SHEET 7.

Exercise 1

Prove that $N = \begin{cases} 0 & \text{a.s. if } \theta(p) = 0, \\ 1 & \text{a.s if } \theta(p) > 0. \end{cases}$

Exercise 2

Define $\chi(p) = \mathbb{E}_p[|C_0|]$, for $p \in [0,1]$

(1) Recall why $\chi(p)$ is non-decreasing in p.

Prove that $\forall\, p \geq p_c$ $\chi(p) = \infty$.

(2) Prove that $\chi(p) \geq \frac{1}{e}\left(\tilde{\xi}(p) + 1\right) \geq \frac{1}{e}\left(\xi(p) + 1\right)$.

In particular, we have $\chi(p) \xrightarrow[p \to p_c]{} \infty$.

(3) [difficult].

Prove that χ is analytic on $[0, p_c)$.

(Hint: One can use the expression

$$\chi(p) = \sum_n \sum_{i,b} n\, a_{n,i,b}\, p^i (1-p)^b$$

where $a_{n,i,b}$ is the number of connected subgraphs containing 0 with n vertices, i (internal) edges and b "boundary" edges)