g) be a fixed (non-necessarily compact) Riemannian manifold (C^4 is sufficient) and for all $k \in \mathbb{N}$ let $\Omega^k(M) \subset \Lambda^k T^* M$ be the vector-space of compactly supported k-differential forms.

Definition 1.2.1. A k-dimensional current of a Riemannian manifold M is an element of the dual of $\Omega^k(M)$, equipped with the weak sequential topology. We denote this space $\mathscr{D}_k(M)$, and we say that a sequence $\{T_j\}_{j \in \mathbb{N}}$ converges towards $T \in \mathscr{D}_k(M)$ if for all $\omega \in \Omega^k(M)$, we have

$$\lim_{k \to \infty} T_k(\omega) = T(\omega).$$

The first operation that we can define is the boundary $\partial T \in \mathscr{D}_{k-1}(M)$ of a current $T \in \mathscr{D}_k(M)$ which is characterised by the requirement that

$$\partial T(\omega) = T(d\omega) \text{ for all } \ \omega \in \Omega^{k-1}(M).$$

The sign convention is made in order to make Stokes theorem true for currents of integration on a C^1 sub-manifold with boundary.

The reference of this paragraph is the monograph of Federer 1.7 and 1.8 ([Fed69])).

Let V be a real vector space of dimension $m \geq 1$ equipped with an inner product $\langle \cdot \, , \cdot \rangle$ and let $|\cdot| = \sqrt{\langle \cdot \, , \cdot \rangle}$ be the associated norm. This scalar product induces norms on $\Lambda_k V$ for all $1 \leq k \leq m$, still denoted by $\langle \cdot \, , \cdot \rangle$. They are characterised by the following property : if $(e_1, \cdots, e_m)$ is any orthonormal basis of V, $1 \leq k \leq m$ is a fixed integer and $v, w \in \Lambda_k V$ are written in the base $(e_1, \cdots, e_m)$ as

$$v = \sum_{1 \leq i_1 < \cdots < i_k \leq m} v_{i_1, \cdots, i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad w = \sum_{1 \leq i_1 < \cdots < i_k \leq m} w_{i_1, \cdots, i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$$

then the scalar product

$$\langle v, w \rangle = \sum_{1 \leq i_1 < \cdots < i_k \leq m} v_{i_1, \cdots, i_k} w_{i_1, \cdots, i_k}$$

is well-defined independently of the orthonormal basis $(e_1, \cdots, e_m)$ of V. We also denote by $|\cdot|$ the associated norm of this scalar product on $\Lambda_k V$. We can define analogously a scalar product on $\Lambda^k V$, but the notion we will need is another norm, called the *comass*. For all $\omega \in \Lambda^k V$, the comass of ω is

$$\|\omega\| = \sup \left\{ \omega(v) : v \in \Lambda_k V, \ v \text{ is simple}, \ |v| \leq 1 \right\}.$$

For all $v \in \Lambda_k V$, the mass of v is

$$\|v\| = \sup\left\{\omega(v) :\ \omega \in \Lambda^k V,\ \|\omega\| \leq 1\right\}.$$

We always have the inequalities

$$|v| \leq \|v\| \leq \binom{m}{k}^{\frac{1}{2}} |v|. \tag{1.2.1}$$

If $(\xi_1, \cdots, \xi_m)$ is the dual base of an orthonormal base $(e_1, \cdots, e_m)$ such that

$$v = \sum_{1 \leq i_1 < \cdots < i_k \leq m} v_{i_1,\cdots,i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \neq 0$$

and $\omega \in \Lambda^k V$ is chosen such that

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \frac{v_{i_1,\cdots,i_k}}{|v|} \xi_{i_1} \wedge \cdots \wedge \xi_{i_k},$$

we have

$$\|\omega\| = \max_{1 \leq i_1 < \cdots < i_k \leq m} \frac{|v_{i_1,\cdots,i_k}|}{|v|} \leq 1$$

and $\omega(v) = |v|$, so we obtain $|v| \leq \|v\|$ and the left-hand side inequality of (1.2.1). For the other inequality, we see that for all $\omega \in \Lambda^k V$ such that

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \omega_{i_1,\cdots,i_k} \xi_{i_1} \wedge \cdots \wedge \xi_{i_k},$$

the condition $\|\omega\| \leq 1$ implies that $|\omega_{i_1,\cdots,i_k}| \leq 1$ for all $1 \leq i_1 < \cdots < i_k \leq m$, so we obtain

$$\begin{aligned}\omega(v) = \sum_{1 \leq i_1 < \cdots i_k \leq m} \omega_{i_1,\cdots,i_k} v_{i_1,\cdots,i_k} &\leq \sum_{1 \leq i_1 < \cdots i_k \leq m} |v_{i_1,\cdots,i_k}| \\ &\leq \binom{m}{k}^{\frac{1}{2}} \left(\sum_{1 \leq i_1 < \cdots < i_k \leq m} |v_{i_1,\cdots,i_k}|^2 \right)^{\frac{1}{2}} = \binom{m}{k}^{\frac{1}{2}} \|v\|\end{aligned}$$

and this yields the right-hand side inequality of (1.2.1).

More generally, if M^m is a Riemannian manifold and $\omega \in \Omega^k(M)$, the comass of ω is

$$\|\omega\| = \sup_{x \in M} \|\omega(x)\|.$$

This finally allows us to define the *mass* of a current $T \in \mathscr{D}_k(M)$ as

$$\mathbf{M}(T) = \sup\left\{T(\omega) :\ \|\omega\| \leq 1\right\}.$$

Remark 1.2.2. This is somewhat unfortunate to use the same letter for the ambient manifold and the mass, but this convention was adopted consistently in the papers which will be the object of study so we chose to keep this terminology to get the reader used to this abusive notation.

We now introduce the most useful classes of currents (our terminology for cycles differs from the one of Federer). To introduce the class of cycles we first need a definition for the admissible sets for cycles.

Definition 1.2.3. We say that a subset $A \subset M$ is a local Lipschitz neighbourhood retract if there exists a neighbourhood $U \subset M$ of A and a *locally* Lipschitzian map $f : U \to A$ such that $f_{|A} = \mathrm{Id}_A$.

Definition 1.2.4. A current $T \in \mathscr{D}_k(M)$ is called rectifiable if there exists a k-rectifiable subset $A \subset M$, a $\mathscr{H}^k \llcorner A$ integrable k-vector field η such that $T = (\mathscr{H}^k \llcorner A) \wedge \eta$ such that for $\mathscr{H}^k$ almost all $x \in A$,

$$\eta(x) \text{ is simple}, \qquad |\eta(x)| \in \mathbb{N} \setminus \{0\}$$
$$T_x^k(\mathscr{H}^k \llcorner A) \text{ is associated with } \eta(x).$$

We denote this space of currents by $\mathscr{R}_k(M)$ and we define

$$\mathscr{I}_k(M) = \mathscr{R}_k(M) \cap \{T : \partial T \in \mathscr{R}_{k-1}(M)\}$$

which are called integral currents.If $B \subset A \subset M$ are local Lipschitz neighbourhood retracts, we define the space of cycles $\mathscr{Z}_k(A, B)$ as

$$\mathscr{Z}_k(A, B) = \mathscr{I}_k(M) \cap \{T : \mathrm{supp}(T) \subset A,\ \mathrm{supp}(\partial T) \subset B\}.$$

If $B = \varnothing$, then we write more simply $\mathscr{Z}_k(A) = \mathscr{Z}_k(A, \varnothing)$.

The appropriate topology to study sequences of currents is the flat topology, first introduced by Whitney. One of it main features is that it makes the boundary operator ∂ continuous. For all $T \in \mathscr{I}_k(M)$,

$$\begin{aligned} \mathcal{F}(T) &= \sup\{T(\omega) : \|\omega\| \leq 1 \text{ and } \|d\omega\| \leq 1\}, \\ &= \inf\{\mathbf{M}(R) + \mathbf{M}(S) : T = R + \partial S,\ R \in \mathscr{R}_k(M), S \in \mathscr{R}_{k+1}(M)\}. \end{aligned}$$

The flat topology on $\mathscr{I}_k(M)$ is the topology induced by the distance $\mathcal{F}$ defined for all $S, T \in \mathscr{I}_k(M)$ by

$$\mathcal{F}(T_1, T_2) = \mathcal{F}(T_1 - T_2).$$

Remark 1.2.5. To my best knowledge (see [Fed69] 4.1.12), the equivalence of the two definitions relies on the axiom of choice (otherwise we would only have the inequality $\leq$).

One of the fundamental contributions of Federer and Fleming in [FF] is to show compactness *within* these special classes of currents.

Theorem 1.2.6. *Let $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{I}_k(M)$ a sequence of integral currents such that*

$$\limsup_{n\to\infty} \mathbf{M}(T_n) + \mathbf{M}(\partial T_n) < \infty,$$

then there exists an integral current $T \in \mathscr{I}_k(M)$ such that $\{T_n\}_{n\in\mathbb{N}}$ converges to T in the flat topology. In particular, we have

$$\mathbf{M}(T) \leq \liminf_{n\to\infty} \mathbf{M}(T_n).$$

1.3 Integral currents modulo ν

In the papers in study, it will be necessary to work with integral currents with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients, and we will recall below the necessary adaptations in the definitions of flat norm and mass.

For all $T \in \mathscr{R}_k(M)$, we define

$$\mathcal{F}_\nu(T) = \inf\{\mathbf{M}(R) + \mathbf{M}(S) : T = R + \partial S + \nu Q,\ R \in \mathscr{R}_k(M), S \in \mathscr{I}_{k+1}(M), Q \in \mathscr{R}_k(M)\}.$$

For all $S, T \in \mathscr{I}_k(M)$, we say that $S = T \mod \nu$ if $\mathcal{F}_\nu(S - T) = 0$, and for all $T \in \mathscr{R}_k(M)$, we denote by $[T]_\nu$ the class of T modulo this equivalence relation. Then we define

$$\mathscr{R}_k(M, \mathbb{Z}_\nu) = \{[T]_\nu : T \in \mathscr{R}_k(M)\}.$$

The boundary operator ∂ still makes sense modulo ν and by keeping the same notation we define

$$\mathscr{I}_k(M, \mathbb{Z}_\nu) = \mathscr{R}_k(M, \mathbb{Z}_\nu) \cap \{[T]_\nu : \text{for all } \; T \in [T]_\nu, \partial T \in \mathscr{R}_k(M)\} \tag{1.3.1}$$

The definitions are rather cumbersome and it would be more natural to define $\mathscr{I}_k(M)$ as

$$\mathscr{I}_k(M, \mathbb{Z}_\nu) = \mathscr{I}_k(M)/\nu\mathscr{I}_k(M), \tag{1.3.2}$$

but the completeness and compactness of $\mathscr{I}_\nu(M, \mathbb{Z}_\nu)$ were not known to hold with these definition. The equivalence of these two definition remained an unanswered question of Federer ([Fed69], 4.2.26) which was only recently solved by Robert Young in 2013 for $M = \mathbb{R}^n$ ([You18] his result is more general and also applies to flat chains modulo ν, a notion which will not be necessary in these lectures). However, when one wishes to localise, the definition (1.3.1), the alternative definition (1.3.2) need not be equivalent.

For an instructive example, we refer to the discussion in [Pau77] about the claimed counter-example of Federer ([Fed69], 4.2.26).

We also one define $\mathscr{Z}_k(M, \mathbb{Z}_\nu)$ in an analogous fashion.

Finally, for $T \in \mathscr{I}_k(M, \mathbb{Z}_\nu)$, we define

$$\mathbf{M}^\nu(T) = \inf_{\varepsilon > 0} \left\{\mathbf{M}(R) : R \in \mathscr{R}_k(M), \mathcal{F}^\nu([R]_\nu, T) < \varepsilon\right\}.$$

The main point of these definitions is to allow one to obtain the compactness theorem for integral chains modulo ν and notably the isoperimetric inequality, which will be one of the main ingredients of the geometrical constructions.

In the next sections, we will drop the ν indices for the flat norm and the mass as all papers in study deal with flat chains modulo 2.

1.4 Deformation theorem and isoperimetric inequality

All proofs of the isoperimetric inequality for currents available in the literature seem to make use in one form or another of the deformation theorem of Federer and Fleming. To introduce it, we first need to define polyhedral chains.

Definition 1.4.1. Let $U \subset \mathbb{R}^n$ be an open subset and $K \subset U$ be a fixed compact subset. We define $\mathscr{P}_{k,K}(U)$ as the additive subgroup of $\mathscr{D}_k(U)$ generated by the oriented simplexes Δ_k of dimension k such that the convex hull of Δ_k be included in K. The abelian group $\mathscr{P}_k(V)$ of *integral polyhedral chains* is the union of the groups $\mathscr{P}_{k,K}(U)$ for all compact subset $K \subset U$.

Remark 1.4.2. An alternative definition (and probably more natural one) of rectifiable currents is the following : let $V \subset \mathbb{R}^n$ be some open subset and L be a compact subset of U. We say that $T \in \mathscr{R}_{k,L}(V)$ if for all $\varepsilon > 0$, there exists an open subset U of $\mathbb{R}^k$, a compact subset K of U, and a Lipschitzian map $f : U \to V$ with $f(K) \subset L$ and an integral polyhedral chain $P \in \mathscr{P}_{k,K}(U)$ such that

$$\mathbf{M}(T - f_\# P) < \varepsilon.$$

Then $\mathscr{R}_k(V)$ is the union of all abelian groups $\mathscr{R}_{k,L}(V)$ corresponding to all compact subsets L of V.

The equivalence is given in ([Fed69], 4.1.28), and gives some intuition on the following theorem, which implies in particular that $\mathscr{R}_k(\mathbb{R}^n)$ is the $\mathcal{F}$ closure (resp. $\mathbf{M}$ closure if $k = n$) of $\mathscr{P}_k(\mathbb{R}^n)$.

Theorem 1.4.3 (Deformation theorem, [FF] 5.5, [Fed69] 4.2.9)**.** *Let $T \in \mathscr{I}_k(\mathbb{R}^n)$, fix some positive number $0 < \varepsilon < \infty$ and let $c_1 = c_1(n,k) = 2n^{2(k+1)}$. There exists an integral polyhedral chain $P \in \mathscr{P}_k(\mathbb{R}^n)$, and integral currents $Q \in \mathscr{I}_k(\mathbb{R}^n)$, $S \in \mathscr{I}_{k+1}(\mathbb{R}^n)$ satisfying the following properties:*

(1) $T = P + Q + \partial S$.

(2) $P \in \mathscr{P}_k(\mathbb{R}^n)$ *is an integral linear combination of disjoint k-dimensional cubes with side length 2ε.*

(3) $\mathrm{supp}\,(P) \cup \mathrm{supp}\,(S) \subset \mathrm{supp}\,(T) + \overline{B}(0, 2n\,\varepsilon)$ *and* $\mathrm{supp}\,(\partial P) \cup \mathrm{supp}\,(Q) \subset \mathrm{supp}\,(\partial T) + \overline{B}(0, 2n\,\varepsilon)$ *if* $k \geq 1$.

(4) *If $1 \leq k \leq n$, we have*

$$\begin{cases} \dfrac{\mathbf{M}(P)}{\varepsilon^k} \leq c_1 \left(\dfrac{\mathbf{M}(T)}{\varepsilon^k} + \dfrac{\mathbf{M}(\partial T)}{\varepsilon^{k-1}} \right), & \dfrac{\mathbf{M}(\partial P)}{\varepsilon^{k-1}} \leq c_1 \dfrac{\mathbf{M}(\partial T)}{\varepsilon^{k-1}}, \\ \dfrac{\mathbf{M}(Q)}{\varepsilon^k} \leq c_1 \dfrac{\mathbf{M}(\partial T)}{\varepsilon^{k-1}}, & \dfrac{\mathbf{M}(S)}{\varepsilon^{k+1}} \leq c_1 \dfrac{\mathbf{M}(T)}{\varepsilon^k}. \end{cases} \tag{1.4.1}$$

(5) *If $k = 0$, then $Q = 0$, $\mathbf{M}(P) \leq \mathbf{M}(T)$, and $\mathbf{M}(S) \leq c_1 \varepsilon \mathbf{M}(T)$.*

The second conclusion is the key feature will allow isoperimetric inequalities to holds for currents.

Corollary 1.4.4. *Let $T \in \mathscr{I}_k(\mathbb{R}^n)$ be such that $\partial T = 0$. Then there exists $S \in \mathscr{I}_{k+1}(\mathbb{R}^n)$ such that $\partial S = T$ and furthermore*

$$\mathbf{M}(S) \leq c_2 \mathbf{M}(T)^{\frac{k+1}{k}}.$$

where $c_2 = c_2(n,k) = c_1(n,k)^{\frac{k+1}{k}} = 2^{\frac{k+1}{k}} n^{\frac{2(k+1)^2}{k}}$.

Proof. Suppose that $T \neq 0$, otherwise there is nothing to prove. Let $\varepsilon > 0$ such that $c_1 \mathbf{M}(T) = \varepsilon^k$, and consider $P \in \mathscr{P}_k(\mathbb{R}^n)$, $Q \in \mathscr{I}_k(\mathbb{R}^n)$, $S \in \mathscr{I}_{k+1}(\mathbb{R}^n)$ given by theorem 1.4.3 such that

$$T = P + Q + \partial S. \tag{1.4.2}$$

As $\partial T = 0$, we have $Q = 0$. Furthermore, by (1.4.1), we have

$$\mathbf{M}(P) \leq c_1 \mathbf{M}(T) = \varepsilon^k.$$

However, by the second conclusion of theorem 1.4.3, $\mathbf{M}(P) \in (2\varepsilon)^k \mathbb{N}$, so $P = 0$.

Finally, we deduce from (1.4.2) that $T = \partial S$ and

$$\mathbf{M}(S) \leq \varepsilon \cdot c_1 \mathbf{M}(T) = \varepsilon^{k+1} = c_1^{\frac{k+1}{k}} \mathbf{M}(T)^{\frac{k+1}{k}}.$$

This concludes the proof of the corollary. ☐

A basic property required in the constructions of the papers of question is to ask for a localisation of support in the isoperimetric inequality, and this will only be possible by making a smallness assumption on the volume.

Theorem 1.4.5. *Let K be a compact set admitting a local Lipschitz retraction from a neighbourhood U of K. There exists constants $0 < \delta_1(n,K,k), c_3(n,K,k) < \infty$ such that for all integral cycle $T \in \mathscr{Z}_k(\mathbb{R}^n)$ such that $\mathrm{supp}\,(T) \subset K$, and $\mathbf{M}(T) \leq \delta_1$ there exists $S \in \mathscr{I}_k(\mathbb{R}^n)$ such that $\mathrm{supp}\,(S) \subset K$ and*

$$\mathbf{M}(S) \leq c_3 \mathbf{M}(T)^{\frac{k+1}{k}}.$$

Proof. Let $\alpha = \alpha(K) > 0$ such that $K + \overline{B}(0, \alpha) \subset U$, and let $f : U \to K$ a local Lipschitz retraction. Choose $\delta_1 = \dfrac{\alpha}{2n}$ and suppose that

$$c_1 \mathbf{M}(T) \leq \delta_1^k$$

Let $0 < \varepsilon < \delta_1$ such that $c_1 M(T) = \varepsilon^k$. By theorem 1.4.3 (applied for ε), there exists $P \in \mathscr{P}_k(\mathbb{R}^n)$, $Q \in \mathscr{I}_k(\mathbb{R}^n)$ and $S_0 \in \mathscr{I}_{k+1}(\mathbb{R}^n)$ such that

$$T = P + Q + \partial S_0.$$

As $\mathbf{M}(P) \leq c_1 \mathbf{M}(T) \leq \varepsilon^k$, and $\mathbf{M}(P)$ is an integral multiple of $(2\varepsilon)^k$, we have $P = 0$, and as $\partial T = 0$, we also obtain $Q = 0$. Furthermore, as $\varepsilon \leq \delta_1$, we have

$$\mathrm{supp}\,(S_0) \subset \mathrm{supp}\,(T) + \overline{B(0, 2n\,\varepsilon)} \subset K + \overline{B}(0, \alpha) \subset U.$$

As $\mathrm{supp}\,(T) \subset K$, and $f|K = \mathrm{Id}_K$, we have $f_\# T = T$, so with $S = f_\# S_0$, we obtain

$$T - \partial S = f_\# T - \partial\,(f_\# S_0) = f_\#\,(T - \partial S_0) = 0$$

and

$$\mathbf{M}(S) \leq \mathrm{Lip}(f)^{k+1} \mathbf{M}(S_0) \leq c_2 \mathrm{Lip}(f)^{k+1} \mathbf{M}(T)^{\frac{k+1}{k}}$$

so we get the conclusion with $\delta_1 = \dfrac{\alpha(K)}{2n}$ and $c_3 = c_2(n,k)\mathrm{Lip}(f)^{k+1}$. □

All these theorems extend to integral currents modulo ν, and to state a general result, we introduce the following definition.

Definition 1.4.6. We say that a group G is admissible if $G = \mathbb{Z}$ or $G = \mathbb{Z}_\nu$ for some integer $\nu \geq 2$. We denote by $\mathscr{I}_k(U, G)$ and $\mathscr{Z}_k(A, B, G)$ the associated sets of currents. We denote $\mathbf{M} = \mathbf{M}^\nu$ and $\mathcal{F} = \mathcal{F}^\nu$ whenever the context is clear.

Remark 1.4.7. This is a sub-class if the groups Almgren calls *admissible* in his set of unpublished notes.

Theorem 1.4.8. *Let M^m be a compact smooth Riemannian manifold and G be an admissible group. There exists a positive numbers $\delta_1 = \delta_1(M), c_3 = c_3(M)$ depending only on M^m with the following property. For all $1 \leq k \leq m$ and all cycle $T \in \mathscr{Z}_k(M, G)$ such that $\mathbf{M}(T) \leq \delta_1$, there exists $S \in \mathscr{I}_{k+1}(M, G)$ such that $\partial S = T$ and*

$$\mathbf{M}(S) \leq c_3 \mathbf{M}(T)^{\frac{k+1}{k}}.$$

Proof. By Nash's isometric embedding theorem, it suffices to check that any C^k compact sub-manifold $M^m \subset \mathbb{R}^n$ is a Lipschitz retraction of some open neighbourhood. This is a classical theorem of Whitney ([Whi57]) that C^k sub-manifolds of $\mathbb{R}^n$ are C^k retract, so we are done (the converse holds and is due to Federer - 3.1.20 [Fed69]). □

Definition 1.4.9. An element $S \in \mathscr{I}_{k+1}(M)$ as in theorem 1.4.8 such that $\partial S = T$ and $\mathbf{M}(S) \leq c_3 \mathbf{M}(T)^{\frac{k+1}{k}}$ is called an $\mathbf{M}$-isoperimetric choice.

The next two corollaries are due to Almgren ([Alm62], corollaries (1.13) and (1.14)).

Corollary 1.4.10. *Let M^m be a compact Riemannian manifold and G be an admissible group. For all $p \in \mathbb{N}$, there exists $\delta_2(M, p) > 0$ such that for all $1 \leq k \leq m$ and for all $T_1, \cdots, T_p \in \mathscr{I}_k(M)$ such that*

$$T_1 + \cdots + T_p \in \mathscr{Z}_k(M), \qquad \sup_{1 \leq j \leq p} \mathbf{M}(T_j) \leq \delta_2(M, p),$$

there exists an $\mathbf{M}$-isoperimetric choice $S \in \mathscr{I}_{k+1}(M)$ of $T_1 + \cdots + T_p$ such that

$$\mathbf{M}(S) \leq \sup_{1 \leq j \leq p} \mathbf{M}(T_j).$$

Proof. Let $\delta_2 \leq \dfrac{\delta_1(M)}{p}$ be some positive real number to be fixed later and let $S \in \mathscr{I}_{k+1}(M)$ be an $\mathbf{M}$-isoperimetric choice of $T = T_1 + \cdots T_p$. Then by the triangle inequality

$$\mathbf{M}(S) \leq c_3 \mathbf{M}(T)^{\frac{k+1}{k}} \leq c_3 \left(\sum_{j=1}^{p} \mathbf{M}(T_j)\right)^{\frac{k+1}{k}} \leq c_3 p^{\frac{k+1}{k}} \left(\sup_{1\leq j\leq m} \mathbf{M}(T_j)\right)^{\frac{k+1}{k}} \leq c_3\, p^{\frac{k+1}{k}} \delta_2^{\frac{1}{k}} \sup_{1\leq j\leq m} \mathbf{M}(T_j).$$

By choosing $\delta_2(M,p) = \min\left\{\dfrac{\delta_1(M)}{p}, \dfrac{1}{c_3(M)^k p^{k+1}}\right\}$ this finishes the proof. $\square$

The next corollary is more important as it will be the basic ingredient in the homotopy arguments as we take families continuous in the flat topology $\mathcal{F}$ (but not in the mass topology $\mathbf{M}$).

Corollary 1.4.11. *Let M^m be a compact Riemannian manifold and G be an admissible group. There exists a constant $\delta_3 = \delta_3(M) > 0$ such that for all cycle $T \in \mathscr{Z}_k(M)$ such that $\mathcal{F}(T) \leq \delta_3$, there exists $S \in \mathscr{I}_{k+1}(M)$ such that $\partial S = T$ and*

$$\mathbf{M}(S) = \mathcal{F}(T). \tag{1.4.3}$$

Proof. Fix $\delta_3 \leq \dfrac{\delta_1(M)}{2}$ be a positive real number to be chosen later (here δ_1 is the constant given by the theorem 1.4.8), assume that $\mathcal{F}(T) \leq \delta_3$ and let $0 < \eta \leq 1$ be any positive real number. By the definition of the flat norm, the set

$$\mathscr{A}_\eta = \mathscr{I}_k(M) \times \mathscr{I}_{k+1}(M) \cap \{(R,S) : T = R + \partial S, \text{ and } \mathbf{M}(R) + \mathbf{M}(S) \leq (1+\eta)\mathcal{F}(T)\}$$

is non empty for all $0 < \eta \leq 1$, and for all $(R,S) \in \mathscr{A}_\eta$, we have

$$T = R + \partial S, \quad \text{and } \mathbf{M}(R) + \mathbf{M}(S) \leq (1+\eta)\mathcal{F}(T) \leq 2\delta_3 \leq \delta_1,$$

so there exists by theorem 1.4.8 some integral current $Q \in \mathscr{I}_{k+1}(M)$ such that $\partial Q = R$ and

$$\mathbf{M}(Q) \leq c_3 \mathbf{M}(R)^{\frac{k+1}{k}} \leq c_3((1+\eta)\mathcal{F}(T))^{\frac{1}{k}}\mathbf{M}(R) \leq c_3(2\delta_3)^{\frac{1}{k}}\mathbf{M}(R) \leq \mathbf{M}(R),$$

for $\delta_3 \leq \dfrac{1}{2c_3^k}$. Therefore, we obtain $T = \partial(Q+S)$ with

$$\mathbf{M}(Q+S) \leq \mathbf{M}(R) + \mathbf{M}(S) \leq (1+\eta)\mathcal{F}(T).$$

By compactness and by letting $\eta \to 0$, we see that there exists $S \in \mathscr{I}_{k+1}(M)$ such that $\partial S = T$ and

$$\mathbf{M}(S) \leq \mathcal{F}(T).$$

which trivially implies by the definition of the flat norm that $\mathbf{M}(S) = \mathcal{F}(T)$. Therefore, choosing $\delta_3 = \min\left\{\dfrac{\delta_1(M)}{2}, \dfrac{1}{2c_3(M)^k}\right\}$ yields the conclusion. $\square$

1.5 Homotopy groups of the space of cycles

We fix some admissible group G in this section and some compact Riemannian manifold (M^m, g), that we suppose isometrically embedded in some euclidean space by Nash's embedding theorem ([Nas56]).

As was noticed by Federer, any cycle $T \in \mathscr{Z}_k(M,G)$ induces a well-defined element $[T] \in H_k(M,G)$. Indeed, such cycle is the limit in the flat topology of a sequence of polyhedral chains by the deformation theorem 1.4.3, and as we can represent any element of $H_k(M,G)$ by polyhedral chains we obtain an isomorphism

$$\Lambda_0 : \pi_0(\mathscr{Z}_k(M,G), \mathcal{F}) \to H_k(M,G). \tag{1.5.1}$$

This prompted Federer to propose the following theorem as a PhD subject for Almgren.

Theorem 1.5.1. *For all $k \geq 1$, there exists a canonical isomorphism*

$$\Lambda_l : \pi_l(\mathscr{Z}_k(M,\mathbb{Z}),\mathcal{F}) \to H_{k+l}(M,\mathbb{Z}).$$

The notation $\pi_l(\mathscr{Z}_k(M,\mathbb{Z}_2),\mathcal{F})$ means that we take equivalence classes of map $f : (I^l,\partial I^l) \to \mathscr{Z}_k(M,\mathscr{Z}_2)$ continuous in the flat topology.

It was pointed out by Larry Guth in [Gut08] that the extension to $\mathbb{Z}_\nu$ coefficients is not available in a published reference. Nevertheless, one can see a proof of this extension in the unpublished mimeographed notes of Almgren ([Alm65] section 13), and we will assume in this set of notes that the following more general result holds.

Remark 1.5.2. A new and simpler proof of the injectivity of the isomorphism for $\mathbb{Z}_2$ cycles in the codimension 1 case by Marques and Neves (using the constancy theorem, see [MN18]) appeared after these lectures were given. However, to our knowledge, [Alm65] remains the only reference (and unpublished) for the general case.

Theorem 1.5.3. *Let (M^m,g) be a compact Riemannian manifold and let G be an admissible group. For all $l \geq 1$, there exists a canonical isomorphism*

$$\Lambda_l : \pi_l(\mathscr{Z}_k(M,G),\mathcal{F}) \to H_{k+l}(M,G).$$

If this theorem holds, it will imply that $\mathscr{Z}_k(M^m,G)$ is an Eilenberg-MacLane space $K(G,m-k)$.

Remarks 1.5.4. We recall that an Eilenberg-MacLane space $K(G,n)$ is a topological space X whose homotopy groups are all trivial, except for $\pi_n(X)$ which is isomorphic to G. One of the basic properties of these spaces is that they do exist for any (necessarily abelian) group G if $n > 1$ in the category of CW-complexes and are unique up to weak homotopy equivalence. It was showed by Serre (in [Ser53]) that one can compute explicitly the cohomology ring $H^*(K(\mathbb{Z}_2,n),\mathbb{Z}_2)$ for any $n \geq 1$.

In particular, we deduce that

$$\pi_{m-k}(\mathscr{Z}_k(M^m,G),\mathcal{F}) \simeq H_m(M^m,G) \simeq G,$$

and as codimension 1 cycles will be the main object in study in the next chapters, we stress out the following corollary.

Corollary 1.5.5. *Let (M^{n+1},g) be a compact Riemannian manifold and G be an admissible group. Then*

$$\pi_1(\mathscr{Z}_n(M^{n+1},G),\mathcal{F}) \simeq G.$$

To make the notation lighter, we will write in the following $=$ instead of $\simeq$ for the equality of homotopy groups.

We recall that the infinite projective space $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}_2,1)$, that is $\pi_1(\mathbb{RP}^\infty) = \mathbb{Z}_2$ and $\pi_i(\mathbb{RP}^\infty) = 0$ for $i \neq 1$. This observation will prove useful when we will introduce the p-sweep-outs.

We will only check directly the surjectivity part of the case $l = 1$ of theorem 1.5.3. The injectivity is the main part of [Alm62] (see section 13 of [Alm65] for the adaptation to $\mathbb{Z}_\nu$ coefficients).

Proof. (of the case $l = 1$ theorem 1.5.3) The proof of surjectivity uses only the isoperimetric inequality, so is valid for any admissible group thanks of corollary 1.4.8.

Step 1 : Construction of an homomorphism $\pi_1(\mathscr{Z}_k(M,G),\mathcal{F}) \to H_{k+1}(M,G)$.

Let $\varphi : [0,1] \to \mathscr{Z}_k(M,G)$ a continuous map in the flat topology such that $\varphi(0) = \varphi(1)$. Introduce for all $j \geq 1$ the cube complex $I(1,j)$ on $I = [0,1]$ whose 0 cells are $[i \cdot 3^{-j}] = [a_i]$ for $0 \leq i \leq 3^j$ and 1-cells are

$$[0,3^{-j}],[3^{-j},2\cdot 3^{-j}],\cdots,[1-3^{-j},3^{-j}].$$

Let j large enough such that

$$\mathcal{F}(\varphi(x),\varphi(y)) \leq \frac{\delta_3}{4}, \quad \text{for all } x, y \in [a_i, a_{i+1}],\ 0 \leq i \leq 3^j - 1, \tag{1.5.2}$$

if $\delta_3 = \delta_3(M)$ is the constant given by corollary 1.4.11 In particular, there exists by corollary 1.4.11 some integral current $S_i \in \mathscr{I}_{k+1}(M,G)$ such that

$$\partial S_i = \varphi(a_{i+1}) - \varphi(a_i) \quad \text{and} \quad \mathbf{M}(S_i) = \mathcal{F}(\varphi(a_i), \varphi(a_{i+1})) \leq \frac{\delta_3}{4}. \tag{1.5.3}$$

As $\varphi(1) - \varphi(0) = 0$, we have

$$\partial\left(\sum_{i=0}^{3^j-1} S_i\right) = \sum_{i=0}^{3^j-1}\Big(\varphi(a_{i+1}) - \varphi(a_i)\Big) = \varphi(1) - \varphi(0) = 0$$

and we define

$$\Lambda_1(\varphi) = \Lambda_0\left(\sum_{i=0}^{3^j-1} S_i\right) \in H_{k+1}(M,G).$$

Step 2 : The homomorphism $\Lambda_1 : \pi_1(\mathscr{Z}_k(M,G),\mathcal{F}) \to H_{k+1}(M,G)$ is well-defined.

To make sure that this is well-defined, let $T_i \in \mathscr{I}_{k+1}(M,G)$ such that

$$\partial T_i = \partial S_i = \varphi(a_{i+1}) - \varphi(a_i)$$

for all $0 \leq i \leq 3^j - 1$. Then $\mathcal{F}(S_i - T_i) \leq \dfrac{\delta_3}{2}$ for all $i = 0, \cdots, 3^j - 1$, so there exists $R_i \in \mathscr{I}_{k+2}(M,G)$ such that

$$\partial R_i = S_i - T_i.$$

Therefore, we obtain

$$\sum_{i=0}^{3^j-1} S_i = \sum_{i=0}^{3^j-1} T_i + \partial\left(\sum_{i=0}^{3^j-1} R_i\right),$$

so that

$$\Lambda_0\left(\sum_{i=0}^{3^{j-1}} S_i\right) = \Lambda_0\left(\sum_{i=0}^{3^j-1} T_i\right) \in H_{k+1}(M,G).$$

Step 3 : The homomorphism $\Lambda_1 : \pi_1(\mathscr{Z}_k(M,G),\mathcal{F}) \to H_{k+1}(M,G)$ does not depend on the subdivision of the unit interval.

Wo check that this construction is independent of all large enough $j \geq 1$. For all $0 \leq i \leq 3^j - 1$, we divide the interval $[a_i, a_{i+1}]$ into three equal pieces $[a_i^1, a_{i+1}^1]$, $[a_i^2, a_{i+1}^2]$ and $[a_i^3, a_{i+1}^3]$ and by (1.5.2) and corollary 1.4.11, we obtain three integral currents S_i^1, $S_i^2, S_i^3 \in \mathscr{I}_{k+1}(M)$ such that

$$\partial S_i^l = \varphi(a_{i+1}^l) - \varphi(a_i^l)$$

and $\mathbf{M}(S_i^l) = \mathcal{F}(\varphi(a_{i+1}^l) - \varphi(a_i^l)) \leq \dfrac{\delta_3}{4}$. Therefore, we obtain by (1.5.3)

$$\mathbf{M}\Big(S_i - (S_i^1 + S_i^2 + S_i^3)\Big) \leq \delta_3,$$

which implies that there exist $R_i \in \mathscr{I}_{k+2}(M)$ such that

$$\partial R_i = S_i - (S_i^1 + S_i^2 + S_i^3).$$

This implies that

$$\Lambda_0\left(\sum_{i=0}^{3^j-1} S_i\right) = \Lambda_0\left(\sum_{i=0}^{3^j-1}\left(S_i^1 + S_i^2 + S_i^3\right)\right) \in H_{k+1}(M,G)$$

and concludes the proof of the third step.

Step 4: The homomorphism $\Lambda_1 : \pi_1(\mathscr{Z}_k(M,G),\mathcal{F}) \to G$ is surjective.

It suffices to construct a map $\Phi_1 : H_{k+1}(M,G) \to \pi_1(\mathscr{Z}_k(M,G),\mathcal{F})$ such that $\Psi_1 \circ \Lambda_1 = \mathrm{Id}$. Here we suppose that (M^m, g) is isometrically embedded in some $\mathbb{R}^n$ and we consider a neighbourhood U of M admitting a Lipschitz retract $f : U \to M$. Assuming without loss of generality that $M \subset [0,1]^n$, we can represent each class $\tau \in H_{k+1}(M,G)$ by an integral current $T = \Lambda_0^{-1}(\tau) \in \mathscr{Z}_{k+1}(U,G)$ (if $\Lambda_0^{-1} : \pi_0(\mathscr{Z}_{k+1}(M,G),\mathcal{F}) \to H_{k+1}(M,G)$ is the Federer isomorphism (1.5.1)) and we consider $\Psi_0 : [0,1] \to \mathscr{Z}_k(U,G)$ defined by

$$\Psi_0(t) = \partial(T \llcorner \{x_n < t\}).$$

If we perturb slightly T so that no face is parallel to the a face of unit cube (actually, the top face suffices), then $\Psi_0(t)$ has finite mass and as $T \llcorner \{x_n < t\} \in \mathscr{R}_k(U)$, we know by the Boundary Rectifiability Theorem (to be proved in 3.2.3) that $\Psi_0(t) \in \mathscr{R}_k(U)$ and we obtain trivially $\Psi_0(t) \in \mathscr{Z}_k(U,G)$.

Then for all $0 < s_1 < t < s_2$, we have

$$\begin{aligned}
\mathcal{F}(\Psi_0(s_1),\Psi_0(t)) &\leq \mathbf{M}(T \llcorner \{s_1 < x_n < t\}) \underset{s_1 \to t}{\longrightarrow} 0 \\
\mathcal{F}(\Psi_0(s_2),\Psi_0(t)) &\leq \mathbf{M}(T \llcorner \{t < x_n < s_2\}) \underset{s_2 \to t}{\longrightarrow} 0
\end{aligned}$$

so Ψ_0 is continuous in the flat topology. Then we define $\Psi_1(\tau) = [f_\# \circ \Psi_0 : [0,1] \to \mathscr{Z}_k(M,G)]$. As $f_\# \circ \Psi_0$ is continuous in the flat topology, and this map does not depend on the representative $T = \Lambda_0^{-1}(\tau)$ as it chosen by the fixed isomorphism $\Lambda_0 : \pi_0(\mathscr{Z}_k(U,G),\mathcal{F}) \to H_{k+1}(U,G)$, we get a well-defined homomorphism $H_{k+1}(M,G) \to \pi_1(\mathscr{Z}_k(M),G),\mathcal{F})$. Then we still have Ψ_1 continuous in the flat topology, and by construction we have

$$\Lambda_1 \circ \Psi_1(\tau) = \Lambda_0(T) = \tau \in H_{k+1}(M,G)$$

which concludes the proof of the theorem.

Step 5 : The homomorphism $\Lambda_1 : \pi_1(\mathscr{Z}_k(M,G),\mathcal{F}) \to G$ is injective. For $G = \mathbb{Z}$ this is the main content of [Alm62], and for $G = \mathbb{Z}_\nu$ the necessarily replacements are described in [Alm65], section 13. ☐

Chapter 2

Almgren-Pitts theory

We fix a compact Riemannian manifold (M^m, g) and an admissible group G for this whole chapter.

2.1 Introduction to varifolds

Definition 2.1.1. A Radon measure on a topological space X is a Borel regular measure locally finite.

Definition 2.1.2. Let $1 \leq k \leq n$. We note $G(n,k)$ the space of non-oriented k dimensional subspaces of $\mathbb{R}^n$. If $M \subset \mathbb{R}^n$ is a smooth submanifold, and $A \subset \mathbb{R}^n$ is any subset of $\mathbb{R}^n$ we define

$$\mathscr{G}_k(A) = (M \cap A) \times G(n,k) \cap \{(x,S) : S \subset T_x M\}\,.$$

A k varifold on M is simply a Radon measure on $\mathscr{G}_k(M)$. We note this set $V_k(M)$, and equip it with the topology of weak star convergence (see lecture 2). Whenever $V \in V_k(M)$, we associate a Radon measure $\|V\|$ on M by

$$\|V\|(A) = V(\mathscr{G}_k(A)) = \int_{\mathscr{G}_k(M)} \mathbf{1}_{\{(x,S)\,:\, x \in A\}} dV(x,S).$$

If $E \subset \mathbb{R}^n$ is a k-rectifiable set, we associate a varifold $|E| \in V_k(\mathbb{R}^n)$ such that for all $A \subset \mathscr{G}_k(\mathbb{R}^n)$,

$$|E|(A) = \mathscr{H}^k \llcorner E\,\{x : (x, T_x E) \in A\}\,,$$

and as the map

$$T.E : \mathbb{R}^n \to G(n,k)$$

is $\mathscr{H}^k \llcorner E$ measurable, if $|E|$ is simply the image measure by the application $(\mathrm{Id}, T.E) : \mathbb{R}^n \to \mathscr{G}_k(\mathbb{R}^n)$ of $\mathscr{H}^k \llcorner E$.

In this special case, we note that $\|\,|E|\,\| = \mathscr{H}^k \llcorner E$. If $\theta : \mathscr{G}_k(\mathbb{R}^n) \to \mathbb{R}_+$, and $\theta \in \mathrm{L}^1(\mathscr{G}_k(\mathbb{R}^n), |E|)$, we also have $|E| \wedge \theta \in V_k(\mathbb{R}^n)$, and the space $RV_k(\mathbb{R}^n)$ of *rectifiable varifolds* is the set of convergent sums of such varifolds, i.e. $V \in RV_k(\mathbb{R}^n)$ if there exists a sequence of k-rectifiable sets $\{E_j\}_{j \in \mathbb{N}}$ and $\theta_j : \mathscr{G}_k(\mathbb{R}^n) \to \mathbb{R}_+$ such that

$$V = \sum_{j=1}^{\infty} |E_j| \wedge \theta_j.$$

If we take instead $\theta : \mathscr{G}_k(\mathbb{R}^n) \to \mathbb{N}$, we obtain the set of *integral varifolds*, denoted by $IV_k(\mathbb{R}^n)$.

Finally, if G is either $\mathbb{Z}$ or $\mathbb{Z}_\nu$ for an integer $\nu \geq 2$ and $T \in \mathscr{I}_k(\mathbb{R}^n, G)$ is a G-valued rectifiable current (see [Fed69], 4.1.22-4.1.31), as $T = (\mathscr{H}^k \llcorner E_T) \wedge \eta_T$ with $E_T \subset \mathbb{R}^n$ a k-rectifiable set, η_T the orienting k-plan, and $\theta_T = |\eta_T| \in \mathrm{L}^1(E_T, \mathscr{H}^k)$, we define

$$|T| \in IV_k(\mathbb{R}^n)$$

by

$$|T| = |E_T| \wedge \theta_T.$$

Now, we define sets of varifolds in a submanifold $M \subset \mathbb{R}^n$ as follows.

$$\begin{aligned}
\mathscr{RV}_k(M) &= RV_k(\mathbb{R}^n) \cap \{V : \operatorname{supp} \|V\| \subset M\} \\
\mathscr{IV}_k(M) &= IV_k(\mathbb{R}^n) \cap \{V : \operatorname{supp} \|V\| \subset M\} \\
\mathscr{V}_k(M) &= \overline{\mathscr{R}_k(M)} \quad \text{for the weak } * \text{ topology.}
\end{aligned}$$

and if $U \subset U$ is any open set, we define

$$\begin{aligned}
\mathscr{RV}_k(M, U) &= \mathscr{V}_k(M) \cap \{V : V \llcorner \mathscr{G}_k(U) \in \mathscr{RV}_k(M)\} \\
\mathscr{IV}_k(M, U) &= \mathscr{V}_k(M) \cap \{V : V \llcorner \mathscr{G}_k(U) \in \mathscr{IV}_k(M)\}
\end{aligned}$$

2.1.1 Mapping varifolds

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ a C^1 proper map, and $V \in V_k(\mathbb{R}^n)$. We define $f_\# V \in V_k(\mathbb{R}^n)$ by the condition

$$f_\#(A) = \int_{\mathscr{G}_k(F^{-1}(A))} J_k^S f(x) dV(x, S)$$

where $J_k^S f(x) = |\wedge_k Df(x) \circ S|$ is the Jacobian associated to the k-plan $S \in G(n,k)$, and $F : \mathbb{R}^n \to \mathscr{G}_k(\mathbb{R}^n)$ is defined by

$$F(x, S) = (f(x), Df(x) \cdot S), \quad \forall (x, S) \in \mathscr{G}_k(\mathbb{R}^n).$$

This requirement is made to ensure compatibility with area formula and mapping of currents. The function $f_\# : V_k(\mathbb{R}^n) \to V_k(\mathbb{R}^n)$ is continuous for the weak $*$ topology. Furthermore, if $\iota : M \to \mathbb{R}^n$ is the inclusion map, then we have a natural identification

$$\iota_\# : V_k(M) \to \mathscr{V}_k(M) \subset \mathbb{R}^n$$

and we shall in the following use the latter definition of varifolds in a submanifold, as $V_k(M)$ is not a subset of $V_k(\mathbb{R}^n)$.

2.1.2 Variations, metrics

Let $U \subset M$ an open subset, $I \subset \mathbb{R}$ an open interval containing 0, and $\varphi : I \times M \to M$ a 1-parameter group of diffeomorphism, such that $\varphi^0 = \mathrm{Id}_M$. If

$$\frac{d}{dt} \varphi^t_{|t=0} = f \in \Gamma(TU),$$

we define for $\nu \in \mathbb{N}$, the ν-variation of V by

$$\delta^\nu V(f) = \frac{d^\nu}{dt^\nu} \left(\|\varphi^t_\# V\|(M) \right)_{|t=0}.$$

and if $\nu = 1$, we simply note $\delta^1 = \delta$.

Definition 2.1.3. We say that $V \in \mathscr{V}_k(M)$ is stationary in U if $\delta V = 0$, *i.e.* for all $f \in \Gamma(TU)$, $\delta V(f) = 0$. We say that V is stable in U if it is stationary and if for all $f \in \Gamma(TU)$, $\delta^2 V(f) \geq 0$.

It is elementary to show that these definitions coincide with the definition of minimal surface and stable minimal surface in the smooth setting. If $g \in \mathscr{D}(M, \mathbb{R}^n)$, we extend the definition of ν-variation by defining

$$\delta^\nu V(f) = \delta^\nu V(\overline{f})$$

if $\overline{f} \in \Gamma(\mathbb{R}^n)$ is any extension of g. Note that in the right-hand member, we see $V \in V_k(M) \subset V_k(\mathbb{R}^n)$. As we easily check (see [All72], 2.4) that for all $f \in \Gamma(TM)$,

$$\delta V(f) = \int_{\mathscr{G}_k(M)} \mathrm{div}_S\, g(x)\, dV(x,S)$$

therefore, for all vector field $f \in \mathscr{D}(M,\mathbb{R}^n)$, if H is the mean curvature tensor, an elementary computation ([All72], 2.5, 4.4) shows that

$$\delta V(f) = \int_{\mathscr{G}_k(M)} \mathrm{div}_S\, f(x)\, dV(x,S) = \int_{\mathscr{G}_k(M)} \mathrm{div}_S\, f^{\top}(x)\, dV(x,S) - \int_{\mathscr{G}_k(M)} f^{\perp}(x)\cdot H(M,x,S)\, dV(x,S)$$

so a varifold $V \in \mathscr{V}_k(M)$ is stationary if and only if for all $f \in \mathscr{D}(M,\mathbb{R}^n)$. We have

$$\int_{\mathscr{G}_k(M)} \mathrm{div}_S\, f(x)\, dV(x,S) = -\int_{\mathscr{G}_k(M)} f(x)\cdot H(M,x,S)\, dV(x,S)$$

where $\top : T\mathbb{R}^n \to TM$ (resp. $\perp\colon T\mathbb{R}^n \to (TM)^{\perp}$) is the orthogonal projection.

Let us see how to derive this formula for the first variation. We have

$$J_k^S f(x) = \sqrt{\det\left((D\varphi^t(x)|S)^*\cdot(D\varphi^t(x)|S)\right)}$$

Let $(e_1,\cdots,e_k)$ orthonormal vectors spanning some $S \in G(n,k)$. As $\varphi^0 = \mathrm{Id}$, we have for all $1 \leq i \leq k$

$$D\varphi^t(x)\cdot e_i + t\, Df(x)\cdot e_i + O(t^2)$$

so that

$$(D\varphi^t(x)|S)^*\cdot(D\varphi^t(x)|S) = (a_{i,j}(t))_{1\leq i,j\leq k}$$

with

$$a_{i,j}(t) = \left(e_i + t\, Df(x)\cdot e_i + O(t^2)\right)\cdot\left(e_j + t\, Df(x)\cdot e_j + O(t^2)\right) = \delta_{i,j} + t\left(D_{e_i}f(x)\cdot e_j + D_{e_j}f(x)\cdot e_j\right) + O(t^2)$$

and as for all $k\times k$ matrix A, there trivially holds

$$\det(\mathrm{Id} + tA) = 1 + t\,\mathrm{Tr}(A) + O(t^2),$$

we deduce as $\sqrt{1+x} = 1 + \dfrac{x}{2} + O(x^2)$ that

$$J_k^S\varphi^t(x) = 1 + t\sum_{i=1}^k D_{e_i}f(x)\cdot e_i + O(t^2) = 1 + t\,\mathrm{div}_S\, f(x) + O(t^2)$$

so that the claimed formula holds true.

Definition 2.1.4. If $U \subset M$ is open, $K \subset M$ is compact, we note $\mathcal{F}_K$ the flat norm for currents and $\mathbf{M}$ the mass. We define a third metric on varifold, the distance function $\mathbf{F} : V_k(\mathbb{R}^n)\times V_k(\mathbb{R}^n) \to \mathbb{R}_+$, by

$$\mathbf{F}(V_1,V_2) = \sup\left\{V_1(f) - V_2(f) : f \in W^{1,\infty}(\mathscr{G}_k(\mathbb{R}^n))\cap C_c(\mathscr{G}_k(\mathbb{R}^n))\ \ \|f\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^n)} \leq 1\right\}$$

for all $V_1, V_2 \in V_k(\mathbb{R}^n)$. For all Borelian $A \subset \mathbb{R}^n$, we define

$$\mathbf{F}_A(V_1,V_2) = \mathbf{F}(V_1 \llcorner G_k(A), V_2 \llcorner G_k(A)).$$

One easily check that for all $0 < C < \infty$, the weak $*$ topology coincides with the d metric on

$$V_k(\mathbb{R}^n)\cap\{V : \|V\|(\mathbb{R}^n) \leq C\}$$

and for all $T_1, T_2 \in \mathscr{R}_k(\mathbb{R}^n, G)$, we have

$$\mathbf{F}(|T_1|, |T_2|) \leq \mathbf{M}(T_1 - T_2).$$

On the space of integral currents $\mathscr{I}_k(M)$ we define the $\mathbf{F}$-metric by

$$\mathbf{F}(T_1, T_2) = \mathcal{F}(T_1, T_2) + d(|T_1|, |T_2|),$$

and for all Borelian set $A \subset \mathbb{R}^n$ the restriction

$$\mathbf{F}_A(T_1, T_2) = \mathbf{F}(|T_1| \llcorner \mathscr{G}_k(A), |T_2| \llcorner \mathscr{G}_k(A)).$$

Remark 2.1.5. The $\mathbf{F}$-metric has for principal advantage over the flat norm $\mathcal{F}$ to make the mass $\mathbf{M}$ continuous in the $\mathbf{F}$ distance. We have the following inclusions for any two rectifiable currents T_1 and T_2

$$\mathcal{F}(T_1, T_2) \leq \mathbf{F}(T_1, T_2) \leq 2\,\mathbf{M}(T_1, T_2).$$

Remark 2.1.6. This is unfortunate that the same notation $\mathbf{F}$ is used for two metrics, but this bad notation was adopted consistently in the subsequent literature.

2.1.3 Compactness

The first theorem is a very useful corollary of Allard's compactness theorem.

Theorem 2.1.7 (Allard's compactness theorem). *Let $0 \leq C < \infty$, and $U \subset M$ an open bounded subset. Then*

$$\mathscr{R}\mathscr{V}_k(M, U) \cap \{V : (\|V\| + \|\delta V\|)\,(M) \leq C\}\,.$$

and

$$\mathscr{I}\mathscr{V}_k(M, U) \cap \{V : (\|V\| + \|\delta V\|)\,(M) \leq C\}\,.$$

are compact in the weak $*$ topology.

2.2 Terminology and definitions of discrete homotopies

This section follows mostly the exposition of [Riv15].

We let I denote the unit interval $[0,1]$ and we define for all $j \geq 1$ a cube complex $I(1,j)$ on $I = [0,1]$ whose 0 cells are $[i \cdot 3^{-j}] = [a_i]$ for $0 \leq i \leq 3^j$ and 1-cells are

$$[0, 3^{-j}], [3^{-j}, 2 \cdot 3^{-j}], \cdots, [1 - 3^{-j}, 3^{-j}].$$

More generally, $I(n,j) = I(1,j) \otimes \cdots \otimes I(1,j)$ is the cube complex of the n-dimensional cube $I^n = [0,1]^n$ whose p-cells are the elements $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n \in I(n,j)$ such that

$$\sum_{i=1}^{n} \dim(\alpha_i) = p.$$

Furthermore, we denote for $0 \leq p \leq n$ by $I(n,j)_q \subset I(n,j)$ the sub-complex p-cells. The most important in the construction is probably $I(n,j)_0$, which identifies to the vertices of $I(n,j)$. Finally, we write $\partial I(n,j)_0$ for the intersection between $I(n,j)_0$ and $\partial[0,1]^n$. This complex comes with a natural boundary operator $\partial : I(n,j) \to I(n,j)$ such that

$$\begin{cases} \partial([a]) = 0 & \text{for all 0-cell } [a] \in I(n,j)_0 \\ \partial([a,b]) = [b] - [a] & \text{for all 1-cell } [a,b] \in I(n,j)_1 \\ \displaystyle\partial\alpha = \sum_{i=1}^{n} (-1)^{\sigma(i)} \alpha_1 \otimes \cdots \otimes \partial\alpha_i \otimes \cdots \otimes \alpha_n & \end{cases}$$

where

$$\alpha(i) = \sum_{p<i} \dim(\alpha_p).$$

The distance between two vertices $x = (x_1 3^{-j}, \cdots, x_n 3^{-j})$ and $y = (y_1 3^{-j}, \cdots, y_n 3^{-j})$ of $I(n,j)$ (that is, between two elements x and y of $I(n,j)_0$) is given by

$$\mathbf{d}_j(x,y) = \sum_{i=1}^{n} |x_i - y_i| = 3^j |x-y|_1$$

if $|x-y|_1$ is the 1-norm on $\mathbb{R}^n$. If $\varphi_j : I(n,j)_0 \to \mathscr{Z}_k(M,G)$ is any map, we define the fineness of φ_j by

$$\mathbf{f}(\varphi_j) = \sup\left\{ \frac{\mathbf{M}(\varphi_j(x) - \varphi_j(y))}{\mathbf{d}_j(x,y)}, x, y \in I(n,j)_0 \text{ and } x \neq y \right\}$$

Remark 2.2.1. This should really be seen of a norm for continuous and *not* for Lipschitz functions.

Then, it we will have to construct a notion of homotopy for maps $I(n,j)_0 \to \mathscr{Z}_k(M,G)$ with *different* j.

Let $\varphi^1_{j_1}$ and $\varphi^2_{j_2}$ two such maps. If $j_1 = j_2$, we say that φ^1_j is homotopic to φ^2_j for the fineness $\delta > 0$ if there exists a map

$$H : I(1,j)_0 \times I(n,j)_0 \to \mathscr{Z}_k(M,G)$$

such that

$$\begin{cases} H(0,\,\cdot\,) = \varphi^1_j \text{ and } H(1,\,\cdot\,) = \varphi^2_j \\ H\big(I(1,j)_0 \times \partial I(n,j)_0\big) = \{0\} \\ \mathbf{f}(H) < \delta. \end{cases}$$

If $j_1 < j_2$, we let $p(j_2,j_1) : I(n,j_2) \to I(n,j_1)$ be the nearest point projection for the d_{j_2} distance[1], and we say that $\varphi^1_{j_1}$ is homotopic to $\varphi^2_{j_2}$ for the fineness δ if $\varphi^1_{j_2} = \varphi^1_{j_1} \circ p(j_2,j_1)$ is homotopic to $\varphi^2_{j_2}$ for the fineness δ.

Definition 2.2.2. Let $n \in \mathbb{N} \setminus \{0\}$ be a fixed integer. We say that a sequence $\varphi = \{\varphi_j\}_{j\in\mathbb{N}}$ is a $\mathscr{Z}_k(M,G)$-valued $(n,\mathbf{M})$-homotopy sequence if for all $j \in \mathbb{N}$, each φ_j is a map from $I(n,j)_0$ to $\mathscr{Z}_k(M,G)$ and the following properties are satisfied:

(1) For all $j \in \mathbb{N}$, φ_j is homotopic to φ_{j+1} for some fineness $\delta_j > 0$,

(2) $\lim\limits_{j\to\infty} \delta_j = 0$,

(3) $\sup\limits_{j\in\mathbb{N}} \sup\limits_{x\in I(n,j)_0} \mathbf{M}(\varphi_j(x)) < \infty$.

This allows us to introduce the notion of discrete class of homotopy.

Definition 2.2.3. Let $\varphi^1 = \{\varphi^1_j\}_{j\in\mathbb{N}}$ and $\varphi^2 = \{\varphi^2_j\}_{j\in\mathbb{N}}$ two $\mathscr{Z}_k(M,G)$-valued $(n,\mathbf{M})$-homotopy sequences. We say that they are homotopic if there exists a sequence of positive real numbers $\{\delta_j\}_{j\in\mathbb{N}}$ such that :

(1) For all $j \in \mathbb{N}$, the map φ^1_j is homotopic to the map φ^2_j for the fineness $\delta_j > 0$,

(2) $\lim\limits_{j\to\infty} \delta_j = 0$.

[1] it is well-defined as we took intervals of length 3^{-j}, but with 2^{-j} we would not be able to define this projection.

The set of equivalence classes of this equivalence relation is denoted by $\pi_n^{\#}(\mathscr{Z}_k(M,G),\mathbf{M})$, and the class of a $(n,\mathbf{M})$-homotopic sequence by $[\varphi]$.

Recall that by admissible groups, we mean $\mathbb{Z}$ or $\mathbb{Z}_\nu$ for some integer $\nu \geq 2$. This is slightly more restrictive the the definition of Almgren, which allows finite direct sums of $\mathbb{Z}_\nu$ (for different values of ν)[2].

Theorem 2.2.4 (Pitts [Pit81])**.** *For all* $1 \leq n \leq m$ *and for all* $1 \leq k \leq m-n$*, the following groups are naturally isomorphic*

$$\pi_n(\mathscr{Z}_k(M,G),\mathcal{F}) \simeq \pi_n^{\#}(\mathscr{Z}_k(M,G),\mathbf{M}) \simeq H_{n+k}(M^m,G).$$

We deduce in particular that the fundamental group of the space of hyper-cycles with G-coefficient is isomorphic to G.

The next definition will allow us to define min-max methods for discrete classes of homotopy.

For any $\mathscr{Z}_k(M,G)$-valued $(n,\mathbf{M})$-homotopy sequence $\varphi = \{\varphi_j\}_{j\in\mathbb{N}}$, we define

$$\mathbf{L}(\varphi) = \limsup_{j\to\infty} \max_{x\in I(n,j)_0} \mathbf{M}(\varphi_j(x))$$

as a replacement of the maximum function for continuous min-max. The next definition finally permits to make sense of the value or *width* given to a class of discrete homotopies.

Definition 2.2.5. Let $n \in \mathbb{N}$, $1 \leq k \leq m$ be fixed integers and let $\Pi \in \pi_n^{\#}(\mathscr{Z}_k(M,G),\mathbf{M})$. The width of the min-max associated to Π is

$$\mathbf{L}(\Pi) = \inf_{\varphi\in\Pi} \mathbf{L}(\varphi).,$$

and we say that φ is critical if $\mathbf{L}(\varphi) = \mathbf{L}(\Pi)$.

Now let $\varphi = \{\varphi_j\}_{j\in\mathbb{N}}$ a $\mathscr{Z}_k(M,G)$-valued $(n,\mathbf{M})$-homotopy sequence. We define the limiting space of φ in $\mathscr{V}_k(M)$ by

$$\mathbf{K}(\varphi) = \mathscr{V}_k(M) \cap \left\{ V : V = \lim_{j\to\infty} |\varphi_{j'}(x_{j'})| \text{ for a sub-sequence } j' \text{ and } \{x_{j'}\}_{j'\in\mathbb{N}} \subset I(n,j')_0 \right\}$$

Finally, we define the critical set of φ as

$$\mathbf{C}(\varphi) = \mathbf{K}(\varphi) \cap \{V : \ \|V\| = \mathbf{L}(\varphi)\},$$

which is a compact non-empty set.

The main theorem in the existence part is to show that width are positive when they correspond to a non-trivial homotopy class and can be attained.

Theorem 2.2.6. *Let* M^m *be a compact Riemannian manifold and* $1 \leq n \leq m$*,* $1 \leq n \leq m$ *and* $1 \leq k \leq m-n$ *such that* $H_{n+k}(M,G) \neq 0$*. Then there exists* $\Pi \in \pi_n^{\#}(\mathscr{Z}_k(M,G),\mathbf{M})$ *such that* $\mathbf{L}(\Pi) > 0$ *and a critical* $\mathscr{Z}_k(M,G)$*-valued* $(n,\mathbf{M})$*-homotopy sequence* φ*,* i.e. *such that* $\mathbf{L}(\varphi) = \mathbf{L}(\Pi)$*. Furthermore,* φ *can be chosen such that every element of* $\mathbf{C}(\varphi)$ *is stationary.*

2.3 Almost minimizing property

2.3.1 Definition

Throughout this section, $M \subset \mathbb{R}^n$ is an embedded C^4 submanifold, $U \subset M$ is a bounded open subset, K is a compact set such that $U \subset K \subset M$, and G is an admissible group. Let ρ a distance on varifold, *i.e.* one of the three distances $\mathbf{M}$ (the mass), $\mathbf{F}$ (the canonical distance on varifolds), and $\mathcal{F}_K$ (the flat norm).

[2]The following theorem would also hold for this definition of admissible groups.

Definition 2.3.1. For all $0 < \varepsilon, \delta < \infty$, we define the

$$G_k(U, \varepsilon, \delta, d, G) \subset \mathscr{Z}_k(M, M \setminus U, G)$$

of the cycles T satisfying the following property. If $T = T_0, T_1, \cdots, T_m \in \mathscr{Z}_k(M, M \setminus U, G)$, and

$$\begin{aligned}
&\operatorname{supp}(T - T_i) \subset U, \quad \text{for } i = 1, \cdots, m \\
&\rho(T_i, T_{i-1}) \leq \delta \quad \text{for } i = 1, \cdots, m \\
&\mathbf{M}(T_i) \leq \mathbf{M}(T) + \delta, \quad \text{for } i = 1, \cdots, m,
\end{aligned}$$

then $\mathbf{M}(T) - \mathbf{M}(T_m) \leq \varepsilon$.

Definition 2.3.2. We say that $V \in \mathscr{V}_k(M)$ is G almost minimizing in U if for all $\varepsilon > 0$, there exists $\delta > 0$, and $T \in G_k(U, \varepsilon, \delta, \mathcal{F}_K, G)$ such that $d_K(V, |T|) < \varepsilon$.

We say that $V \in \mathscr{V}_k(M)$ is G almost minimizing at $x \in M$ if it almost minimizing in a neighbourhood of x.

The possibility to localize the first part of this definition is obvious as the property of almost-minimizing is independent of the compact K. In the following, we shall write the distance d and the group G only if necessary, as they will be in general $d = \mathcal{F}_K$, and $G = \mathbb{Z}$.

Proposition 2.3.3. *We have the following properties.*

(1) $G_k(U, \varepsilon, \delta, \mathcal{F}_K) \subset G_k(U, \varepsilon, \delta, \mathbf{F}) \subset G_k(U, \varepsilon, \frac{\delta}{2}, \mathbf{M})$ *for all positive real numbers* ε *and* δ.

(2) If $T \in G_k(U, \varepsilon, \delta)$*, then* $S \in G_k(U, \varepsilon, \delta)$ *if* $S \in \mathscr{Z}_k(M, M \setminus U)$ *and* $T \llcorner U = S \llcorner U$.

Proof. The only assertion non-trivial is the first one, and it is easily checked by looking as the inclusion of the topologies induces by the three metrics on currents. ☐

The next theorem is fundamental to understand the motivations hidden behind these definitions.

Theorem 2.3.4. *Almost minimizing varifolds are stationary and stable.*

Proof. Indeed, let $g \in \Gamma(U)$, then if $\{\varphi^t\}_{t \in I}$ is the local 1-parameter group generated by the vectorfield g, if we define for all $t \in I$

$$\begin{aligned}
\Phi^t : \mathscr{G}_k(M) &\to \mathbb{R} \\
(x, S) &\mapsto \left| \wedge_k D\varphi^t(x) \cdot S \right|
\end{aligned}$$

Then

$$\delta V(g) = \frac{d}{dt} \|\varphi^t_{\#} V\|(M)_{|t=0} = \frac{d}{dt} V(\Phi^t)_{|t=0}.$$

We argue by contradiction. If V is not stationary, then there exists $g \in \Gamma(U)$ such that $\delta V(g) < 0$. Then there exists an open interval $J \subset I$ containing 0, $\tau \in J$ a positive number, and $0 < \varepsilon < \infty$, such that for all $W \in \mathscr{V}_k(M)$, if $d_U(V, W) < \varepsilon$ then $t \mapsto W(\Phi^t)$ is strictly decreasing, and

$$\begin{aligned}
&W\left(\frac{d}{dt}\Phi^t\right) < 0, \quad \forall t \in J \\
&W(\Phi^0) - W(\Phi^\tau) > \varepsilon.
\end{aligned}$$

If $T \in \mathscr{Z}_k(M, M \setminus U)$ and $d_U(V, |T|) < \varepsilon$, for all $\delta > 0$, there exists a subdivision of $0 = t_0 < t_1 < \cdots < t_m = \tau$ such that

$$\begin{aligned}
&\mathcal{F}_K((\varphi_{t_i \#} - \varphi_{t_{i-1}\#})T) < \delta, \quad i = 1, \cdots, m \\
&\mathbf{M}(\varphi_{t_{i-1}\#} T) > \mathbf{M}(\varphi_{t_i \#} T), \quad i = 1, \cdots, m \\
&\mathbf{M}(T) - \mathbf{M}(\varphi_{m \#} T) > \varepsilon,
\end{aligned}$$

which contradicts the fact that V is almost minimizing.

If V is stationary but not stable, let $g \in \Gamma(U)$ such that $\delta^2 V(g) < 0$. We choose $\eta > 0$ such that $(-\eta, \eta) \subset I$, $\varepsilon > 0$, and $\tau \in (0, \eta)$ such that for all $W \in \mathscr{V}_k(M)$, if $d_U(V, W) < \varepsilon$, then

$$
\begin{aligned}
&t \mapsto W\left(\frac{d}{dt}\Phi^t\right) \quad \text{is strictly decreasing on } (-\eta, \eta)\\
&W\left(\frac{d^2}{dt^2}\Phi^t\right) < 0\\
&W(\Phi^0) - W(\Phi^\tau) > \varepsilon\\
&W(\Phi^0) - W(\Phi^{-\tau}) > \varepsilon,
\end{aligned}
$$

hence $t \mapsto W(\Phi^t)$ is strictly decreasing on $[0, \tau]$ if $\delta W(g) \leq 0$, and strictly increasing on $[-\tau, 0]$, so we get contradiction by the same argument of the first part of the proof. $\square$

2.3.2 Construction of the pseudo harmonic replacement

In this subsection we construct a class of comparison surfaces, in an analogous spirit as the harmonic replacement of Colding and Minicozzi [CM11] (although this remark may seem anachronistic). The existence of such construction is provided by the almost minimizing property.

Let $L \subset U \subset K$ a compact set, $V \in \mathscr{V}_k(M)$ a G almost minimizing varifold in U. We shall not write the group G in the following procedure.

Step 1: By definition, there exists sequences $\{\delta_i\}_{i\in\mathbb{N}}, \{\varepsilon_i\}_{i\in\mathbb{N}} \subset (0, \infty)$ converging to 0, and $\{T_i\}_{i\in\mathbb{N}} \subset \mathscr{Z}_k(M, M \setminus U)$, such that for all $i \in \mathbb{N}$, $T_i \in G_k(U, \varepsilon_i, \delta_i)$, and $\mathbf{F}_U(V, |T_i|) < \varepsilon_i$.

Step 2: For each $j \in \mathbb{N}$, let μ_j the infimum of all numbers $\mathbf{M}(S)$ corresponding to all $S \in \mathscr{Z}_k(M, M \setminus U)$ for which there exists a sequence $T_j = T_j^1, \cdots, T_j^m = S \in \mathscr{Z}_k(M, M \setminus U)$ with

$$
\begin{aligned}
&\bigcup_{i=1}^m \operatorname{supp}\left(T_j^i - T_j\right) \subset L\\
&\max_{1\leq i\leq m} \mathbf{M}(T_j^i) \leq \mathbf{M}(T_j) + \delta_j\\
&\max_{1\leq i< m} \mathcal{F}_K(T_j^i - T_j^{i+1}) \leq \delta_j.
\end{aligned}
$$

Then we choose a finite sequence $T_j = T_j^1, \cdots, T_j^m = T_j^*$ in $\mathscr{Z}_k(M, M \setminus U)$ with the above properties such that $\mathbf{M}(T_j^*) = \mu_j$. Then the six following properties are true.

(1) $T_j^* \in G_k(U, \varepsilon_j, \delta_j)$

(2) $0 \leq \mathbf{M}(T_j) - \mathbf{M}(T_j^*) \leq \varepsilon_j$

(3) $T_j \llcorner (\mathbb{R}^n \setminus L) = T_j^* \llcorner (\mathbb{R}^n \setminus L)$

(4) $\mathbf{M}(T_j^*) \leq \mathbf{M}(S)$ for all $S \in \mathscr{Z}_k(M, M \setminus U)$ with $\operatorname{supp}(S - T_j) \subset L$ and $\mathcal{F}_K(S - T_j^*) \leq \delta_j$.

(5) $|T_j^*|$ is stable in $\operatorname{Int} L$.

(6) For each $p \in \operatorname{Int} L$, there exists a positive number r such that $\mathbf{M}(S) \geq \mathbf{M}(T_j^*)$ whenever $S \in \mathscr{Z}_k(M, M \setminus U)$ with $\operatorname{supp}(S - T_j^*) \subset \overline{B}(p, r)$.

As almost minimizing varifolds are stable, we have (5), and as $T_j \in G_k(U, \varepsilon_j, \delta_j)$ (2) follows immediately, and (1), (3) and (4) are obvious. So we need only prove the final assertion.

Let $p \in \operatorname{Int}(L)$, and $r > 0$ small enough such that $M \subset \overline{B}(p, r) \subset \operatorname{Int}(L)$, and

$$\|T_j^*\|\overline{B}(p, r) \leq \frac{\delta_j}{2}.$$

Then if $\mathbf{M}(S) < \mathbf{M}(T_j^*)$ for some $S \in \mathscr{Z}_k(M, M \setminus U)$ with $\operatorname{supp}(S - T_j^*) \subset \overline{B}(p,r)$, we have

$$\begin{aligned} &T_j^* \llcorner (\mathbb{R}^n \setminus \overline{B}(p,r)) = S \llcorner (\mathbb{R}^n \setminus \overline{B}(p,r)) \\ &\|S\|\overline{B}(p,r) < \|T_j^*\|\overline{B}(p,r) \\ &\mathcal{F}_K(S - T_j^*) \leq \left(\|S\| + \|T_j^*\|\right)\overline{B}(p,r) < \delta_j, \end{aligned}$$

which contradicts the fact (4).

Step 3 : For each $j \in \mathbb{N}$, we define

$$V_j^* = |T_j^*| \llcorner \mathscr{G}_k(U) + V \llcorner \mathscr{G}_k(\mathbb{R}^n \setminus U)$$

Step 4 : We say $V^* \in \mathscr{B}(V, U, L)$ if $V^* \in \mathscr{V}_k(M)$ is the limit of a sequence $\left\{V_j^*\right\}_{j \in \mathbb{N}}$ constructed from steps 2 and 3, *i.e.* is the closure of the sets of $\left\{V_j^*\right\}_{j \in \mathbb{N}}$ in the weak $*$ topology. By Banach-Alaoglu theorem, $\mathscr{B}(V, U, L)$ is compact and non-empty.

Theorem 2.3.5. *Let $V \in \mathscr{V}_k(M)$, V is almost minimizing in U, $K \subset U$ is a compact set, and $V^* \in \mathscr{B}(V, U, K)$. Then the following five properties are true.*

(1) $V \llcorner \mathscr{G}_k(\mathbb{R}^n \setminus K) = V^* \llcorner \mathscr{G}_k(\mathbb{R}^n \setminus K)$.

(2) *V^* is almost minimizing in U.*

(3) $\|V\|(M) = \|V^*\|(M)$.

(4) *For all $\varepsilon > 0$, there exists $T \in \mathscr{Z}_k(M, M \setminus U)$ such that $\mathbf{F}_U(V^*, |T|) < \varepsilon$ and $T \llcorner Z$ is locally area minimizing for all compact Lipschitz neighbourhood retract of* $\operatorname{Int}(K)$.

(5) $V^* \in \mathscr{I}\mathscr{V}_k(M, \operatorname{Int}(K))$.

Proof. The claims **(1), (2)** are trivial consequences of the construction.

Claim (3). As we used the varifold metric $\mathbf{F}$ under which the mass is continuous and thanks of **Step 1** and property (2) of **Step 2**, we obtain the point **(3)** of the theorem.

Claim (4). This follows from the property **(6)** of **Step 2**.

Claim (5). If suffices to show that for all open subset $Z \subset U$ such that $\overline{Z} \subset \operatorname{Int}(K)$, and $\|V^*\|(\partial Z) = 0$, we have $V^* \in \mathscr{I}\mathscr{V}_k(M, Z)$.

Let $\{V_i^*\}_{i \in \mathbb{N}} \subset \mathscr{I}\mathscr{V}_k(M, Z)$ such that $V_i^* \underset{i \to \infty}{\longrightarrow} V$. By assertion (5) of step 2 of the construction V_j^* is stable in Z so it is stationary in Z, and $V_j^* \in \mathscr{I}\mathscr{V}_k(M, Z)$ (*i.e.* V_j^* is integral in Z), so by the compactness theorem, as

$$\lim_{i \to \infty} V_i^* \llcorner \mathscr{G}_k(Z) = V^* \llcorner \mathscr{G}_k(Z)$$

we have $V^* \in \mathscr{I}\mathscr{V}_k(M, Z)$. $\square$

Finally, we close this section with the important theorem, independent of the unquoted theorems of chapter 3 of [Pit81].

Theorem 2.3.6. *Let $V \in \mathscr{V}_k(M)$ a stationary varifold in an open subset $U \subset M$. If for all $p \in U$, there exists a finite positive number s such that V is almost minimizing in $A(p, r, s)$ for all $0 < r < s$, then $V \in \mathscr{I}\mathscr{V}_k(M, U)$. If V is almost minimizing in U, the condition of stationariness is redundant.*

2.4 Existence theorems

We simply quote the main result of chapter 4 of [Pit81], to understand why one needs to have a regularity theory in the case of annuli and not simply balls (contrasting with the classical results of Allard [All72]). The first result shows that one can choose the critical sequence to be almost minimizing in all annuli.

Theorem 2.4.1. *Let M^m a compact Riemannian manifold and $1 \leq n \leq m$ such that $1 \leq n \leq m$ and $1 \leq j \leq m-n$ and $H_{n+k}(M,G) \neq 0$. Then there exists $\Pi \in \pi_n^{\#}(\mathscr{Z}_k(M,G), \mathbf{M})$ such that $\mathbf{L}(\Pi) > 0$ and a critical $\mathscr{Z}_k(M,G)$-valued $(n, \mathbf{M})$-homotopy sequence φ,* i.e. *such that $\mathbf{L}(\varphi) = \mathbf{L}(\Pi)$. Furthermore, φ can be chosen such that every element of $\mathbf{C}(\varphi)$ is stationary and almost minimizing in all sufficiently small annulus.*

From this one deduce the main existence result.

Theorem 2.4.2. *Let M a compact C^4 Riemannian manifold such that $H_m(M,G) \neq \{0\}$. Then for all $1 \leq k \leq m$, there exists $V \in \mathscr{V}_k(M)$ such that the following four statements are true.*

(1) V is stationary in M.

(2) For all $p \in M$, there exists a positive number s such that V is G almost minimizing in $A_M(p,r,s)$ for all $0 < r < s$.

(3) The set of points of M where V is not G almost minimizing is finite.

(4) $V \in \mathscr{I}\mathscr{V}_k(M)$.

2.5 Regularity of stationary almost minimizing varifolds

This is the topic of chapter 4. The main statement which is proven is the following.

Theorem 2.5.1 (Pitts)**.** *If $2 \leq k \leq 5$, $\max\{4,k\} \leq \nu \leq \infty$, N^{k+1} a $C^{\nu+1}$ submanifold of $\mathbb{R}^n$, $p \in N$, $0 < r < \infty$, $V \in \mathscr{V}_k(N)$, V is stationary in $N \cap B(p,r)$, and V is almost minimizing in $N \cap A(p,s,r)$ for all $0 < s < r$, then $\operatorname{supp}\|V\| \cap B(p,r)$ is a k dimensional minimal C^ν submanifold of $N \cap B(p,r)$.*

This theorem of Pitts combined with the existence theory and the extension for $k = 6$ of theorem 2.5.1 by Schoen and Simon yields the following result.

Theorem 2.5.2. *Let M^m a compact Riemannian manifold of dimension $m \leq 7$. Every stationary integral almost minimizing varifold V on every open annulus is the varifold associated to the current of integration of an embedded smooth minimal hypersurface.*

Finally, we obtain the first major conclusion of Almgren-Pitts min-max theory.

Theorem 2.5.3. *If $2 \leq n \leq 7$, $\max\{5,n\} \leq \nu \leq \infty$, M^n a compact C^ν Riemannian manifold, then there exists a non-empty embedded $C^{\nu-1}$ minimal submanifold of M.*

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