# SHEET 4.

## Exercice 1

Prove that $\theta : [0,1] \longrightarrow [0,1]$ is right continuous.

## Exercice 2

Let $p < p_c$. Prove that $E_p[|C_0|] < \infty$.

## Exercice 3 (Percolation with long-range interactions.)

Let $(J_{x,y})_{x,y \in \mathbb{Z}^d}$ be a family of non-negative translation-invariant numbers : ie $J_{x,y} \geq 0$ and $\forall x, y, z$ $J_{x,y} = J_{x+z,y+z}$.

Let $\mathbb{P}_\beta$ be the bond percolation measure on $\mathbb{Z}^d$ defined as follows : for $x, y \in \mathbb{Z}^d$ (not necessarily neighbours), $\{x,y\}$ is open with probability $1 - e^{-\beta J_{x,y}}$, closed with probability $e^{-\beta J_{x,y}}$.

1. Define the analogues $\beta_c$, $\tilde{\beta}_c$, $\varphi_\beta(S)$ of $p_c$, $\tilde{p}_c$, $\varphi_p(S)$ in this context.

2. Show that there exists $c > 0$ s.t. $\forall \beta \geq \tilde{\beta}_c$ $\mathbb{P}_\beta[0 \longleftrightarrow \infty] \geq c(\beta - \tilde{\beta}_c)$

3. If the interactions are finite-range (ie. $\exists R$ s.t. $J_{x,y} = 0$ if $|x-y| \geq R$) then show that $\forall \beta < \tilde{\beta}_c$ $\exists c > 0$ s.t. $\mathbb{P}_\beta[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}$.

4. In the general case, show that for any $\beta < \tilde{\beta}_c$ $\sum_{x \in \mathbb{Z}^d} \mathbb{P}_\beta[0 \longleftrightarrow x] < \infty$.

<u>Hint</u>: Consider $S$ s.t. $\varphi_\beta(S) < 1$ and show that for $n \geq 1$

$$\forall x \in \Lambda_n \quad \sum_{y \in \Lambda_n} \mathbb{P}_\beta[x \longleftrightarrow y] \leq \frac{|S|}{1 - \varphi_\beta(S)}.$$