PERCOLATION

INTRODUCTION

1 STATISTICAL PHYSICS.

<u>General idea</u>: Study physical systems with a very large number of elements using tools from probability theory.

<u>Examples</u>: • population dynamics ($\simeq 10^9$ individuals)

• a glass of water ($\gg 10^{23}$ molecules.)

• a piece of Iron ($\gg 10^{23}$ atoms)

• Cars on a high way.

• percolation systems (the topic of this course!).

...

. Giving an exact description of such system is very difficult (e.g. for water, one needs to understand a system of $\gg 10^{23}$ equations!).

. Instead, we give a probabalistic description. Each element has a random behaviour, and the system is described by very <u>few parameters</u>.

. We are interested in the large-scale behaviour of such system.

<u>Examples</u>: • population dynamics $\longrightarrow$ survival / extinction?

• water $\longrightarrow$ solid / liquid / gaz?

• iron $\longrightarrow$ magnetization / no magnetization?

• cars $\longrightarrow$ fluid traffic / traffic jam?

For such systems, we often observe a <u>sharp phase transition</u>: a small change in the parameters may give rise to completely different macroscopic behaviours.

Modelisation.

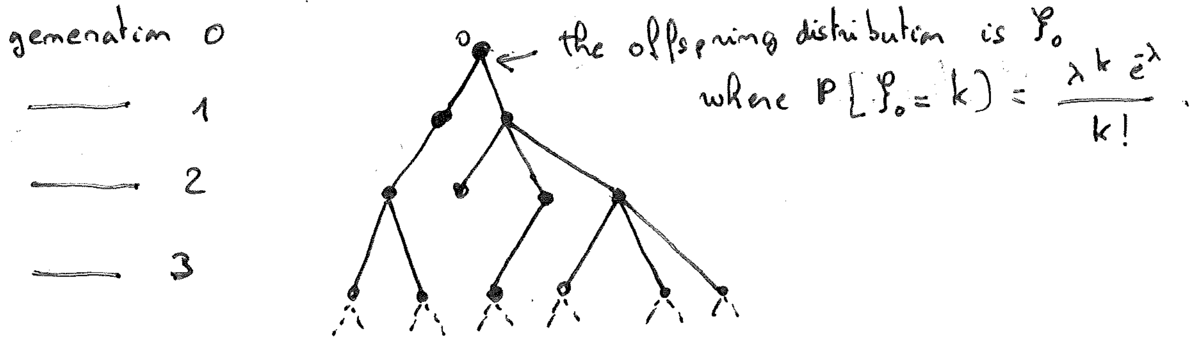
- $\Omega = \{$ "possible states for the system" $\}$.
- P_β = probability measure on Ω, indexed by a parameter β.

First example: Galton-Watson trees

Fix a parameter $\lambda \in [0,1]$, and write $p_\lambda(\{k\}) = \dfrac{\lambda^k e^{-\lambda}}{k!}$.

We construct a population as follows:

- A generation 0, there is 1 individual.
- At generation n, each n-th generation individual produces (independently) a random number of individuals, according to p_λ, in the $(n+1)$-generation.

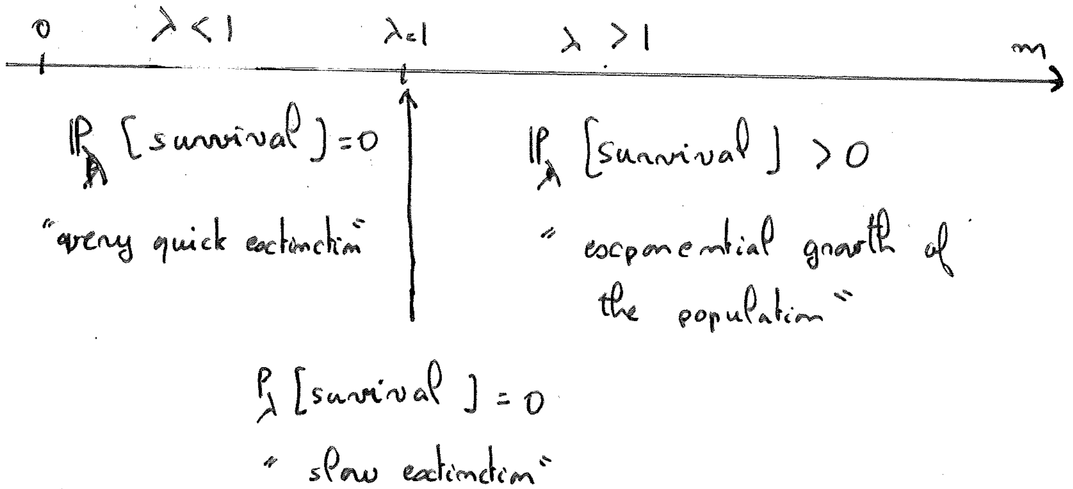


In this case, we have

Ω = { locally finite genealogical trees }.

$\mathbb{P}_\lambda$ = law on Ω that depends on λ.

We observe a phase transition w.r.t. λ (the expectation of the offspring distribution)

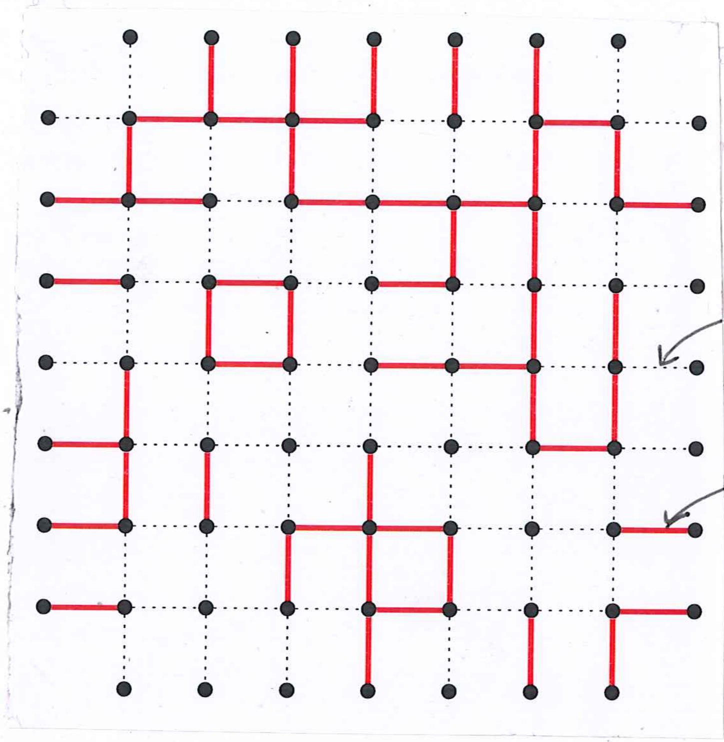


2. WHAT IS PERCOLATION ?

→ See Slides.

3. BOND PERCOLATION ON $\mathbb{Z}^d$ ($d \geq 1$ fixed)

Graph $(\mathbb{Z}^d, E)$ where $E = \{ \{x,y\} \subset \mathbb{Z}^d : \|x-y\|_1 = 1 \}$.



$\omega(e) = 0$ "closed"

$\omega(e) = 1$ "open"

<u>parameter</u> $p \in [0,1]$

Each edge is declared open ("red") with probability p, closed with probability $1-p$, independently of the other edges.

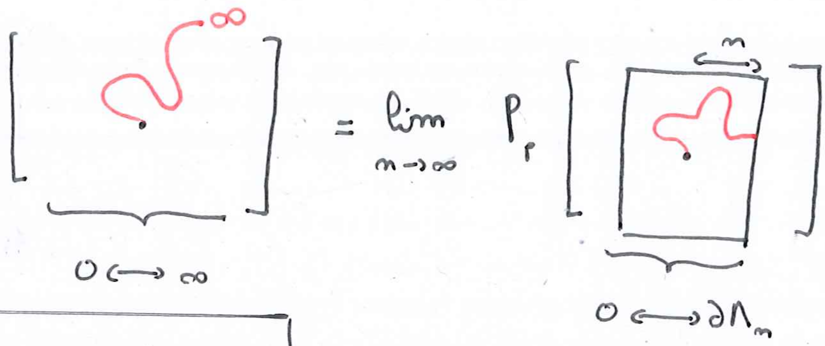
<u>state space</u> $\Omega = \{0,1\}^E$

<u>percolation configuration</u> : $\omega \in \{0,1\}^E$ $\to \omega(e) = 0$ "edge closed"

$\omega(e) = 1$ "edge opened"

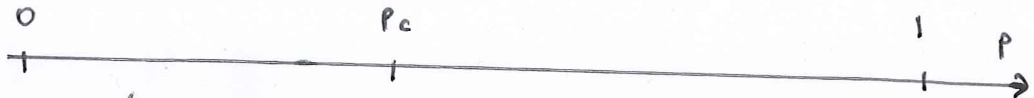
<u>percolation measure</u> $P_p = (\text{Bernoulli}(p))^{\otimes E}$.

Phase transition:



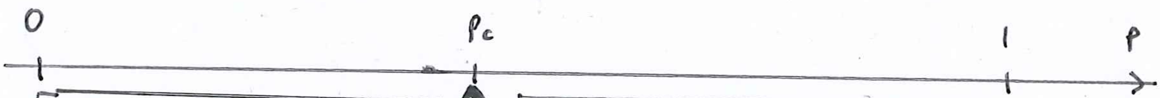
$$\theta(p) := P_p\Big[\underbrace{\quad\quad}_{0 \leftrightarrow \infty}\Big] = \lim_{n\to\infty} P_p\Big[\underbrace{\quad\quad}_{0 \leftrightarrow \partial\Lambda_n}\Big]$$

$$\boxed{p_c = p_c(d) = \sup\{p : \theta(p) = 0\}.}$$



The subcritical $(p < p_c)$ and supercritical $(p > p_c)$ regimes are now well understood. In these lectures, we will first introduce the fundamental tools in percolation theory and proves (among others) the theorems below.

percolation from a point.



$$\boxed{\begin{array}{l}\underline{\text{Thm}}: \text{ for } p < p_c \ \exists c > 0 \\ \mathbb{P}_p\left[0 \leftrightarrow \partial\Lambda_n\right] \leq e^{-cn}\end{array}}$$

$$\boxed{\begin{array}{l}\underline{\text{Thm}}: \text{ for } p > p_c \ \exists c > 0 \\ \mathbb{P}_p\left[0 \leftrightarrow \partial\Lambda_n, 0 \not\leftrightarrow \infty\right] \leq e^{-cn}\end{array}}$$

$$\boxed{\begin{array}{l}\underline{\text{Conjecture}}: \text{ For } p = p_c, \ d \geq 2, \\ \mathbb{P}_{p_c}\left[0 \leftrightarrow \infty\right] = 0\end{array}}$$

↳ known for $d = 2$ and $d \geq 11$

Box-crossing probabilities

$0 \qquad\qquad p_c \qquad\qquad 1 \quad p$

Thm: for $p < p_c$ $\exists c > 0$

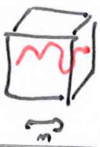
$\mathbb{P}_p\Big[$  $\Big] \leq e^{-cn}$

Thm: for $p > p_c$ $\exists c > 0$

$\mathbb{P}_p\Big[$  $\Big] \geq 1 - e^{-cn^{d-1}}$

Conjecture: for $p = p_c$

If $d < 6$ $\quad 0 < \lim_{n \to \infty} \mathbb{P}_p\Big[$  $\Big] < 1$

If $d \geq 6$ $\quad \lim_{n \to \infty} \mathbb{P}_p\Big[$  $\Big] = 1$

↳ known for $d = 2$ and $d \geq 11$.

BERNOULLI PERCOLATION ON $\mathbb{Z}^d$

I DEFINITIONS

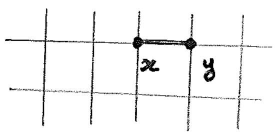
1.1. GRAPH TERMINOLOGY.

For $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, $\|x\|_1 := \sum_{i=1}^{d} |x_i|$ (L^1 norm)

Graph structure on $\mathbb{Z}^d$:

$$E = \{ \{x,y\} \subset \mathbb{Z}^d : \|x - y\|_1 = 1 \} \qquad \text{"edge set"}$$

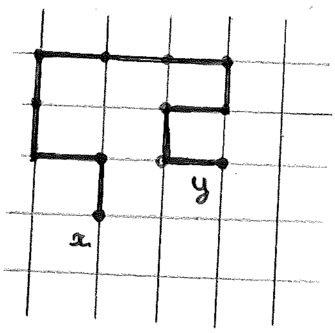
Notation:
- For $x, y \in \mathbb{Z}^d$, write $xy = \{x, y\}$.
- If $xy \in E$ we say that x and y are neighbours and we write $x \sim y$.



→ $(\mathbb{Z}^d, E)$ is an infinite graph of degree $2d$.

Def: A *path* of length l from a vertex x to a vertex y is a sequence $\gamma = (\gamma_0, \dots, \gamma_l)$ of *distinct* vertices s.t.

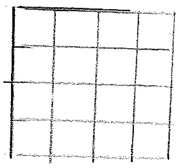
$$\gamma_0 = x, \quad \gamma_1 = y \quad \text{and} \quad \forall i \in \{1, \dots, l\} \ \gamma_i \gamma_{i-1} \in E.$$



Remark: $\|x-y\|_1 = \min\{\text{length}(\gamma),\ \gamma \text{ path from } x \text{ to } y\}$

"graph distance between x and y."

Not. $\Lambda_n = \{-n,\dots,n\}^d$ "box of size n around 0"



Λ_2 on $\mathbb{Z}^2$

Def: Let $S \subset \mathbb{Z}^d$. We define

. $\partial S := \{x \in S : \exists y \in \mathbb{Z}^d \setminus S \quad y \sim x\}$ "vertex boundary of S"

. $\Delta S := \{xy \in E : x \in S,\ y \notin S\}$ "edge boundary of S".

1.2. PERCOLATION CONFIGURATIONS

(bond) percolation configuration : $\omega = (\omega(e))_{e\in E} \in \{0,1\}^E$.

Rk : $\{0,1\}^E \xleftrightarrow{\text{bij}} \mathcal{P}(E)$

$\omega \longleftrightarrow E_\omega = \{e : \omega(e) = 1\}$

We often identify ω with the subgraph $G_\omega = (\mathbb{Z}^d, E_\omega)$.

Def: Let $\omega \in \{0,1\}^E$

. An edge $e \in E$ is said to be - open if $\omega(e) = 1$,
- closed if $\omega(e) = 0$.

. A path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_p)$ is said to be open if $\forall i\ \omega(\gamma_{i-1}\gamma_i) = 1$.

. A cluster is a connected component of G_ω.

Notation : $C_x(\omega)$ = cluster containing x

1.3. PERCOLATION SPACE

$p \in [0,1]$.

We consider the probability space $(\{0,1\}^E, \mathcal{F}, \mathbb{P}_p)$

- $\mathcal{F}$ is the product σ-algebra (it is generated by the events depending on finitely many edges)
- $\mathbb{P}_p = \prod_{e \in E} (p\,\delta_1 + (1-p)\,\delta_0)$ "percolation measure with density p"

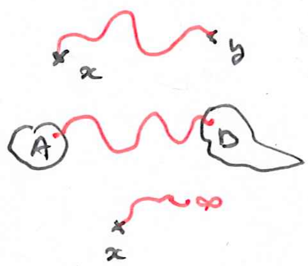
(it is characterized by $\forall e_1, \dots, e_k \in E$ $\forall w_1, \dots, w_k \in \{0,1\}$

$$\mathbb{P}_p[\omega(e_1) = w_1, \dots, \omega(e_k) = w_k] = p^{\sum w_i} (1-p)^{\sum 1-w_i} \;)$$

Rk: $(X \sim \mathbb{P}_p) \iff \big((X(e))_{e \in E}$ are iid Bernoulli(p).$\big)$

Some events: $x, y \in \mathbb{Z}^d$, $A, B \subset \mathbb{Z}^d$.

- $(x \leftrightarrow y) = \{\exists$ open path from x to $y\}$
- $(A \leftrightarrow B) = \{\exists x \in A\ \exists y \in B : x \leftrightarrow y\}$
- $(x \leftrightarrow \infty) = \{x$ belongs to an infinite cluster$\}$



2. PROPERTIES

2.1. MONOTONICITY

Question: It is natural to expect $\mathbb{P}_p[x \leftrightarrow y]$ increasing in p.
What is the property of $A = x \leftrightarrow y$ implying this fact?
How to prove it?

Equip $\{0,1\}^E$ with the product ordering:

$$\omega \leq \eta \iff \forall e \in E \quad \omega(e) \leq \eta(e)$$

Def: An event $A \in \mathcal{F}$ is <u>increasing</u> if

$$\left.\begin{array}{l}\omega \leq \eta \\ \omega \in A\end{array}\right\} \Rightarrow \eta \in A.$$

A is decreasing if A^c decreasing.

Ex: $\{x \leftrightarrow y\}$, $\{|C_x| \geq 10\}$ are increasing.

$\{|C_x| = 10\}$ is neither increasing or decreasing.

Rk: if A, B are increasing then $A \cap B$, $A \cup B$ are increasing.

Def: A function $f : \{0,1\}^E \longrightarrow \mathbb{R}$ is <u>increasing</u> if

$$\omega \leq \eta \Rightarrow f(\omega) \leq f(\eta).$$

It is decreasing if

$$\omega \leq \eta \Rightarrow f(\omega) \geq f(\eta).$$

Ex: $f(\omega) = |C_x(\omega)|$ is increasing.

Rk: A increasing $\Leftrightarrow$ $\mathbb{1}_A$ increasing.

Prop (i) Let $A \in \mathcal{F}$ be an increasing event, then

$$p \longmapsto \mathbb{P}_p[A] \text{ is non-decreasing.}$$

(ii) Let $f : \{0,1\}^E \longrightarrow \mathbb{R}$ measurable increasing, bounded or ≥ 0. Then

$$p \longmapsto \mathbb{E}_p[f] \text{ is non decreasing.}$$

Proof: It suffices to prove (ii). (i) follows by applying (ii) with $f = \mathbb{1}_A$.

We use a monotone coupling. Let $(U_e)_{e \in E}$ be iid uniform in $[0,1]$.

Define for every $p \in [0,1]$ $X_p(e) = \mathbb{1}_{U_e \leq p}$.

We have $p \le p' \Rightarrow X_p \le X_{p'}$ a.s.

$$\Rightarrow f(X_p) \le f(X_{p'}) \text{ a.s.}$$

$$\Rightarrow \underbrace{\mathbb{E}[f(X_p)]}_{E_p[f]} \le \underbrace{\mathbb{E}[f(X_{p'})]}_{E_{p'}[f]}$$

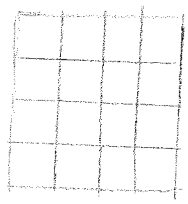
$\blacksquare$

Appli: $P_p[x \leftrightarrow y]$, $P_p[x \leftrightarrow \infty]$, $E_p[|C_0|]$ are non decreasing in p.

2.2 RUSSO'S FORMULA

We consider percolation on a finite graph $G = (V, E)$.
(same def. as on $(\mathbb{Z}^d, E)$)

Ex: $G =$ subgraph induced by Λ_n.



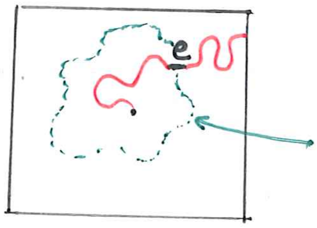
Def. Let $A \subset \{0,1\}^E$ be an increasing event. We say that $e \in E$ is pivotal for A in ω if

$$\omega_e \notin A \text{ and } \omega^e \in A,$$

where $\omega_e(f) = \begin{cases} \omega(f) & f \neq e \\ 0 & f = e \end{cases}$ and $\omega^e(f) = \begin{cases} \omega(f) & f \neq e \\ 1 & f = e \end{cases}$

Rk: The event $\{e \text{ is piv. for } A\}$ $\left(= \{\omega : e \text{ is piv. for } A \text{ in } \omega\}\right)$ is independent of $\omega(e)$.

Ex: on Λ_n, $A = 0 \longleftrightarrow \partial\Lambda_n$.



Diagrammatic representation of the event $\{e \text{ is pivotal for } A\}$.

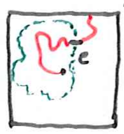
Prop: Let $A \in \{0,1\}^E$ increasing (recall that $|E| < \infty$). Then

$$\boxed{\frac{d}{dp} \mathbb{P}_p[A] = \sum_{e \in E} \mathbb{P}_p[e \text{ is piv. for } A]}$$

Rk: $\mathbb{P}_p[A] = \underbrace{\sum_{\omega \in A}}_{\text{"finite sum"}} p^{|\omega|} (1-p)^{|E| - |\omega|}$ polynomial in p, in particular C^∞.

(where $|\omega| = \sum_{e \in E} \omega(e)$)

Rk: Russo's formula gives a "geometric" interpretation of the derivative of connection probabilities. For example

$$\frac{d}{dp} \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] = \sum_{e \subset \Lambda_n} \mathbb{P}_p \Bigg[$$  $$\Bigg].$$

Proof: Let $E = \{e_1, \dots, e_k\}$.

Define $\forall p_1, \dots, p_k \in [0,1]$ $\forall \omega \in \{0,1\}^E$

$$\mathbb{P}_{p_1, \dots, p_k}[\{\omega\}] = \prod_{i=1}^{k} p_i^{\omega_i} (1-p_i)^{1-\omega_i}.$$

(Rk: $\mathbb{P}_p = \mathbb{P}_{p, \dots, p}$)

Define $f(p_1,\dots,p_k) := \mathbb{P}_{p_1,\dots,p_k}[A]$.

$$\frac{d}{dp}\mathbb{P}_p[A] = \frac{d}{dp} f(p,\dots,p) = \sum_{i=1}^{k} \frac{\partial}{\partial p_i} f(p,p,\dots,p).$$

We compute the right derivative $\lim_{\varepsilon\searrow 0} \dfrac{f(p,\dots,\overset{i}{p+\varepsilon},\dots,p) - f(p,\dots,p)}{\varepsilon}$ for $p<1$. The left derivative for $p=1$ is established analogously.

We couple a random variable ω_p with law $\mathbb{P}_p$ and $\omega_{p+\varepsilon}$ with law $\mathbb{P}_{p,\dots,p+\varepsilon,\dots,p}$.

Let $U_1,\dots,U_k$ iid uniform on $[0,1]$. Define for $q\in[0,1]$

$$\omega_q(e_j) = \begin{cases} \mathbb{1}_{U_j\le p} & \text{if } j\neq i, \\ \mathbb{1}_{U_i\le q} & \text{if } j=i. \end{cases}$$

Notice that $\omega_q \sim \mathbb{P}_{p,\dots,\overset{i}{q},\dots,p}$.

$$\begin{aligned} f(p,\dots,p+\varepsilon,\dots,p) - f(p,\dots,p) &= \mathbb{P}[\omega_{p+\varepsilon}\in A] - \mathbb{P}[\omega_p\in A] \\ &\overset{A \text{ is } \uparrow}{=} \mathbb{P}[\omega_{p+\varepsilon}\in A, \omega_p\notin A] \\ &= \mathbb{P}[e_i \text{ is piv for } A \text{ in } \omega_p, \ U_i\in[p,p+\varepsilon]] \\ &= \varepsilon\, \mathbb{P}[e_i \text{ is piv. for } A \text{ in } \omega_p]. \end{aligned}$$

Hence $\dfrac{\partial f}{\partial p_i}(p,\dots,p) = \mathbb{P}_p[e_i \text{ is piv. for } A]$ ■

2.3 HARRIS - FKG INEQUALITY. (FKG: Fortuin, Kasteleyn, Ginibre).

$G=(\mathbb{Z}^d, E)$ $\quad \Omega=\{0,1\}^E$

Intuition. Let $e\in E$. If we know that e is open, this should help the event, say, $x\leftrightarrow y$ to occur. We expect

$$\mathbb{P}_p[x\leftrightarrow y \mid e \text{ open}] \geq \mathbb{P}_p[x\leftrightarrow y]$$

Also we expect $\mathbb{P}_p[x\leftrightarrow y \mid z\leftrightarrow t] \geq \mathbb{P}_p[x\leftrightarrow y]$ and more generally $\mathbb{P}_p[A\mid B]\geq \mathbb{P}_p[A]$ $\forall A,B\uparrow$.

Prop. i) Let $A, B \uparrow$ events, then $\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\,\mathbb{P}_p[B]$

ii) Let $X, Y : \Omega \to \mathbb{R}$ be two bounded increasing random variables. Then

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\,\mathbb{E}_p[Y].$$

Rk: Holds in more general contexts $\to$ dependent models, general product space ...

Proof. It suffices to prove (ii). (i) follows by considering $X = \mathbb{1}_A$ and $Y = \mathbb{1}_B$. Order $E = \{e_1, e_2, \dots\}$ and write $\omega_i = \omega(e_i)$

Finite volume.

We first prove by induction on $n \geq 1$ that

$$(\mathcal{P}_n) \qquad \forall f, g : \{0,1\}^n \longrightarrow \mathbb{R} \text{ increasing}$$

$$\mathbb{E}_p[f(\omega_1, \dots, \omega_n)\, g(\omega_1, \dots, \omega_n)] \geq \mathbb{E}_p[f(\omega_1, \dots, \omega_n)]\,\mathbb{E}_p[g(\omega_1, \dots, \omega_n)].$$

$n=1$ Let $f, g : \{0,1\} \to \mathbb{R}$ increasing. WLOG, we can assume that $f(0) = g(0) = 0$ since adding a constant to f and/or g does not change the inequality. In such case, we have $f(1) \geq 0$ and $g(1) \geq 0$ since f, g are increasing.

Hence $\mathbb{E}_p[f(\omega_1)\, g(\omega_1)] - \mathbb{E}_p[f(\omega_1)]\,\mathbb{E}_p[g(\omega_1)]$

$$= p\, f(1)\, g(1) - p^2 f(1)\, g(1)$$

$$\geq 0$$

Now let $n \geq 1$ and assume that $(\mathcal{P}_n)$ holds.

Let $f, g : \{0,1\}^{n+1} \longrightarrow \mathbb{R}$ increasing.

$$E_p[f(\omega_1,\dots,\omega_{m+1})\, g(\omega_1,\dots,\omega_{m+1})]$$

$$= p\, E_p[f(\omega_1,\dots,\omega_m,1)\, g(\omega_1,\dots,\omega_m,1)] + (1-p)\, E_p[f(\omega_1,\dots,\omega_m,0)\, g(\omega_1,\dots,\omega_m,0)]$$

$$\overset{\mathcal{P}_m}{\geq} p\, \underbrace{E_p[f(\omega_1,\dots,\omega_m,1)]}_{=:f_1(1)}\, \underbrace{E_p[g(\omega_1,\dots,\omega_m,1)]}_{=:g_1(1)} + (1-p)\, \underbrace{E_p[f(\omega_1,\dots,\omega_m,0)]}_{=:f_1(0)}\, \underbrace{E_p[g(\omega_1,\dots,\omega_m,0)]}_{=:g_1(0)}.$$

$$= E_p[f_1(\omega_1)\, g_1(\omega_1)] \overset{\mathcal{P}_1}{\geq} E_p[f_1(\omega_1)]\, E_p[g_1(\omega_1)].$$

This proves $(\mathcal{P}_{m+1})$ since

$$\begin{aligned} E_p[f_1(\omega_1)] &= p\, f_1(1) + (1-p)\, f_1(1) \\ &= p\, E_p[f(\omega_1,\dots,\omega_m,1)] + (1-p)\, E_p[f(\omega_1,\dots,\omega_m,0)] \\ &= E_p[f(\omega_1,\dots,\omega_{m+1})]. \end{aligned}$$

and equivalently $E_p[g_1(\omega_1)] = E_p[g(\omega_1,\dots,\omega_{m+1})]$

Infinite volume.

Let $X, Y : \{0,1\}^E \to \mathbb{R}$ be two increasing bounded random variables.

Let $X_m = E_p[X \mid \omega_1,\dots,\omega_m]$, $Y_m = E_p[Y \mid \omega_1,\dots,\omega_m]$.

For every $m \geq 1$, we have, by $\mathcal{P}_m$,

$$E_p[X_m Y_m] \geq E_p[X_m]\, E_p[Y_m].$$

By the martingale convergence theorem*, we have

$$X_m \to X \text{ and } Y_m \to Y \text{ in } L^2 \text{ and } L^1,$$

and we obtain

$$E_p[XY] \geq E_p[X]\, E_p[Y]$$

by taking the limit in the equation above as m tends to infinity.

* see e.g. GRIMMETT STIRZAKER p. 484 (3rd edition)
or WALTERS (Probability with martingales) p. 134.

Corollary:

If A, B are decreasing, then
$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\,\mathbb{P}_p[B].$$
If A is increasing and B is decreasing, then
$$\mathbb{P}_p[A \cap B] \leq \mathbb{P}_p[A]\,\mathbb{P}_p[B].$$

Corollary: (square-root trick)

Let $A_1, \dots, A_k$ be k increasing events, $k \geq 1$. Let $\varepsilon > 0$.

If $\mathbb{P}_p[A_1 \cup \dots \cup A_k] \geq 1 - \varepsilon$,

Then $\max\limits_{1 \leq i \leq k} \mathbb{P}_p[A_i] \geq 1 - \varepsilon^{1/k}$.

__Proof:__
$$\mathbb{P}_p[A_1 \cup \dots \cup A_k] = 1 - \mathbb{P}_p[A_1^c \cap \dots \cap A_k^c]$$
$$\underset{+\text{induction}}{\overset{\text{FKG}}{\leq}} 1 - \mathbb{P}_p[A_1^c] \dots \mathbb{P}_p[A_k^c]$$
$$\leq 1 - \left(1 - \max_{1 \leq i \leq k} \mathbb{P}_p[A_i^c]\right)^k$$
ie
$$\max_{1 \leq i \leq k} \mathbb{P}_p[A_i] \geq 1 - \left(1 - \mathbb{P}_p[A_1 \cup \dots \cup A_k]\right)^{1/k} \qquad \blacksquare$$

__Application:__ For percolation on $\mathbb{Z}^2$

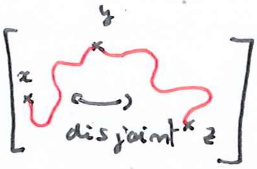
$\mathbb{P}_p\Big[$  $\Big] \geq 1 - \varepsilon \implies \mathbb{P}_p\Big[$  $\Big] \geq 1 - \sqrt{\varepsilon}$.

open path from left to right.

open path from left to right that ends on the upper half segment of the right side.

2.4. BK-REIMER INEQUALITY.

$G = (V, E)$ finite graph.

Motivation: We expect $\mathbb{P}_p\Big[$  $\Big] \leq \mathbb{P}_p[x \leftrightarrow y]\,\mathbb{P}_p[y \leftrightarrow z]$.

Def: Let A be an event. Let $\omega \in \{0,1\}^E$.

A set $I \subset E$ is a <u>witness of A in ω</u> (I wit. A in ω) if $\omega \in A$ and $\forall \omega' \in \{0,1\}^E$, $(\omega_{|I} = \omega'_{|I}) \Rightarrow (\omega' \in A)$.

Ex. • An open path from x to y in ω is a witness of $x \leftrightarrow y$ in ω.

• If $\omega \in A$, then we always have E wit. A in ω.

Question: What would be a witness for $|C_0| = 5$? $|C_0| \geq 5$? $|C_0| \leq 5$?

Exercise: Let $A \uparrow$, $\omega \in A$. Prove that there exists a wit. I for A in ω s.t. $\forall e \in I \quad \omega(e) = 1$.

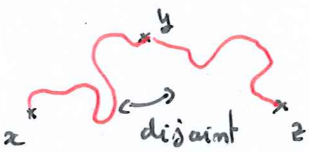
Def: Let A, B be two events. Define

"may depend on ω"

$$A \circ B = \left\{ \omega \in \{0,1\}^E : \exists\, I, J \text{ disjoint s.t. } \begin{array}{l} I \text{ wit. } A \text{ in } \omega \\ J \text{ wit. } B \text{ in } \omega \end{array} \right\}.$$

When $A \circ B$ occurs, we say that A and B <u>occur disjointly</u>.

Examples • $\{e \text{ is open}\} \circ \{f \text{ is open}\} = \begin{cases} \emptyset & \text{if } e = f \\ \{e, f \text{ open}\} & \text{if } e \neq f \end{cases}$.

• $\{x \leftrightarrow y\} \circ \{y \leftrightarrow z\}$ " = " 

Rk: We always have $A \circ B \subset A \cap B$.

Sometimes, we may have $A \circ B = A \cap B$. For example, this equality holds if A and B depend on disjoint set of edges, or if A is $\uparrow$ and B is $\downarrow$. (exercice).

Exercise: Let $A, B \uparrow$. prove that

$$A \circ B = \left\{ \omega : \exists I, J \text{ disjoint } \underline{\text{open}} \text{ s.t. } \begin{array}{l} I \text{ wit. } A \text{ in } \omega \\ J \text{ wit. } B \text{ in } \omega \end{array} \right\}.$$

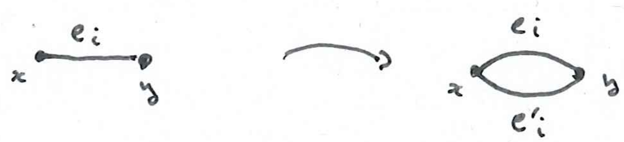
"a set $I \subset E$ is open if all its edges are open"

Thm: (BK - Reimer inequality)

Let A, B be two events (depending on finitely many edges)
Then
$$\boxed{\mathbb{P}_p[A \circ B] \leq \mathbb{P}_p[A]\, \mathbb{P}_p[B].}$$

Rk: Proved by Van den Berg and Kesten for increasing events, extended to general events by Reimer using a different approach. In this course, we present the proof for increasing events.

Proof: Let $A, B \subset \{0,1\}^E$ increasing*. Write $E = \{e_1, \dots, e_m\}$.
We use a construction where the edges are "duplicated": for each edg e_i we add a parallel edge e_i'



* in all the proof the sets witnessing A or B are always assumed to be open.

We then consider independent percolation on the resulting graph. This amounts to consider two copies of the space,

$$\omega = (\omega_1, \dots, \omega_m) \quad \text{and} \quad \omega' = (\omega'_1, \dots, \omega'_m)$$

where $\omega_i = \omega(e_i)$, $\omega'_i = \omega'(e_i)$. We write $\bar{\omega} = (\omega, \omega')$ and $\bar{\mathbb{P}}_p$ the corresponding product measure. Introduce for $0 \leq i \leq m$

$$\omega^{(i)} = (\omega'_1, \dots, \omega'_i, \omega_{i+1}, \dots, \omega_m).$$

interpolating between $\omega^{(0)} = \omega$ and $\omega^{(m)} = \omega'$.

Let

$$\bar{A}_i = \{\bar{\omega} : \omega^{(i)} \in A\} \quad \text{and} \quad \bar{B} = \{\bar{\omega} : \omega \in B\}.$$

We write $\bar{E} = E \cup E'$ for the set of all duplicated edges. The disjoint occurrence is well define on $\{0,1\}^{\bar{E}}$ and we have

$$\bar{\mathbb{P}}_p[\bar{A}_m \circ \bar{B}] = \bar{\mathbb{P}}_p[\bar{A}_m]\, \bar{\mathbb{P}}_p[B] = \mathbb{P}_p[A]\, \mathbb{P}_p[B]$$

(since $\bar{A}_0$ and $\bar{B}$ depend on disjoint set of edges)

and

$$\bar{\mathbb{P}}_p[\bar{A}_0 \circ \bar{B}] = \mathbb{P}_p[A \circ B]$$

(since $\bar{A}_0$ and $\bar{B}$ are only defined in term of ω).

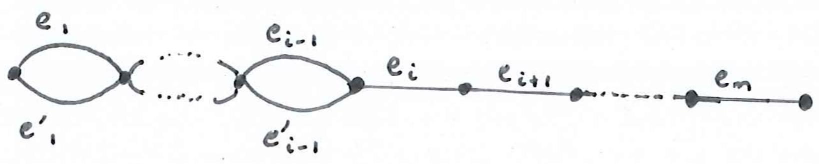
Hence, BK inequality can be rewritten as.

$$\bar{\mathbb{P}}_p[\bar{A}_0 \circ \bar{B}] \leq \mathbb{P}_p[\bar{A}_m \circ \bar{B}].$$

It suffices to show that for every $1 \leq i \leq m$

$$\bar{\mathbb{P}}_p[\bar{A}_{i-1} \circ \bar{B}] \leq \mathbb{P}[\bar{A}_i \circ \bar{B}].$$

$\overline{A}_{i-1} \circ \overline{B}$



$\overline{A}_i \circ \overline{B}$

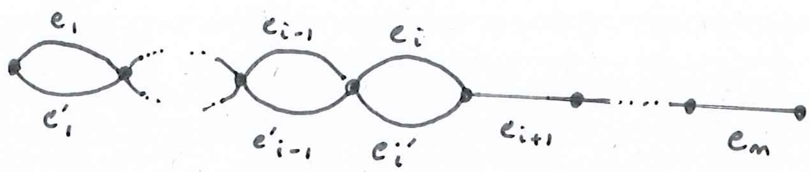


Illustration of the edges "used" by $\overline{A}_{i-1} \circ \overline{B}$ and $\overline{A}_i \circ \overline{B}$:

We wish to decompose the event $\overline{A}_{i-1} \circ \overline{B}$. Define

$$C_1 = \left\{ \exists\, I, J \subset \overline{E} \setminus \{e_i, e'_i\} \text{ disj open s.t. } \begin{array}{l} I \text{ wit. } \overline{A}_{i-1} \text{ in } \overline{\omega} \\ J \text{ wit. } \overline{B} \text{ in } \overline{\omega} \end{array} \right\}$$

" $A_{i-1} \circ B$ occurs regardless of ω_i "

$$C_2 = C_1^c \cap \left\{ \exists\, I, J \subset \overline{E} \setminus \{e_i, e'_i\} \text{ disj. open s.t. } \begin{array}{l} I \text{ wit } \overline{A}_{i-1} \text{ in } \overline{\omega} \\ J \cup \{e_i\} \text{ wit. } \overline{B} \text{ in } \overline{\omega}^{e_i} \end{array} \right\}$$

" if we set $\omega(e_i) = 1$, then one can find a witness for $\overline{B}$ using e_i and a disjoint witness for $\overline{A}_{i-1}$.
if we set $\omega(e_i) = 0$ then $\overline{A}_{i-1} \circ \overline{B}$ does not hold "

$$C_3 = C_1^c \cap C_2^c \cap \left\{ \exists\, I, J \subset \overline{E} \setminus \{e_i, e'_i\} \text{ disj. open s.t. } \begin{array}{l} I \cup \{e_i\} \text{ wit. } \overline{A}_{i-1} \text{ in } \overline{\omega}^{e_i} \\ J \text{ wit. } \overline{B} \text{ in } \overline{\omega} \end{array} \right\}$$

" the only way $\overline{A}_{i-1} \circ \overline{B}$ can occur is that $\omega(e_i) = 1$ and the witness for $\overline{A}_{i-1}$ uses e_i ".

Observe that C_1, C_2, C_3 are independent of ω_i and ω_i'.

Furthermore, $\overline{A}_{i-1} \circ \overline{B}$ can be decomposed as the disjoint union

$$\overline{A}_{i-1} \circ \overline{B} = C_1 \sqcup C_2 \cap \{\omega_i = 1\} \sqcup C_3 \cap \{\omega_i = 1\}$$

And therefore

$$\overline{\mathbb{P}}_p[\overline{A}_{i-1} \circ \overline{B}] = \overline{\mathbb{P}}_p[C_1] + p\left(\overline{\mathbb{P}}_p[C_1] + \overline{\mathbb{P}}_p[C_2]\right).$$

In the same way, one can check that

$$\left(C_1 \sqcup C_2 \cap \{\omega_i = 1\} \sqcup C_3 \cap \{\omega_i' = 1\}\right) \subset \overline{A}_i \circ \overline{B}.$$

(This is only an inclusion here because we do not consider the case when both e_i and e_i' are used to realize $\overline{A}_i \circ \overline{B}$.)

Hence

$$\mathbb{P}_p[C_1] + p\left(\mathbb{P}_p[C_2] + \mathbb{P}_p[C_3]\right) \leq \mathbb{P}_p[\overline{A}_i \circ \overline{B}],$$

which concludes the proof. ∎

<u>Applications for percolation on $(\mathbb{Z}^d, E)$</u>

<u>Appl. 1</u> $\mathbb{P}_p\Big[$ 

$\Big] \leq \mathbb{P}_p[x \longleftrightarrow y]\, \mathbb{P}_p[y \longleftrightarrow z]$

<u>Not</u>: for $S \subset \mathbb{Z}^d$ write $A \overset{S}{\longleftrightarrow} B$ for the event that there exists an open path from A to B, all the vertices of which belong to S.

"proof": Let $n \geq 1$. By BK-inequality,

$$P_p\left[\{x \overset{\Lambda_n}{\longleftrightarrow} y\} \circ \{y \overset{\Lambda_n}{\longleftrightarrow} z\}\right] \leq P_p\left[x \overset{\Lambda_n}{\longleftrightarrow} y\right] P_p\left[y \overset{\Lambda_n}{\longleftrightarrow} z\right]$$

and we obtain the result by letting n tend to infinity. ∎

2.5 INVARIANCE, MIXING PROPERTY AND ERGODICITY.

$G = (\mathbb{Z}^d, E)$.

$\mathbb{Z}^d$ (additive group) acts on . $\mathbb{Z}^d$ by translation $z \cdot x = z + x$

. E : $z \cdot \{x, y\} = \{z+x, z+y\}$.

. $\{0,1\}^E$: $(z \cdot \omega)(\{x,y\}) = \omega(\{x-z, y-z\})$

. $\mathcal{F}$: $z \cdot A = \{z \cdot \omega, \omega \in A\}$.

NB: (e open in ω) $\Leftrightarrow$ ($z \cdot e$ open in $z \cdot \omega$)

Ex: If $A = x \leftrightarrow y$, then $z \cdot A = \{z + x \longleftrightarrow z + y\}$.

Prop. For every event A and every $z \in \mathbb{Z}^d$, we have

$$P_p[z \cdot A] = P_p[A].$$ "P_p is invariant"

Proof: True for cylinder events. Conclude with monotone class theorem.

Appli: $P_p[0 \longleftrightarrow \infty] = P[z \longleftrightarrow \infty]$

Prop. [MIXING PROPERTY]

Let A, B be two events. Then

$$\lim_{|z|\to\infty} P_p[A \cap z.B] = P_p[A]\,P_p[B]$$

Proof: Let $\varepsilon > 0$. Choose $A_\varepsilon, B_\varepsilon$ depending on finitely many edges such that

$$P_p[A \,\Delta\, A_\varepsilon] \leq \varepsilon \quad \text{and} \quad P_p[B \,\Delta\, B_\varepsilon] \leq \varepsilon.$$

"symmetric difference"

By independence, if $|z|$ is large enough, we have

$$P_p[A_\varepsilon \cap z.B_\varepsilon] = P_p[A_\varepsilon]\cdot P_p[z.B_\varepsilon]$$

$$= P_p[A_\varepsilon]\,P_p[B_\varepsilon]$$

invariance.

Therefore, if $|z|$ large enough

$$P_p[A \cap z.B] \leq P_p[A_\varepsilon \cap z.B_\varepsilon] + 2\varepsilon$$

$$= P_p[A_\varepsilon]\,P_p[B_\varepsilon] + 2\varepsilon$$

$$\leq P_p[A]\,P_p[B] + 4\varepsilon.$$

Equivalently, $P_p[A \cap z.B] \geq P_p[A]\,P_p[B] - 4\varepsilon$, which concludes the proof. ■

<u>Application</u> : $\mathbb{P}[0 \leftrightarrow \infty, z \leftrightarrow \infty] \xrightarrow[|z| \to \infty]{} \mathbb{P}_p[0 \leftrightarrow \infty]^2 \quad (= \theta(p)^2)$

<u>Prop.</u> [ERGODICITY]

Let A be an invariant event (ie $\forall z \in \mathbb{Z}^d \quad z.A = A$).
Then
$$\mathbb{P}_p[A] \in \{0, 1\}.$$

<u>Proof</u>: By invariance of A, $\mathbb{P}_p[A] = \mathbb{P}[A \cap z \cdot A]$. Hence

$$\mathbb{P}_p[A] = \lim_{|z| \to \infty} \mathbb{P}_p[A \cap z.A] \underset{\substack{\uparrow \\ \text{Mixing}}}{=} \mathbb{P}_p[A]^2 \qquad \blacksquare$$

<u>Application</u>: Let $N(\omega)$ be the number of disjoint infinite clusters in ω. Then $\forall k \in \mathbb{N} \cup \{\infty\}$

$$\mathbb{P}_p[N = k] \in \{0, 1\}.$$

CHAPTER 2 :

SUBCRITICAL PERCOLATION.

$G = (\mathbb{Z}^d, E)$, $d \geq 2$.

1. PHASE TRANSITION

Not. : $\theta(p) = \mathbb{P}_p[0 \longleftrightarrow \infty]$, $\theta_n(p) = \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n]$.

Rk: $\theta : [0,1] \to [0,1]$ is non-decreasing.

Exercice: Prove that θ is right continuous. (Hint: use $\theta = \lim_{n\to\infty} \theta_n$)

Def. The critical parameter for Bernoulli percolation is defined by

$$p_c = \sup\{p \in [0,1] : \theta(p) = 0\}.$$

Rk: We have already seen in the introduction that $0 < p_c < 1$.

Question: We know that for $p < p_c$ $\theta_n(p) \xrightarrow[n\to\infty]{} 0$. At which speed?

2. EXPONENTIAL DECAY

In this section, our goal is to prove the following theorem.

Thm [AIZENMAN-BARSKY, MENSHIKOV, '87]

(i) $\forall p < p_c$ $\exists c = c(p)$ s.t. $\forall n \geq 1$

$$\theta_n(p) \leq e^{-cn}.$$

(ii) $\forall p \geq p_c$ $\theta(p) \geq \frac{1}{2}(p - p_c)$. "mean field lower bound".

The bound (ii) is sharp in the following sense. For Bernoulli percolation on a tree or on $\mathbb{Z}^d$, $d \geq 6$, we expect $\theta(p) \underset{p \to p_c}{\sim} C(p-p_c)$. This is known for the tree, and the upper bound $\theta(p) \leq C(p-p_c)$ is known for $\mathbb{Z}^d$, $d \geq 11$. On $\mathbb{Z}^2$ we will see that $\theta(p) \geq c(p-p_c)^\gamma$ $\gamma < 1$.

$\boxed{\text{Def: Let } S \subset \mathbb{Z}^d \text{ finite s.t. } 0 \in S. \text{ Introduce } \quad \phi_p(S) = \sum_{xy \in \Delta S} p \cdot \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x]}$

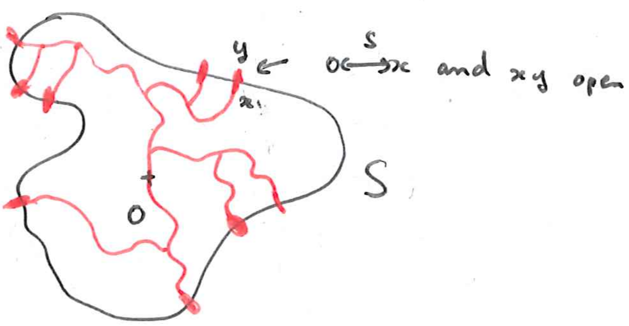
Convention: if $0 \notin S$ we set $\phi_p(S) = 0$.

<u>Geometric interpretation:</u>

$$\phi_p(S) = \sum_{xy \in \Delta S} \mathbb{P}_p[xy \text{ open}]\, \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x]$$

$$\overset{\text{indep.}}{=} \sum_{xy \in \Delta S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x,\ xy \text{ open}]$$

$$= \mathbb{E}\Big[\sum_{xy \in \Delta S} \mathbb{1}_{0 \overset{S}{\longleftrightarrow} x,\ xy \text{ open}} \Big]$$



$$\phi_p(S) = \mathbb{E}_p\Big[\text{"number of open edges through which one can exit } S \text{ starting at } 0\text{"} \Big]$$

Lemma 1. Let $S \subset \mathbb{Z}^d$ finite s.t. $0 \in S$. Assume that

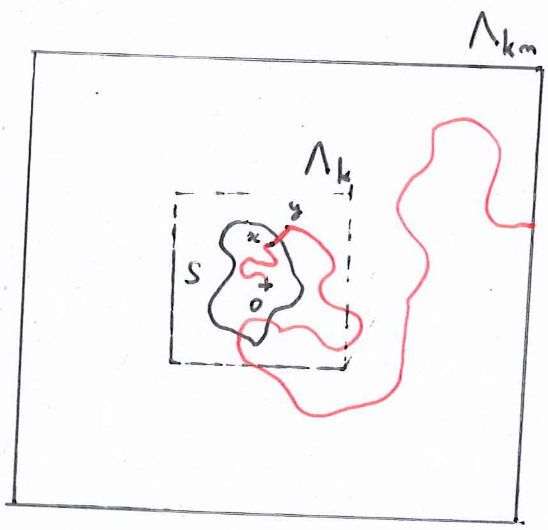
$$\phi_p(S) < 1$$

Then there exists $c > 0$ s.t.

$$\forall n \geq 1 \quad \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}.$$

Proof: Let k large enough s.t. $S \subset \Lambda_k$.

If $0 \longleftrightarrow \partial\Lambda_{kn}$ occurs, then there exists an edge xy at the boundary of S s.t. $\{0 \longleftrightarrow x, xy \text{ open}\}$ and $\{y \longleftrightarrow \partial\Lambda_{kn}\}$ occur disjointly. (To see this, consider the first traversed edge xy at the boundary of S, when following an open path from 0 to $\partial\Lambda_{kn}$).



$\{0 \longleftrightarrow x, xy \text{ open}\}$ and $\{y \longleftrightarrow \partial\Lambda_{kn}\}$ occur disjointly when there exists an open path from 0 to $\partial\Lambda_{kn}$.

By the union bound and BK-inequality, we find

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{km}] \leq \sum_{xy \in \Delta S} \mathbb{P}_p[\{0 \overset{S}{\longleftrightarrow} x, xy \text{ open}\} \circ \{y \longleftrightarrow \partial\Lambda_{km}\}]$$

$$\overset{BK}{\leq} \sum_{xy \in \Delta S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x, xy \text{ open}] \underbrace{\mathbb{P}_p[y \longleftrightarrow \partial\Lambda_{km}]}_{\leq \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{k(m-1)}]}$$

"translation invariance" (see 2.5)

$$\leq \phi_p(S) \cdot \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{k(m-1)}]$$

By induction, we obtain, for every $n \geq 1$

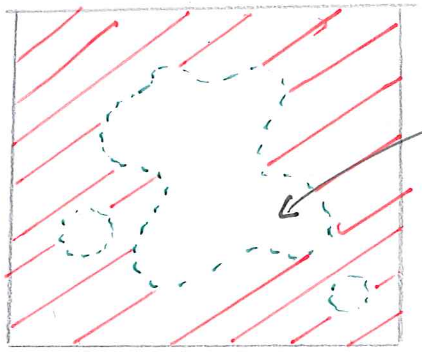
$$P[0 \longleftrightarrow \partial\Lambda_{km}] \leq \phi_p(S)^m$$

$\blacksquare$

<u>Rk</u>: The percolation cluster C_0 is "smaller" than a subcritical branching process when $\phi_p(S) < 1$. $\phi_p(S)$ can be interpreted as the "progeny" of C_0 at the boundary of S.

<u>Lemma 2</u>. Consider the random set $\mathcal{S}_m = \{x \in \Lambda_m : x \not\longleftrightarrow \partial\Lambda_m\}$.
Then for every fixed $m \geq 1$ and every $p \in [0,1]$,

$$\theta_m'(p) \geq \frac{1}{p(1-p)} \mathbb{E}_p[\phi_p(\mathcal{S}_m)].$$



$\mathcal{S}_n$ is the the set of points that are not connected to $\partial\Lambda_n$; it can be seen as the complement of all the clusters touching $\partial\Lambda_n$

<u>Proof:</u> Let E_n be the edges between vertices in Λ_n. Apply Russo's to $A = 0 \leftrightarrow \partial\Lambda_n$ to get.

$$\theta_n'(p) = \sum_{e\in E_n} \mathbb{P}_p[e \text{ is piv. for } A]$$

$$\underset{\uparrow}{=} \frac{1}{1-p}\sum_{e\in E_n} \mathbb{P}_p[e \text{ piv. for } A, e \text{ closed}]$$

$\{e \text{ is piv}\}$ indep. of $\omega(e)$

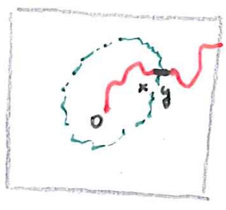
$$= \frac{1}{1-p}\sum_{e\in E_n} \mathbb{P}_p[\{e \text{ piv. for } A\}\cap A^c]$$

Now we use the partition $A^c = \bigsqcup_{\substack{S\subset\Lambda_n:\\ 0\in S}} \{\mathcal{S}_n = S\}$.

$$\theta_n' = \frac{1}{1-p}\sum_{\substack{S\subset\Lambda_n\\ 0\in S}}\sum_{e\in E_n} \mathbb{P}_p[e \text{ piv. for } A, \mathcal{S}_n = S]$$

Observation: An edge e is pivotal for A if

- one of its extremity x is connected to 0
- the other extremity y is connected to $\partial\Lambda_n$
- 0 is not connected to Λ_n in ω_e.



On the event $\mathcal{S} = S$, one sees that an edge e is piv if and only if

- $e \in \Delta S$
- one extremity of e is connected to 0 in S.

Hence,

$$\theta_n'(p) = \frac{1}{1-p} \sum_{\substack{S \subset \Lambda_n \\ 0 \in S}} \sum_{xy \in \Delta S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x, \mathcal{S}_n = S]$$

The event $\mathcal{S}_n = S$ is measurable with respect to the edges adjacent to at least one edge in S^c, while the event $0 \overset{S}{\longleftrightarrow} x$ is measurable with respect to the edges with both extremities in S. Hence, these two events are independent, and we obtain.

$$\theta_n'(p) = \frac{1}{1-p} \sum_{\substack{S \subset \Lambda_n \\ 0 \in S}} \underbrace{\Big(\sum_{xy \in \Delta S} \mathbb{P}_p[0 \longleftrightarrow x] \Big)}_{= \frac{1}{p} \phi_p(S)} \mathbb{P}_p[\mathcal{S}_n = S]$$

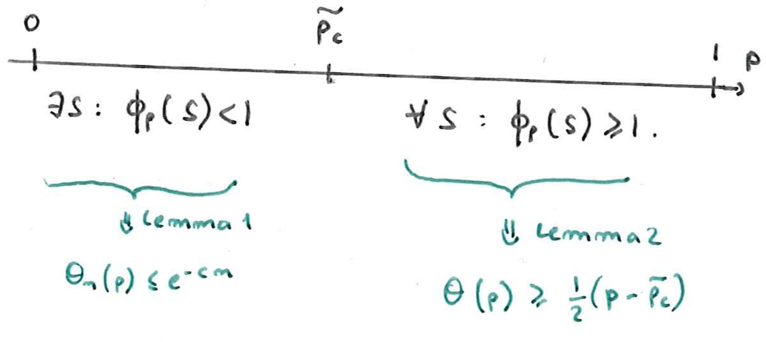
$$= \frac{1}{p(1-p)} \mathbb{E}_p[\phi_p(\mathcal{S}_n)] .$$

■

<u>Proof of the theorem.</u>

A set S is always assumed to satisfy $|S| < \infty$ $0 \in S$.

Introduce $\tilde{p}_c = \sup\{p \in [0,1] : \exists S \subset \mathbb{Z}^d \ \phi_p(S) < 1\}$.



By lemma 1, we have $\forall p < \tilde{p_c}$ $\exists c > 0 : \forall n$ $\theta_n(p) \leq e^{-cn}$.

By lemma 2, we have $\forall p \geq \tilde{p_c}$ and $\forall n$

$$\theta_n' \geq \frac{1}{p(1-p)} \mathbb{E}_p[\mathbb{1}_{0 \in \mathcal{S}_n}]$$
$$\geq 1 - \theta_n$$

Fix $p > \tilde{p_c}$. If $\theta_n(p) \geq \frac{1}{2}$ we also have $\theta_n(p) \geq \frac{1}{2}(p - \tilde{p_c})$
Otherwise we have $\forall q \in [\tilde{p_c}, p]$ $\theta_n' \geq \frac{1}{2}$ and we obtain $\theta_n(p) \geq \frac{1}{2}(p - p_c)$ by integrating between $\tilde{p_c}$ and p.

Conclusion: $\forall n$ $\forall p > \tilde{p_c}$ $\theta_n(p) \geq \frac{1}{2}(p - \tilde{p_c})$ and we obtain $\forall p > \tilde{p_c}$ $\theta(p) \geq \frac{1}{2}(p - \tilde{p_c})$ by letting n tend to infinity. This concludes that $p_c = \tilde{p_c}$, and finishes the proof of the theorem.

Remarks:

1. Lemma 2 actually gives $\forall p > \tilde{p_c}$ $\theta_n' \geq \frac{1}{p(1-p)}(1 - \theta_n)$, which can be integrated between $\tilde{p_c}$ and p to prove the stronger bound $\theta(p) \geq \frac{1}{p(1-\tilde{p_c})}(p - \tilde{p_c})$.

2. $\{p : \exists S \ \phi_p(S) < 1\} = \bigcup_{\substack{S \subset \mathbb{Z}^d \text{ finite} \\ 0 \in S}} \{p : \phi_p(S) < 1\}$ is open.

In particular p_c does not belong to this set and we have $\forall S \subset \mathbb{Z}^d$ finite s.t. $0 \in S$, $\phi_{p_c}(S) \geq 1$.

This implies $\mathbb{E}_{p_c}[|C_0|] \geq \sum_n \phi_p(\Lambda_n) = +\infty$.

3. Since $\phi_p(\{0\}) = 2dp$, we see that $p_c(d) \geqslant \frac{1}{2d}$.

3. Correlation length.

We have seen $\forall p < p_c \quad p^n \leq \theta_n(p) \leqslant e^{-cn} \quad (c > 0 \text{ constant})$

→ can we obtain a more precise estimate?

Thm [Definition of the correlation length]

Let $e_1 = (1, 0, \dots, 0)$. Let $p \in [0,1]$. The quantity

$$\boxed{\xi(p) = \left(\lim_{n \to \infty} -\frac{1}{n} \log\left(\mathbb{P}_p[0 \longleftrightarrow n e_1]\right)\right)^{-1}}$$

is well defined and finite for $p < p_c$.

Def: $\xi(p)$ is called the correlation length.

Lemma (Fekete's lemma).

Let $(u_n)_{n \geq 0}$ be a sequence of numbers in $[-\infty, \infty)$ satisfying

$$\forall m, n \geqslant 0 \quad u_{m+n} \leqslant u_m + u_n \qquad \text{"subadditivity"}$$

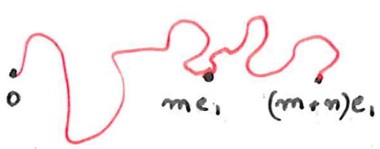
Then the limit of $\left(\frac{u_n}{n}\right)$ exists in $[-\infty, \infty)$ and

$$\lim_{n \to \infty} \frac{u_n}{n} = \inf_{n \geqslant 0} \left(\frac{u_n}{n}\right).$$

Proof of the theorem.

By FKG inequality, we have $\forall m, n \geqslant 0$

$$\mathbb{P}[0 \longleftrightarrow (m+n)e_1] \geqslant \mathbb{P}_p[0 \longleftrightarrow m e_1] \times \underbrace{\mathbb{P}_p[m e_1 \longleftrightarrow (m+n)e_1]}_{\substack{= \mathbb{P}[0 \longleftrightarrow n e_1] \\ \text{translation invariance}}}$$



Hence, $u_n = -\log\left(\mathbb{P}_p[0 \longleftrightarrow n e_1]\right)$ is subadditive and Fekete's lemma concludes that $\left(\frac{u_n}{n}\right)$ converges towards $\inf\limits_{n \geq 0} \left(\frac{u_n}{n}\right)$. ■

Rk: The definition of $\xi(p)$ can be also rewritten as

$$\mathbb{P}_p[0 \longleftrightarrow n e_1] = e^{-n/\xi(p) + o(n)}.$$

Prop: Let $p < p_c$, $\exists c, C > 0$ s.t. $\forall n \geq 1$

$$\boxed{\frac{1}{C n^{d-1}} e^{-\frac{n}{\xi(p)}} \leq \theta_n(p) \leq C n^{d-1} e^{-n/\xi(p)}}$$

Proof: [Upper bounds] Fix $p < p_c$.

$$\forall n \quad \frac{u_n}{n} \geq \inf_{n \geq 0} \frac{u_n}{n} = \frac{1}{\xi(p)}$$

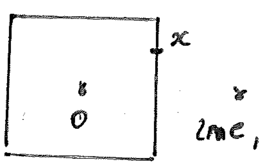
ie $\forall n \quad \mathbb{P}_p[0 \longleftrightarrow n e_1] \leq e^{-n/\xi(p)}$.

By symmetry we can pick $x \in \partial\Lambda_n$ s.t. $x_1 = n$ and

$$\mathbb{P}_p[0 \longleftrightarrow x] = \max_{y \in \partial\Lambda_n} \mathbb{P}_p[0 \longleftrightarrow y].$$

By invariance of $\mathbb{P}_p$ with respect to the reflexion in $\{n\} \times \mathbb{Z}^{d-1}$,

$$\mathbb{P}_p[0 \longleftrightarrow x] = \mathbb{P}_p[x \longleftrightarrow 2n e_1]$$



By FKG inequality

$$\mathbb{P}_p[0 \longleftrightarrow 2ne_1] \geqslant \mathbb{P}_p[0 \longleftrightarrow x]\,\mathbb{P}_p[x \longleftrightarrow 2ne_1]$$
$$= \mathbb{P}_p[0 \longleftrightarrow x]^2$$

Finally

$$\theta_n(p) = \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n]$$
$$\leqslant \sum_{y\in\partial\Lambda_n} \mathbb{P}_p[0 \longleftrightarrow y]$$
$$\leqslant |\partial\Lambda_n|\, \mathbb{P}_p[0 \longleftrightarrow x]$$
$$\leqslant |\partial\Lambda_n|\, \mathbb{P}_p[0 \longleftrightarrow 2ne_1]^{\frac{1}{2}}$$
$$\leqslant C n^{d-1} e^{-n/\xi(p)}.$$

[Lower bound.]

For $1 \leq m \leq n$, we have

$$\theta_{m+n}(p) \overset{\text{indep.}}{\leqslant} \underbrace{\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_m]}_{=\theta_m(p)}\ \underbrace{\mathbb{P}_p[\partial\Lambda_m \longleftrightarrow \partial\Lambda_{m+n}]}_{\leq \sum_{x\in\partial\Lambda_m} \mathbb{P}_p[x \longleftrightarrow \partial\Lambda_{m+n}] \leqslant 3^{d-1} m^{d-1}\theta_n(p)}$$

Set $C = 6^{d-1}$

$$C(m+n)^{d-1}\theta_{m+n}(p) \leq C \times (2n)^{d-1} \times \theta_m(p) \times 3^{d-1} m^{d-1} \theta_n(p)$$
$$\leqslant \left(C\, m^{d-1}\theta_m(p)\right)\left(C\, n^{d-1}\theta_n(p)\right)$$

Hence the sequence $v_n = \log\left(C\, n^{d-1}\theta_n(p)\right)$ is subadditive. By Fekete's lemma, we have for every n

$$\frac{v_n}{n} \geq \lim_{n\to\infty} \frac{v_n}{n} = -\frac{1}{\xi(p)}$$

$$\mathbb{P}_p[0 \leftrightarrow n e_1] \leq \theta_n(p) \leq Cte\; n^{d-1}\, e^{-n/\xi(p)}$$

Hence, for every $n \geq 1$

$$\theta_n(p) \geq \frac{1}{C n^{d-1}}\, e^{-n/\xi(p)}.$$

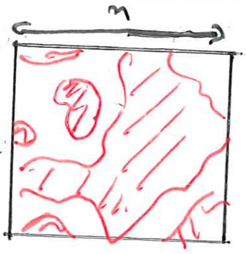
<u>Exercise:</u> Let $p < p_c$. Prove that $\exists\, c > 0$ s.t.

$$\frac{c\, e^{-\frac{\|x\|_\infty}{\xi(p)}}}{\|x\|_\infty^{4d(d-1)}} \leq \mathbb{P}_p[0 \longleftrightarrow x] \leq e^{-\frac{\|x\|_\infty}{\xi(p)}}$$

<u>Rk:</u> More precise estimates, known as Ornstein-Zernike estimates state that $\exists\, c = c(p) > 0$ s.t.

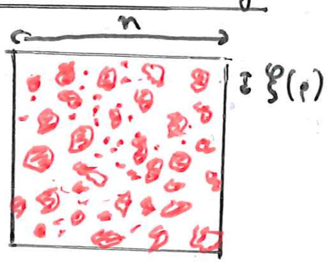
$$\mathbb{P}_p[0 \longleftrightarrow n e_1] = \frac{c}{n^{\frac{d-1}{2}}}\, e^{-(n/\xi(p))}\left(1 + o_{n\to\infty}(1)\right).$$

<u>Geometric intuition concerning the correlation length.</u>



$n \leq \xi(p)$

→ looks critical.



$n \gg \xi(p)$

→ really looks subcritical.

Prop: (Analytic properties of ξ)

$\xi : [0, p_c] \longrightarrow [0, \infty]$ is continuous non-decreasing and satisfies $\xi(0) = 0$, $\xi(p_c) = +\infty$.

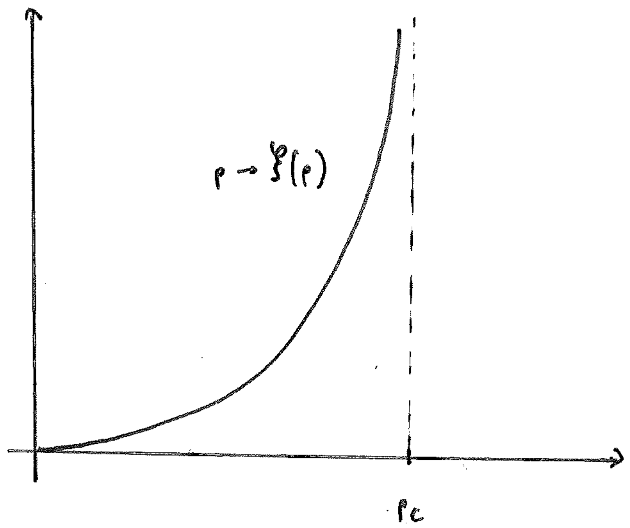
Proof:

$$\frac{1}{\xi(p)} = \sup_{n \geq 1} \frac{-\log\left(C n^{d-1} \theta_n(p)\right)}{n} \Rightarrow \frac{1}{\xi} \text{ is lower semi continuous.}$$

$$\frac{1}{\xi(p)} = \inf_{n \geq 1} \frac{-\log\left(\theta_n(p) / C n^{d-1}\right)}{n} \Rightarrow \frac{1}{\xi} \text{ is upper semi-continuous.}$$

ξ is nondecreasing as a limit of nondecreasing functions.

$\xi(p_c) = +\infty$ follows from the fact that $\theta_n(p_c) \geq \frac{c}{n^{d-1}}$. ∎



Exercise: Prove that ξ is (strictly) increasing on $[0, p_c]$.

Def: (alternative definition of the correlation length.)

For $p \in [0, p_c]$, define

$$\overline{\xi}(p) = \min\left\{ k \geq 1 : P_p\left[\Lambda_k \longleftrightarrow \partial \Lambda_{2k-1}\right] \leq (2de)^{-3^d} \right\}$$

Prop: There exists $c = c(d) > 0$ s.t. $\forall p \in [0, p_c]$

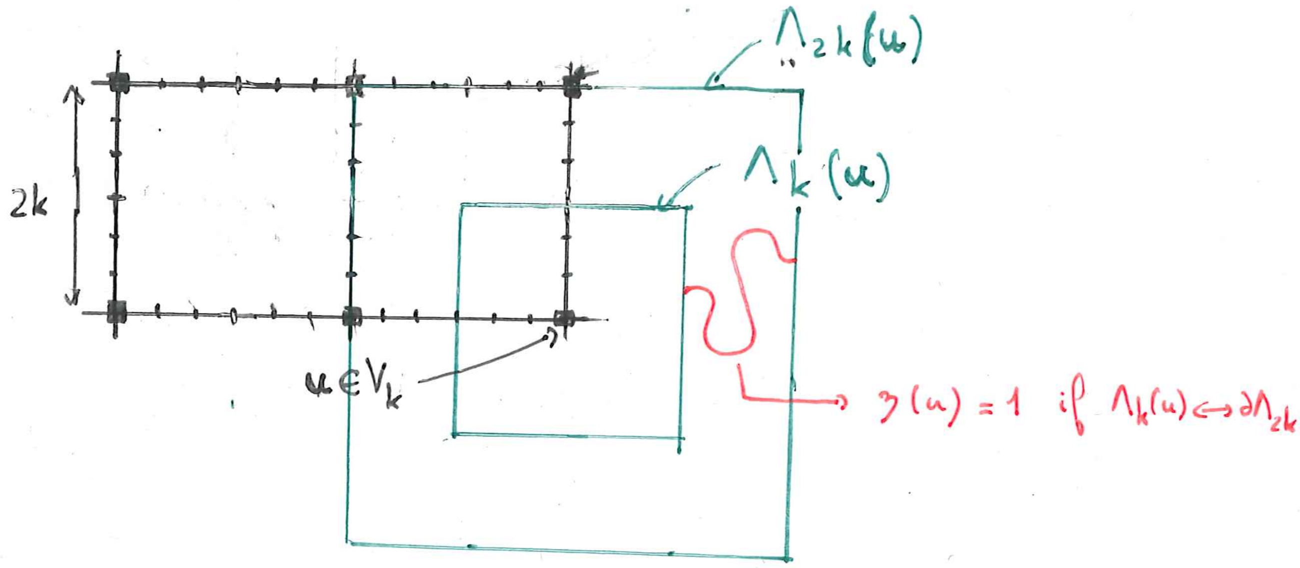
$$\boxed{\frac{1}{2}\xi(p) \leq \overline{\xi}(p) \leq 1{+}C\,\xi(p)\log(2+\xi(p))}$$

Proof: Set $k = \overline{\xi}(p)$. In particular, we have

$$\mathbb{P}_p[\Lambda_k \longleftrightarrow \partial\Lambda_{2k-1}] \leq (2de)^{-3^d}.$$

Consider the graph G_k with vertex set $V_k = 2k.\mathbb{Z}^d$ and edge set

$$E_k = \{\{2kx, 2ky\},\ \|x-y\|_1 = 1\}.$$



Consider the random variable $\eta \in \{0,1\}^{V_k}$ defined by

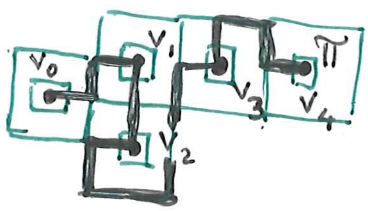
$$\eta(u) = \begin{cases} 0 & \text{if } \Lambda_k(u) \not\longleftrightarrow \partial\Lambda_{2k}(u), \\ 1 & \text{if } \Lambda_k(u) \longleftrightarrow \partial\Lambda_{2k}(u). \end{cases}$$

If $0 \longleftrightarrow \partial\Lambda_{2k(N+1)}$, then there exists a path $\pi = (u_0, u_1, \dots, u_N)$ in G_k from 0 s.t.

$$\forall i \in \{0, \dots, N\}\quad \eta(u_i) = 1. \quad (\pi \text{ is } \eta\text{-open})$$

Fix π a path of length N in G_k. There exists $m \geq \frac{N}{3^d}$ and $v_1, \dots, v_m$ vertices of π such that

$$i \neq j \Rightarrow \|v_i - v_j\|_\infty \geq 4k.$$



To see this, one can for example set $v_0 = u_0$, then, by induction, we set v_{i+1} to be the first vertex along π that does not belong to

$$\bigcup_{j \leq i} \Lambda_{2k}(v_j).$$

$$\mathbb{P}_p[\pi \text{ is } \eta\text{-open}] \leq \mathbb{P}_p[\forall i \leq m \quad \eta(v_i) = 1]$$

$$\overset{\text{independence}}{=} \prod_{i \leq m} \mathbb{P}_p[\eta(v_i) = 1]$$

$$\leq \left(\frac{1}{2de}\right)^{3^d m} \leq \left(\frac{1}{2d}\right)^N e^{-N}.$$

Finally, $\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{2k(N+1)}] \leq \sum_{\substack{\pi \text{ path of length } N \\ \text{in } G_k \text{ from } 0.}} \mathbb{P}_p[\pi \text{ is } \eta\text{-open}]$

$$\leq (2d)^N \times \frac{1}{(2d)^N} e^{-N}$$

$$\leq e^{-N}$$

Therefore $\forall n \geq 1$ $\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-\left(\left\lfloor \frac{n}{2k} \right\rfloor - 2\right)}$

Which directly concludes $\xi \leq 2k$ ie $\frac{1}{2}\xi(p) \leq \overline{\xi}(p)$.

We now prove the upper bound. Let $k \geq 1$.

$$\mathbb{P}_p[\Lambda_k \leftrightarrow \partial\Lambda_{2k}] \leq \sum_{x \in \partial\Lambda_k} \underbrace{\mathbb{P}[x \leftrightarrow \partial\Lambda_k(x)]}_{= \theta_k(p)}$$

$$\leq \underbrace{C_0}_{\text{"constant depending only on } d\text{"}} k^{d-1} \cdot k^{d-1} e^{-k/\xi(p)}$$

Then, one can choose $C = C(d)$ large enough such that $\forall k \geq C \xi \log(2+\xi)$

$$C_0 k^{2(d-1)} e^{-k/\xi(p)} \leq \frac{1}{(2de)^{3^d}}.$$

This concludes that $\tilde{\xi} \leq 1 + C \xi \log(2+\xi)$

■

Exercise (another possible definition of the correlation length.)

For $p \in [0, p_c]$, let $\tilde{\xi}(p) = \min\{n \geq 1 : \phi_p(\Lambda_n) \leq \frac{1}{e}\}$

Prove that $\exists C = C(d)$ s.t.

$$\forall p \in [0, p_c] \quad \xi(p) \leq \tilde{\xi}(p) \leq 1 + C \xi(p) \log(2 + \xi(p)).$$

4 EXPONENTIAL DECAY IN VOLUME.

Rk: If $|C_0| \geq n$ then $\underset{\text{"}\ell^\infty\text{"}}{\operatorname{diam}}(C_0) \geq n^{1/d}$.

Therefore $\forall p < p_c \quad \exists c > 0$ s.t.

$$\forall n \geq 1 \qquad \mathbb{P}_p[|C_0| \geq n] \leq e^{-cn^{1/d}}$$

Q: Can we do better?

Thm: Let $p < p_c$. There exists $c > 0$ s.t.

$$\forall n \geq 1 \quad \mathbb{P}_p\left[|C_0| \geq n\right] \leq e^{-cn}.$$

Lemma: Let $\mathcal{A}_n = \{C \subset \mathbb{Z}^d \text{ s.t. } 0 \in C,\ C \text{ connected},\ |C| = n\}$. "animals of size n"
Then $\forall n \geq 0$

$$\#(\mathcal{A}_n) \leq 16^{dn}$$

Proof: For every $C \in \mathcal{A}_n$, we have

$$\mathbb{P}_p\left[C_0 = C\right] \geq p^{\#\{xy \in E : x \in C, y \in C\}} (1-p)^{|\Delta C|}$$

$$\geq p^{2dn} (1-p)^{2dn}.$$

Now, for $p = \frac{1}{2}$,

$$1 \geq \mathbb{P}_{\frac{1}{2}}\left[|C_0| = n\right] = \sum_{C \in \mathcal{A}_n} \mathbb{P}_{\frac{1}{2}}\left[C_0 = C\right] \geq \#(\mathcal{A}_n) \times 16^{-dn} \quad \blacksquare$$

Proof of the theorem.

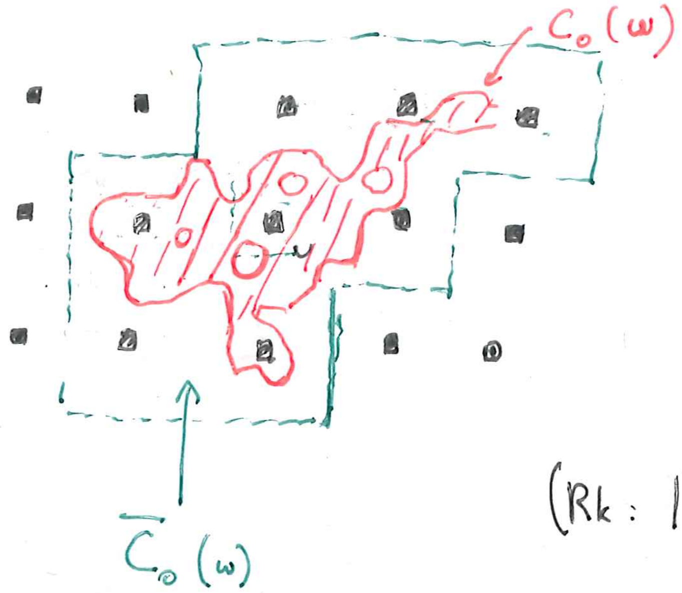
For $p < p_c$, we can choose $k \geq 1$ large enough s.t.

$$\mathbb{P}_p\left[\Lambda_k \longleftrightarrow \partial \Lambda_{2k}\right] \leq \left(\frac{1}{16^d e}\right)^{3^d}.$$

(k can be chosen "of the order" of the correlation length).

Consider $G_k = (V_k, E_k)$ and $\eta \in \{0,1\}^{V_k}$ as in Section 3.

Define $\overline{C_0}(\omega) = \{ u \in V_k : C_0(\omega) \cap \Lambda_k(u) \neq \emptyset \}$



(Rk: $|C_0| \leq |\Lambda_k| \times |\overline{C_0}|$)

Fix a set $\bar{C} \subset V_k$. One can find $v_1, \dots, v_m \in \bar{C} \setminus \{0\}$ s.t. $m \geq \frac{|C|}{3^d}$ and $i \neq j \Rightarrow \|v_i - v_j\| \geq 4k$.

If $|\bar{C}| \geq 2$, we have

$$\mathbb{P}_p[\overline{C_0} = \bar{C}] \leq \mathbb{P}_p[\forall u \in \bar{C} \ \ \eta(u) = 1]$$

$$\leq \mathbb{P}_p[\forall i \leq m \ \ \eta(v_i) = 1]$$

$$\underset{\text{independ.}}{=} \prod_{i=1}^{m} \underbrace{\mathbb{P}[\eta(v_i) = 1]}_{\substack{\leq 1 \\ \text{"translation invariance"}}}$$

$$\leq \left(\frac{1}{16^d e}\right)^{3^d m} \leq 16^{-d|C|} \times e^{-|C|}$$

Let $n \geq |\Lambda_k|$.

Then $\mathbb{P}_p[|C_0| \geq n] \leq \mathbb{P}_p\left[|\bar{C}_0| \geq \frac{n}{|\Lambda_k|}\right]$

$$= \sum_{N \geq n} \sum_{|\bar{c}| = N} \mathbb{P}_p[|\bar{C}_0| = \bar{c}]$$

$$\leq \sum_{N \geq \frac{n}{|\Lambda_k|}} \underbrace{\# \mathcal{A}_N \times 16^{-dN}}_{\leq 1} e^{-N}$$

$$\leq \frac{e}{e-1} \exp\left(- \frac{n}{|\Lambda_k|}\right)$$

■

CHAPTER 3 -

UNIQUENESS OF THE INFINITE CLUSTER.

$G = (\mathbb{Z}^d, E)$ $d \geq 2$

For $\omega \in \{0,1\}^E$, write $N(\omega)$ for the number of infinite clusters in the configuration ω.

Thm: Let $p \in [0,1]$.
Either $\mathbb{P}_p[N=0]=1$ or $\mathbb{P}_p[N=1]=1$.

Exercise: Deduce that.

$$N = \begin{cases} 0 \text{ a.s.} & \text{if } \theta(p) = 0 \\ 1 \text{ a.s} & \text{if } \theta(p) > 0. \end{cases}$$

1. PROOF OF THE THEOREM.

Let $p \in (0,1)$. By ergodicity $\exists k = k(p) \in \mathbb{N} \cup \{+\infty\}$ s.t.
$\mathbb{P}_p[N=k] = 1$.

Lemma 1: $\boxed{k \in \{0, 1, \infty\}.}$

Proof: Assume $1 < k < \infty$.
Let $\mathcal{F}_n = \left\{ \begin{array}{l} \Lambda_n \longleftrightarrow \infty \text{ and all the infinite clusters} \\ \text{intersect the box } \Lambda_n \end{array} \right\}$

For n large enough $\mathbb{P}_p[\mathcal{F}_n] \geq \frac{1}{2}$. (since $1 \leq N < \infty$ a.s.).

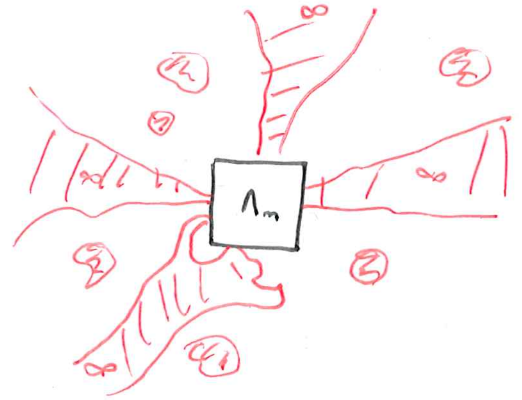


illustration of $\mathcal{F}_n$ for $N=4$.
(Notice that $\mathcal{F}_n$ is independent of the configuration in Λ_n).

$$P_p[N=1] \geq P_p[\mathcal{F}_n \cap \{\text{all the edges in } \Lambda_n \text{ are open}\}]$$

$$\underset{\text{independ.}}{=} P_p[\mathcal{F}_n] \cdot P_p[\text{all the edges in } \Lambda_n \text{ are open}]$$

$$> 0 \qquad \text{Contradiction to } N=k \text{ a.s.!} \qquad \blacksquare$$

<u>Definition</u>:

Let $\omega \in \{0,1\}^E$. A vertex $x \in \mathbb{Z}^d$ is called a <u>trifurcation</u> (in ω) if
- x has exactly 3 adjacent open edges.
- C_x splits into 3 disjoint infinite clusters if we close the edges adjacent to x.

<u>Not</u>: $T_x = \{x \text{ is a trifurcation}\}$.

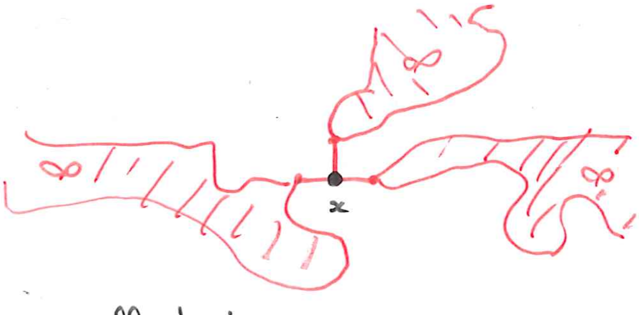


illustration of T_x.

Lemma 2 If $\mathbb{P}_p[N \geq 3] > 0$, then

$$\mathbb{P}_p[T_0] > 0.$$

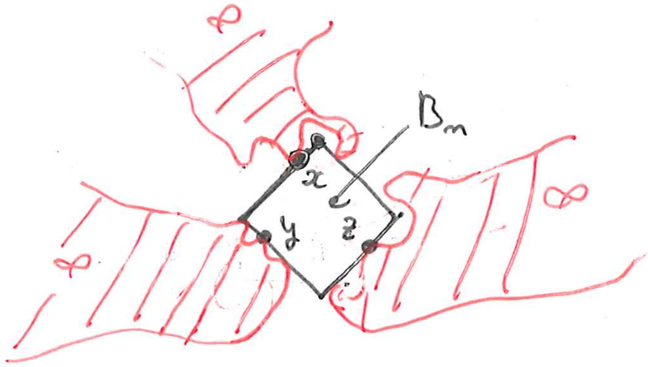
Proof: Let B_n be the ball of radius n centered at 0 for the L^1 distance in $\mathbb{Z}^d$. Pick $n \geq 3$ large enough s.t.

$$\mathbb{P}_p[E_n] > 0$$

where E_n is the event that at least 3 disjoint infinite clusters intersect B_n. We have

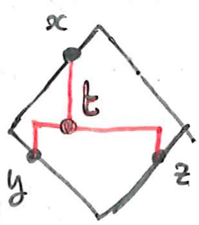
$$0 < \mathbb{P}_p[E_n] \leq \sum_{x,y,z \in \partial B_n} \mathbb{P}_p[E_n(x,y,z)]$$

where $E_n(x,y,z)$ is the event that outside B_n, the clusters of x, y and z are disjoint and infinite.



Let $x,y,z \in \partial B_n$ s.t. $\mathbb{P}_p[E_n(x,y,z)] > 0$.

On can check that there exist a deterministic vertex $t \in B_n \setminus \partial B_n$ and three disjoint paths γ_x, γ_y and γ_z in B_n s.t. γ_i connects t to i for every i



Let $F_n(x,y,z) = \{\gamma_x, \gamma_y, \gamma_z$ are open and all the other edges of B_n are closed$\}$

Proof of the theorem.

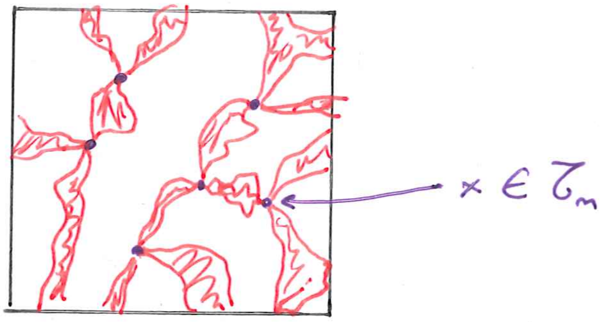
Assume for contradiction that $\mathbb{P}_p[N = \infty] = 1$.

By Lemma 2, we have $c := \mathbb{P}[T_0] > 0$.

Define $\mathcal{T}_m(\omega) = \{x \in \Lambda_m : x \text{ is a trifurcation}\}$

By translation invariance, we have

$$\mathbb{E}[|\mathcal{T}_m|] = \sum_{x \in \Lambda_m} \mathbb{P}[T_x] = c \cdot |\Lambda_m|$$



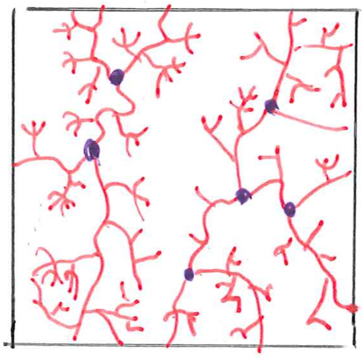
We claim that for every configuration ω, $|\mathcal{T}_m(\omega)| \leq |\partial \Lambda_m|$.

To see this, consider the subgraph of Λ_m obtained by the following peeling procedure.

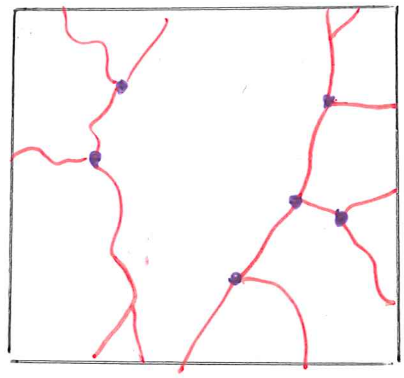
Let $F_0 = \{e_1, \dots, e_n\}$ be the set of open edges in Λ_m

For $i = 1, \dots, n$ set $F_i = \begin{cases} F_{i-1} \setminus \{e_i\} & \text{if } e_i \text{ belongs to a cycle of } F_i. \\ F_{i-1} & \text{otherwise} \end{cases}$

After this first step the graph induced by F_n is a forest.



Then, remove all the vertices of degree 1 in the graph induced by F_n, except the vertices on $\partial\Lambda_m$. Repeat this operation until the time there is no more vertex of degree 1, except those on $\partial\Lambda_m$. Consider the graph induced by the remaining edges.



Write N_1 for the vertices of degree 1 in this graph, and

$N_{\geq 3}$ ________________ ≥ 3 ________ .

Notice that $N_1 \leq |\partial\Lambda_m|$ and $N_{\geq 3} \geq |\mathcal{T}_m|$ (because the trifurcations have not been deleted during the "peeling" procedure. By applying Lemma 3 to each of the connected component of the graph above, we obtain

$$|\mathcal{T}_m| \leq N_{\geq 3} \leq N_1 \leq |\partial\Lambda_m|$$

Taking the expectation, we obtain

$$c\,|\Lambda_m| \leq |\partial\Lambda_m|$$

which is a contradiction to $\dfrac{|\partial\Lambda_m|}{|\Lambda_m|} \xrightarrow[n\to\infty]{} 0$ ∎

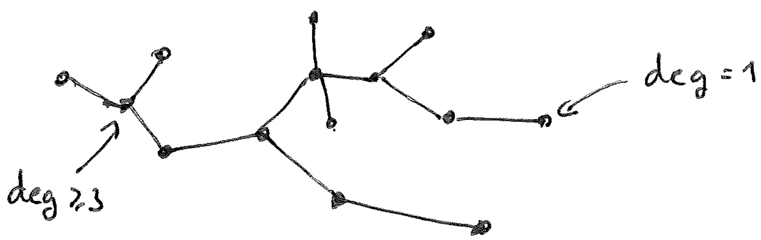
Finally by independence.

$$
\begin{aligned}
0 &< \mathbb{P}_p\left[E_m(x,y,z)\right]\mathbb{P}_p\left[F_m(x,y,z)\right] \\
&\leq \mathbb{P}_p\left[E_m(x,y,z) \cap F_m(x,y,z)\right] \\
&\leq \mathbb{P}_p\left[T_t\right] \underset{\substack{\uparrow \\ \text{translation invariance.}}}{=} \mathbb{P}_p\left[T_0\right]
\end{aligned}
$$

Lemma 3. Let (T,F) be a finite tree. (a finite connected graph with no cycle.)
Let $N_1 = |\{x \in T : \deg(x) = 1\}|$, $N_{\geq 3} = |\{x \in T : \deg(x) \geq 3\}|$

Then

$$N_1 \geq 2 + N_{\geq 3}.$$



a tree with $N_1 = 7$, $N_{\geq 3} = 4$.

Proof: We have $|T| = |F| - 1$ (by induction on $|T|$).
Write $N_2 = |\{x \in T : \deg(x) = 2\}|$
By counting the edges of the tree in two different ways, we find $2|F| = \sum_{x \in T} \deg(x) \geq N_1 + 2N_2 + 3N_{\geq 3}$.

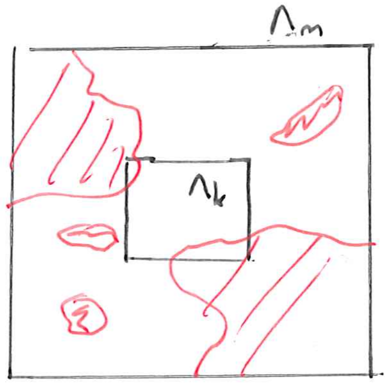
Since $2|F| = 2|T| + 2 = 2(N_1 + N_2 + N_{\geq 3}) + 2$,
we obtain $N_1 \geq N_{\geq 3} + 2$ ∎

2 UNIQUENESS ZONE.

For $1 \leq k \leq n < \infty$, let

$$\boxed{U_{k,n} = \{ K \leq 1 \}}$$

where K counts the number of disjoint clusters <u>in Λ_n</u> intersecting Λ_k and $\partial \Lambda_n$.



Above, $k=2$

<u>Rk</u>: • $U_{k,n}$ is neither an increasing event nor a decreasing one.

• $\mathbb{P}_p [U_{k,n}]$ is increasing in n, decreasing in k.

$$\boxed{\begin{array}{l} \underline{\text{Prop}}. \text{ For every } \varepsilon > 0 \text{ and } k \geq 1, \ \exists \ n = n(\varepsilon, k) \text{ s.t.} \\ \qquad \forall p \in [0,1] \quad \mathbb{P}_p [U_{k,n}] > 1 - \varepsilon . \end{array}}$$

<u>Proof</u>: Fix $\varepsilon > 0$ and $k \geq 1$. Define

$$\mathcal{O}_m = \{ p \in [0,1] \text{ s.t. } \mathbb{P}_p [U_{k,m}] > 1 - \varepsilon \}. \quad \text{(open)}$$

By uniqueness of the infinite cluster (when it exists), we have $\forall p \in [0,1] \ \exists \ m(p) \geq k$ s.t. $\mathbb{P}_p [U_{k,m(p)}] > 1 - \varepsilon$.

Hence $$[0,1] = \bigcup_{m \geq k} \mathcal{O}_m \underset{\substack{\uparrow \\ \text{compactness}}}{=} \bigcup_{1 \leq i \leq i_0} \mathcal{O}_{m_i}.$$

Choosing $n = \max_{1 \leq i \leq i_0} m_i$ concludes the proof. ■

3 APPLICATION 1: CONTINUITY OF θ IN THE SUPERCRITICAL REGIME

$\boxed{\text{Prop: } p \mapsto \theta(p) \text{ is continuous on } (p_c, 1].}$

Proof: Let $p_1 > p_c$. We prove that θ is the uniform limit of θ_m on $[p_1, 1]$.

Let $\varepsilon > 0$

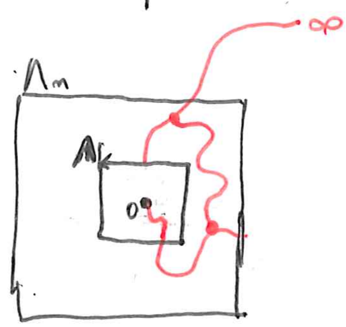
Pick $k \geq 1$ large enough s.t. $\mathbb{P}_{p_1}[\Lambda_k \leftrightarrow \infty] > 1-\varepsilon$.

Pick $m \geq k$ s.t. $\forall p \in [0,1]$ $\mathbb{P}_p[U_{k,m}] > 1-\varepsilon$.

Since for every p,

$$\theta(p) \geq \mathbb{P}_p[(0 \leftrightarrow \partial\Lambda_m) \cap (\Lambda_k \leftrightarrow \infty) \cap U_{k,m}]$$

$$\uparrow$$



We have

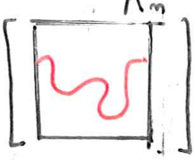
$$\forall p \in [p_1, 1] \quad \theta_m(p) \geq \theta(p) \geq \theta_m(p) - 2\varepsilon$$

And therefore

$$|\theta_m - \theta| \leq 2\varepsilon \text{ on } [p_1, 1] \qquad \blacksquare$$

4. APPLICATION 2 : BOX-CROSSING

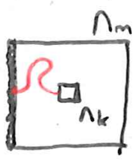
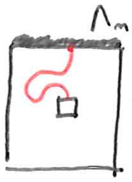
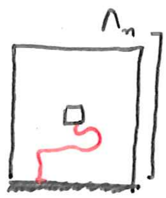
Prop. Let $p \in [0,1)$ s.t. $\theta(p) > 0$. Then

$$\lim_{m \to \infty} P_p \Big[$$  $$\Big] = 1.$$

Proof: Let $\varepsilon > 0$. Let $k \geq 1$ s.t.

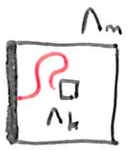
$$P_p[\Lambda_k \longleftrightarrow \infty] > 1 - \varepsilon^{2d}.$$

This implies, for every $m \geq k$.

$$P_p\Big[$$  $$\cup$$  $$\cup \cdots \cup$$  $$\Big] > 1-\varepsilon^{2d}.$$

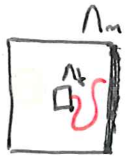
"Λ_k is connected inside Λ_m to one of the 2d-facets of $\partial\Lambda_m$"

The square-root trick and rotation invariance imply

$$P_p\Big[$$  $$\Big] > 1-\varepsilon.$$

Choose $m \geq k$ s.t. $P_p[U_{k,m}] > 1-\varepsilon$.

This implies

$$P_p\Big[$$  $$\Big] \geq P_p\Big[$$  $$\cap$$  $$\cap\, U_{k,m}\Big] > 1-\varepsilon.$$

CHAPTER 4:

PERCOLATION ON $\mathbb{Z}^2$

In this chapter, we fix $d=2$. The graph $(\mathbb{Z}^2, E)$ is planar, which provides several useful tools for the study of percolation: planar graphs satisfy duality relations, that will have deep consequences for percolation, also, it will be easy to "force" open paths to intersect.

1.) $p_c = \frac{1}{2}$

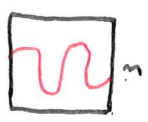
Thm [Kesten '80]

$$\boxed{p_c = \frac{1}{2} \text{ and } \theta(p_c) = 0}$$

Duality for planar percolation.

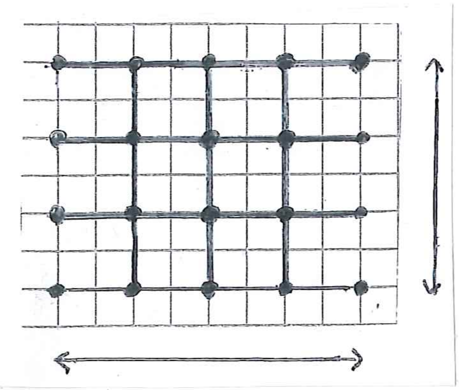
	primal	dual.
graphs.	$(\mathbb{Z}^2, E)$	$((\mathbb{Z}^2)^*, E^*) = (\mathbb{Z}^2, E)$ translated by $(\frac{1}{2}, \frac{1}{2})$.
percolation	$\omega \in \{0,1\}^E$	$\omega^* \in \{0,1\}^{E^*}$ $(\omega^*(e^*) = 1 - \omega(e))$
	$\omega \sim \mathbb{P}_p$	$\omega^* \sim \mathbb{P}_{1-p}$.

Lemma: For $p = \frac{1}{2}$

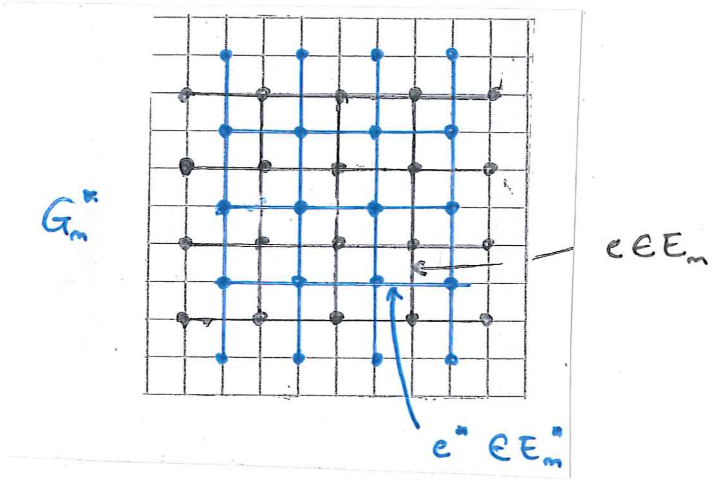
$\forall n \geq 0$ $\mathbb{P}_p\Big[$  $\Big] = \frac{1}{2}$

Proof:

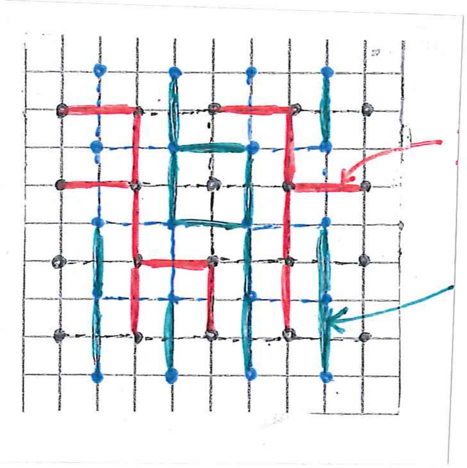
For $m \geq 1$, consider the graph $G_m = (V_m, E_m)$ defined by

$G_m =$ 

Let $G_m^* = (V_m^*, E_m^*)$ be the blue graph below



Notice that
- an edge e of E_m crosses exactly one edge $e^* \in E_m^*$
- G_m^* is a rotated and translated version of G_m.

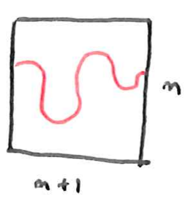
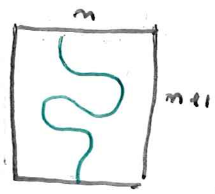


Admitted combinatorial result:

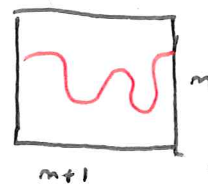
$$(\exists \text{ left-right open path in } \omega) \Leftrightarrow (\exists \text{ no top-down path in } \omega^*)$$

For a proof of this result, see e.g. [Bollobás, Riordan, Chap. 3]

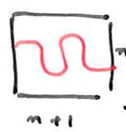
This implies

$\mathbb{P}_p\Big[$  $\Big] + \mathbb{P}_p\Big[$  $\Big] = 1$

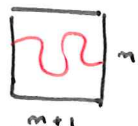
If $p = \frac{1}{2}$, the two probabilities are equal and we have

$\forall n \geq 0 \quad \mathbb{P}_{\frac{1}{2}}\Big[$  $\Big] = \frac{1}{2}.$

Proof of the theorem:

For $p = \frac{1}{2}$ we do not have $\mathbb{P}_p\Big[$  $\Big] \xrightarrow[n\to\infty]{} 1.$

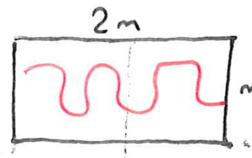
Hence $\theta(\frac{1}{2}) = 0$ and $p_c \geq \frac{1}{2}$.

For $p = \frac{1}{2}$ we do not have $\mathbb{P}_p\Big[$  $\Big] \xrightarrow[n\to\infty]{} 0.$

(In particular there is not exponential decay of the connection probabilities.

Hence $p_c \leq \frac{1}{2}$. $\blacksquare$

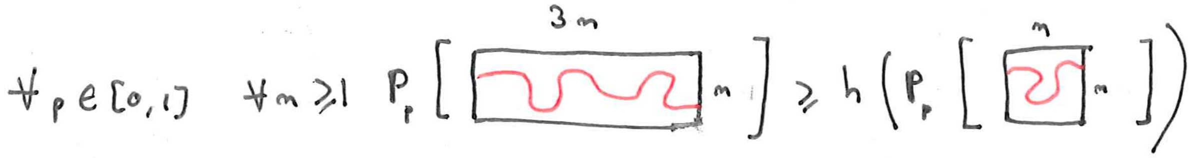
Question:

For $p = \frac{1}{2}$, do we have $\inf_{n \geq 0} \mathbb{P}_p\Big[$  $\Big] > 0$?

2. Russo-Seymour-Welsh Theorem.

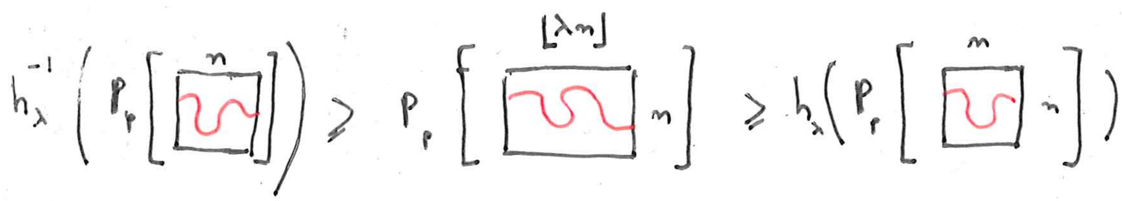
Thm [RSW '78]

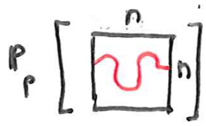
There exists $h : [0,1] \longrightarrow [0,1]$ continuous, (strictly) increasing s.t. $h(0) = 0$, $h(1) = 1$ and



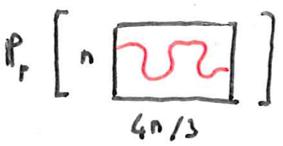
Exercise:

Let $\lambda > 0$. Prove that there exists $h_\lambda : [0,1] \to [0,1]$ as above such that $\forall p \in [0,1]$ $\forall n \geq 1/\lambda$

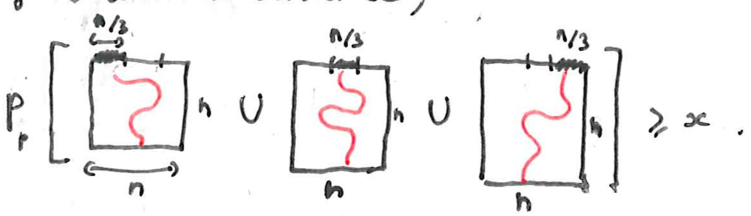


Proof: Fix $p \in [0,1]$, $n \geq 1$. Set $x =$ 

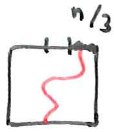
Assume for simplicity that $n \in 6\mathbb{N}$.

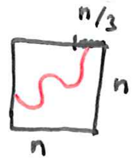
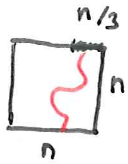
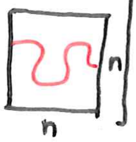
Step 1:  $\geq g(x)$ where $g(x) := x^2 \left(1 - (1-x)^{1/6}\right)^2$.

First, by rotation invariance,



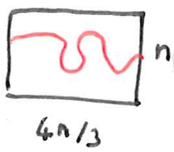
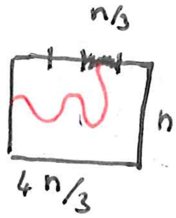
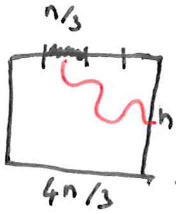
Hence, by reflection invariance and the square-root trick,

case 1: $\mathbb{P}_p\Big[$  $\Big] \geq 1-(1-x)^{\frac{1}{3}}$.

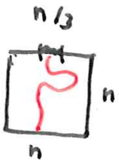
Then $\mathbb{P}_p\Big[$  $\Big] \geq \mathbb{P}_p\Big[$  $\cap$  $\Big]$

$$\overset{FKG}{\geq} x\left(1-(1-x)^{\frac{1}{3}}\right) \overset{\text{trivial}}{\geq} x\left(1-(1-x)^{\frac{1}{6}}\right),$$

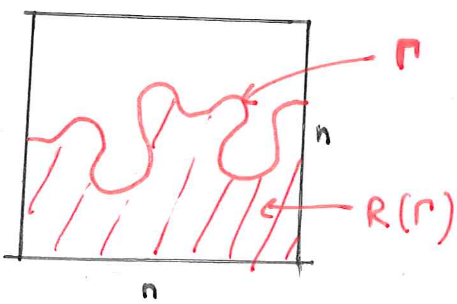
and

$\mathbb{P}_p\Big[$  $\Big] \geq \mathbb{P}_p\Big[$  $\cap$  $\Big]$

$$\overset{FKG}{\geq} x^2\left(1-(1-x)^{\frac{1}{6}}\right)^2.$$

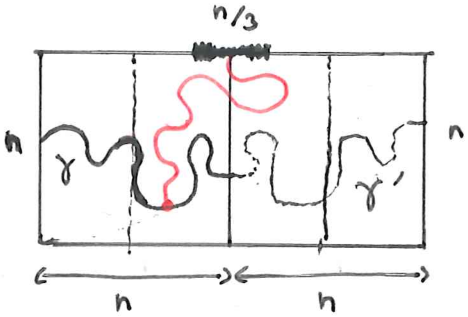
Case 2: $\mathbb{P}_p\Big[$  $\Big] \geq 1-(1-x)^{1/3}$.

If there exists a left-right open path in $[0,n]^2$, define Γ to be the lowest left-right open path in $[0,n]^2$. Set $\Gamma = \emptyset$ if there is no such path.



To check: Γ is well defined, and for every $\gamma \neq \emptyset$ admissible, the event $\{\Gamma = \gamma\}$ is measurable w.r.t the configuration in the region $R(\gamma)$ below γ.

Fix $\gamma \neq \emptyset$ a left-right admissible path. Let γ' be the image of γ by the reflection in the line $\{n\} \times \mathbb{Z}$.

E_γ = 

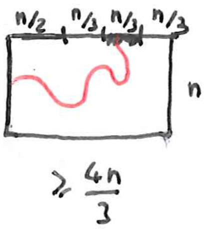
Let E_γ (resp. $E_{\gamma'}$) be the event that $[\frac{5n}{6}, \frac{7n}{6}] \times \{n\}$ is connected to γ (resp. to γ') in the region of $[\frac{n}{2}, \frac{3n}{2}] \times [0, n]$ above $\gamma \cup \gamma'$.

Notice that $\mathbb{P}_p[E_\gamma \cup E_{\gamma'}] \geq \mathbb{P}[$  $] \geq 1 - (1-x)^{1/3}$.

Since $\mathbb{P}_p[E_\gamma] = \mathbb{P}_p[E_{\gamma'}]$, the square-root trick implies.

$$\mathbb{P}_p[E_\gamma] \geq 1 - (1-x)^{1/6}.$$

Notice that $\{\Gamma = \gamma\}$ and E_γ are independent. Hence

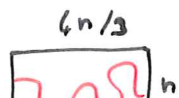
$\mathbb{P}[$  $]$

$$\geq \sum_{\gamma \neq \emptyset} \mathbb{P}_p[\{\Gamma = \gamma\} \cap E_\gamma]$$

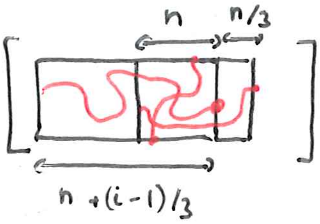
$$\overset{\text{indep.}}{=} \sum_{\gamma \neq \emptyset} \mathbb{P}_p[\Gamma = \gamma]\, \mathbb{P}_p[E_\gamma]$$

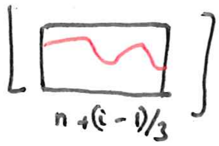
$$\geq \underbrace{\sum_{\gamma \neq \emptyset} \mathbb{P}_p[\Gamma = \gamma]}\, \left(1 - (1-x)^{1/6}\right).$$

$= \mathbb{P}_p[$  $] = x$

We conclude as in the first case that $\mathbb{P}_p[$  $] \geq x^2 (1 - (1-x)^{1/6})^2$.

Step 2: Iteration

$\forall i \geq 2$ $\quad P_p\Big[$  $\Big] \geq P\Big[$  $\Big]$

FKG
$\geq P\Big[$  $\Big] \times x \times g(x)$

induction
$$\geq g(x) \times [x\, g(x)]^{i-1}.$$

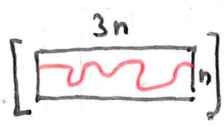
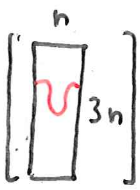
This concludes the proof by setting $h(x) = g(x)\,[x\, g(x)]^5$. ∎

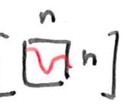
3 CRITICAL BEHAVIOUR.

In this section, we fix $p = p_c = \frac{1}{2}$, and write $P = P_{\frac{1}{2}}$.

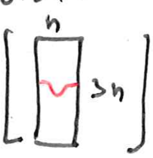
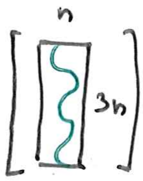
Thm (Box-crossing property).

For $p = p_c$, there exists $c > 0$ s.t.

$\forall n \geq 1$ $\quad c \leq P\Big[$  $\Big] \leq P\Big[$  $\Big] \leq 1 - c.$

Proof: First inequality: use $P\Big[$  $\Big] = \frac{1}{2}$ + RSW.

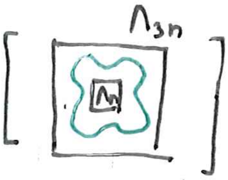
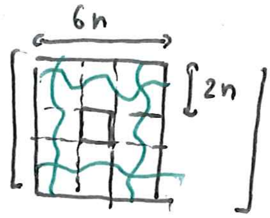
Second inequality: trivial.

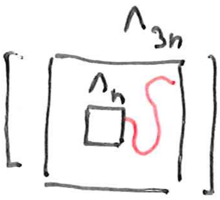
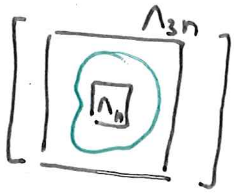
Third inequality: $P\Big[$  $\Big] = 1 - P\Big[$  $\Big] \leq 1 - c.$

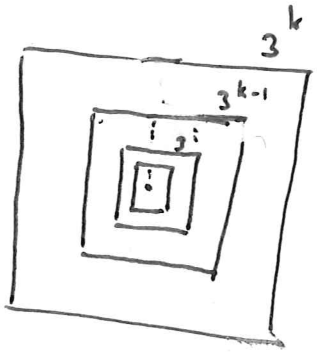
$\uparrow$ duality $\qquad\qquad \uparrow$ first inequality.

Corollary 1: [Polynomial bound on the 1-arm probability.]

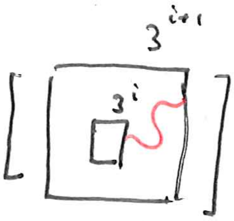
$$\exists c>0 \text{ s.t. } \forall n\geq 1 \quad \theta_n\left(\tfrac{1}{2}\right) \leq \frac{1}{n^c}.$$

Pf: $P\Big[$  $\Big] \geq P\Big[$  $\Big] \overset{FKG}{\geq} c^4 =: c_0$

Hence $P\Big[$  $\Big] = 1 - P_p\Big[$  $\Big] \leq 1 - c_0$



By independence

$\theta_{3^k}\left(\tfrac{1}{2}\right) \leq \prod_{0\leq i<k} P\Big[$  $\Big] \leq (1-c_0)^k$

Choosing $k = \lfloor \log_3 n \rfloor$ for $n \geq 3$ concludes the proof.

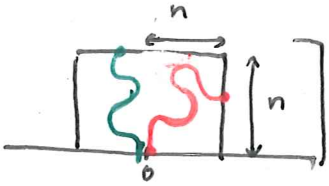
Conjecture:

$$\exists c>0 \text{ s.t. } \theta_n(p) \underset{n\to\infty}{\sim} \frac{c}{n^{5/48}}$$

Rk: the exponent $5/48$ has been proved for site percolation on the triangular lattice.

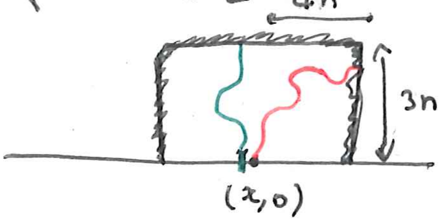
Corollary 2: (A universal arm exponent).

$\exists c > 0$ s.t. $\forall n \geq 1$

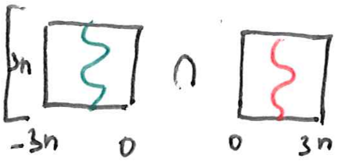
$$\frac{c}{n} \leq \mathbb{P}\Big[$$ 

$$\Big] \leq \frac{1}{c} \cdot \frac{1}{n}$$

Proof (sketch) Define for $x \in \mathbb{Z}$

A_x 

Lower bound.

$$\frac{1}{4} \underset{\text{indep.}}{\leq} \mathbb{P}\Big[$$ 

$$\Big] \underset{\substack{\text{union}\\ \text{bound}}}{\leq} \sum_{|x| \leq 3n} \mathbb{P}[A_x]$$

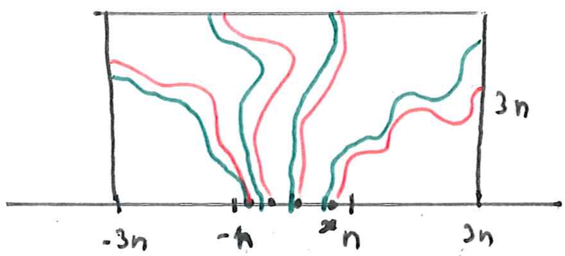
$$\underset{\text{invariance}}{=} (6n+1)\, \mathbb{P}[A_0]$$

Hence $\mathbb{P}[A_0] \geq \dfrac{1}{28n}$

Upper bound

Let N be the number of disjoint open paths from $[-n,n] \times 0$ to $\partial \Lambda_{3n}$ in the half plane $\mathbb{Z} \times [0,\infty]$.

Observe that $\sum_{|x| \leq n} \mathbb{1}_{A_x} \leq N$.



Hence $\quad 2n+1 \ \mathbb{P}_p[A_0] \le \mathbb{E}[N] = \sum_{k\ge 0} P[N\ge k]$

$$= \sum_{k\ge 0} P[\underbrace{(N\ge 1)\circ\cdots\circ(N\ge 1)}_{k \text{ times}}]$$

$$\overset{BK}{\le} \frac{1}{1-P[\quad]}$$

$$\le \frac{1}{c_0} \quad \text{(box-crossing)}$$

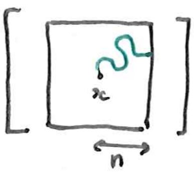
Finally $\quad \mathbb{P}_p[A_0] \le \frac{1}{c_0} \times \frac{1}{2n+1}.$

4 SUPERCRITICAL PERCOLATION.

Key remark: $\quad p > p_c(\mathbb{Z}^2) \iff 1-p < p_c((\mathbb{Z}^2)^*)$

Hence $\forall p > p_c \quad \forall x \in (\mathbb{Z}^2)^*$

$\forall n \ge 1 \quad \mathbb{P}_p[\quad] \le e^{-cn}.$

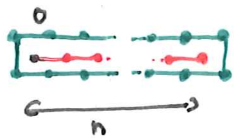


Thm [exponential decay of the radius of a finite cluster]

Let $p_c<p<1$. There exist $c_0, c_1 > 0$ s.t.

$$\forall n\ge 1 \quad e^{-c_0 n} \le \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n,\ 0 \not\longleftrightarrow \infty] \le e^{-c_1 n}.$$

Proof: Lower bound

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n ,\ 0 \not\longleftrightarrow \infty] \geq \mathbb{P}_p\Big[$$  $$\Big]$$

$$= p^n (1-p)^{2n+4}$$

Upper bound.

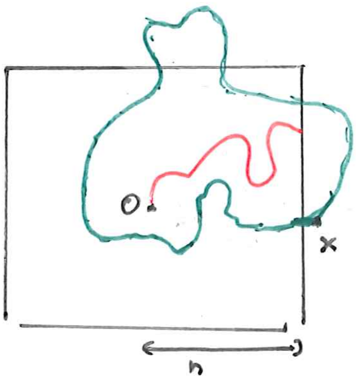


Fig: the event $\{0 \longleftrightarrow \partial\Lambda_n ,\ 0 \not\longleftrightarrow \infty\}$.

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n ,\ 0 \longleftrightarrow \infty] \leq \mathbb{P}_p\left[\begin{array}{l}\exists \text{ dual open circuit with diameter} \\ \text{longer than } n,\ \text{intersecting } \partial\Lambda_n\end{array}\right]$$

$$\leq \mathbb{P}_p\left[\exists x \in (\mathbb{Z}^2)^* \text{ s.t. } \|x\|_\infty \geq n ,\ x \longleftrightarrow \partial\Lambda_{\|x\|}(x)\right]$$

$$\leq C n e^{-cn} \qquad \leq e^{-\frac{c}{2}n} \quad \text{for } n \text{ large enough.} \blacksquare$$

Exercise: (Supercritical correlation length.)

Let $p > p_c$. Using duality, prove that

$$\xi(p) = \lim_{n\to\infty}\left(-\frac{\log\left(\mathbb{P}_p[0 \longleftrightarrow ne_1 ,\ 0 \not\longleftrightarrow \infty]\right)}{n}\right)^{-1}$$

and show that $\xi(p) = 2\,\underset{\uparrow}{\xi}(1-p)$

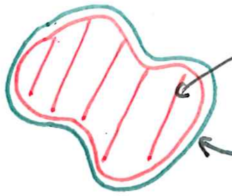
"subcritical correlation length"

Thm [stretch exponential decay in volume]

Let $p_c < p < 1$. There exist $c_0, c_1 > 0$ s.t.

$$\forall n \geq 1 \qquad e^{-c_0\sqrt{n}} \leq \mathbb{P}_p\left[\,|C_0| \geq n,\ 0 \not\leftrightarrow \infty\,\right] \leq e^{-c_1\sqrt{n}}.$$

idea:



Proof: upper bound.

If $|C_0| \geq n$ then $0 \longleftrightarrow \partial\Lambda_{\frac{\sqrt{n}}{3}}$. Hence,

$$\mathbb{P}_p\left[|C_0| \geq n,\ 0 \not\leftrightarrow \infty\right] \leq \mathbb{P}_p\left[0 \longleftrightarrow \partial\Lambda_{\frac{\sqrt{n}}{3}},\ 0 \not\leftrightarrow \infty\right]$$
$$\leq e^{-c\sqrt{n}/3}.$$

lower bound. $\theta = \theta(p)$

Let $k = \left\lceil \sqrt{\frac{n}{\theta}} \right\rceil$ Define $N = \sum_{x \in \Lambda_k} \mathbb{1}_{x \longleftrightarrow \partial\Lambda_k}$.

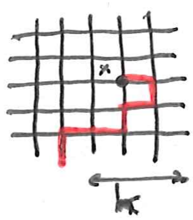


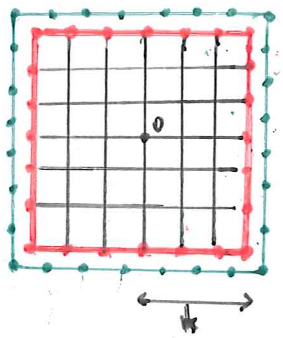
Fig: the event $x \leftrightarrow \partial\Lambda_k$

We have $\mathbb{E}_p[N] = \sum_{x \in \Lambda_k} \mathbb{P}_p\left[x \longleftrightarrow \partial\Lambda_k\right] \geq |\Lambda_k| \cdot \theta$

Hence, by Markov inequality $\mathbb{P}_p\left[N \geq \frac{\theta}{2}|\Lambda_k|\right] \geq \frac{\theta}{2}$.

$\left(\text{indeed } \mathbb{P}_p\left[\,|\Lambda_k| - N \geq \left(1 - \frac{\theta}{2}\right)|\Lambda_k|\,\right] \leq \frac{1-\theta}{1-\theta/2} = 1 - \frac{\theta/2}{1-\theta/2} \leq 1 - \theta/2\right)$

Now, let F be the event that all the edges with both extremities in $\partial\Lambda_k$ are open and all the edges of $\Delta\Lambda_k$ are closed.

$F =$ 

$$\mathbb{P}_p[F] \geq [p(1-p)]^{|\Delta\Lambda_k|} \geq e^{-c_0 k}.$$

$$\mathbb{P}_p[|C_0| \geq n] \geq \mathbb{P}_p\left[|C_0| \geq \frac{\theta}{2}|\Lambda_k|\right]$$

$$\geq \mathbb{P}_p\left[\{0 \longleftrightarrow \partial\Lambda_k,\ N \geq \frac{\theta}{2}|\Lambda_k|\} \cap F\right]$$

$$\overset{\text{indep.}}{=} \mathbb{P}_p\left[0 \longleftrightarrow \partial\Lambda_k,\ N \geq \frac{\theta}{2}|\Lambda_k|\right] \mathbb{P}_p[F]$$

$$\overset{\text{FKG}}{\geq} \frac{\theta^2}{2} e^{-c_0 k}.$$

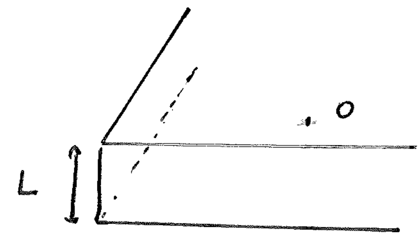
■

Chapter 5 : Supercritical Percolation on $\mathbb{Z}^d$, $d \geq 3$.

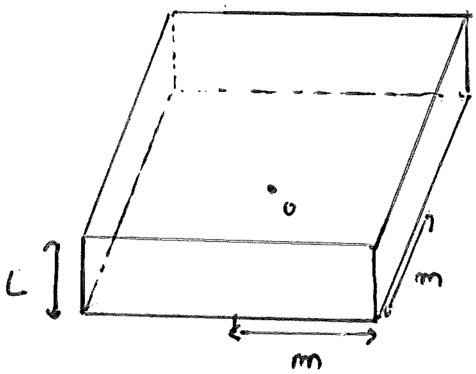
In the whole chapter, we fix $d \geq 3$, and consider percolation on $\mathbb{Z}^d$.

1 Slab percolation.

Not: For $L \geq 1$, set $\mathbb{S}_L = \mathbb{Z}^2 \times \{0, \dots, L\}^{d-2}$. "slab of thickness L".



For $m, N \geq 1$, set $\Lambda_{m,N} = \{-m, \dots, m\}^2 \times \{0, \dots, L\}^{d-2}$.



Def: $p_c(\mathbb{S}_L) = \inf \{ p \geq 0 : \mathbb{P}_p [0 \xleftrightarrow{\mathbb{S}_L} \infty] \}$.

Thm: [Grimmett-Marstrand '90]

$$\boxed{\lim_{L \to \infty} p_c(\mathbb{S}_L) = p_c(\mathbb{Z}^d)}$$

Equivalently: For every $p > p_c(\mathbb{Z}^d)$, (which implies $\mathbb{P}_p[0 \leftrightarrow \infty] > 0$) there exists $L \geq 1$ s.t. $\mathbb{P}_p[0 \xleftrightarrow{S_L} \infty]$.

Rks: • This implies $p_c(\mathbb{Z}^{d-1} \times \mathbb{N}) = p_c(\mathbb{Z}^d)$.

• It is known that $\forall L$ $p_c(S_L) > p_c(\mathbb{Z}^d)$. $\forall L < \infty$.

Thm 2 [Finite volume version of Thm 1]:

Let $p > p_c(\mathbb{Z}^d)$. Then there exists $L \geq 1$ and $\delta > 0$ s.t.

$$\forall m \geq 1 \ \forall x, y \in \Lambda_{m,N} \quad \mathbb{P}_p[x \xleftrightarrow{\Lambda_{m,N}} y] \geq \delta.$$

Exercise: Prove that Thm 2 $\Rightarrow$ Thm 1.

2 RADIUS OF A FINITE CLUSTER.

Thm: Let $p > p_c$. $\exists c_1, c_2 > 0$ s.t.

$$\forall n \geq 1 \qquad e^{-c_1 n} \leq \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n, 0 \not\leftrightarrow \infty] \leq e^{-c_2 n}.$$

Proof: Lower bound:

Let $L_n = \{0, \dots, n\} \times \{0_{\mathbb{Z}^{d-1}}\}$

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n, 0 \not\leftrightarrow \infty] \geq \mathbb{P}_p[\forall e \subset L_n \ \omega(e) = 1, \forall e \in \Delta L_n \ \omega(e) = 0]$$
$$= p^n (1-p)^{|\Delta L_n|} \geq e^{-c_1 n}.$$

Upper bound: Let $L \geq 1$ s.t. $\delta := \mathbb{P}_p[0 \xleftrightarrow{S_L} \infty] > 0$.

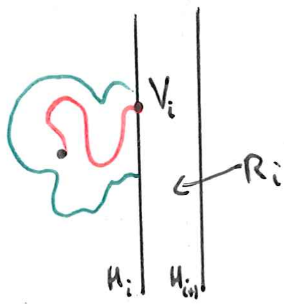
assume that $n = kL$ for some $k \in \mathbb{N}$.

Write $H_i = \{x \in \mathbb{Z}^d : x_1 \leq iL\}$.

By symmetry, we have

$$P_p[0 \leftrightarrow \partial\Lambda_n ,\ 0 \not\leftrightarrow \infty] \leqslant 2d\, P_p[0 \overset{H_k}{\leftrightarrow} \partial H_k ,\ 0 \overset{H_k}{\not\leftrightarrow} \infty]$$

For $0 \le i \le k$, let $A_i = \{0 \leftrightarrow H_i ,\ 0 \overset{H_i}{\not\leftrightarrow} \infty\}$.



Notice that $A_{i+1} \subset A_i$ $\forall\ 0 \le i < k$. Hence,

$$P_p[A_k] = P_p[A_0] \cdot \prod_{0 \le i < k} P_p[A_{i+1} \mid A_i] .$$

For every $\omega \in A_i$, we can pick $V_i(\omega) \in \partial H_i$ s.t. 0 is connected to $V_i(\omega)$ in H_i without using the edges $e \subset \partial H_i$.

$$\therefore P_p[A_{i+1} \mid A_i] \leqslant P_p[V_i \overset{R_i}{\not\leftrightarrow} \infty \mid A_i] \quad (\text{where } R_i := (H_{i+1} \setminus H_i) \cup \partial H_i)$$

$$\leqslant 1 - \delta . \quad (\text{by independence and translation invariance})$$

Therefore $P_p[A_k] \leqslant (1-\delta)^k \leqslant (1-\delta)^{n/L}$ ■

Rk: (Def. of the supercritical correlation length).

Let $p > p_c$. It is possible to prove that

$$\xi(p) = \left(\lim_{n\to\infty} -\frac{1}{n} \log\left(P_p[0 \leftrightarrow \partial\Lambda_n, 0 \not\leftrightarrow \infty]\right)\right)^{-1}$$

is well defined, and

$$\xi(p) = \left(\lim_{n\to\infty} -\frac{1}{n} \log\left(P_p[0 \leftrightarrow n e_1, 0 \not\leftrightarrow \infty]\right)\right)^{-1}.$$

3. UNIQUENESS ZONE

Thm: Let $p > p_c$. There exists $c>0$ s.t.

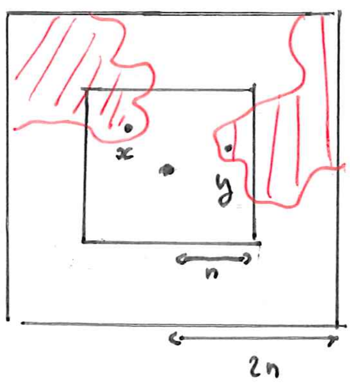
$$\forall n \geq 1 \quad P_p[U(n,2n)] \geq 1 - e^{-cn}.$$

Proof (sketch).

Let $L \geq 1$, $\delta > 0$ s.t. $\forall m \geq 1$ $\forall x,y \in \Lambda_{m,L}$ $P_p[x \overset{\Lambda_{m,L}}{\longleftrightarrow} y] \geq \delta$. (*)

Assume $n = kL$ for some $k \in \mathbb{N}$.

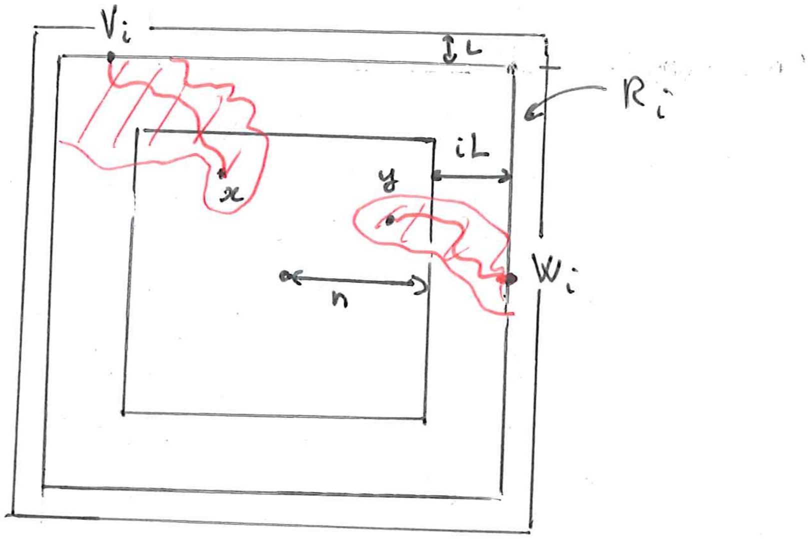
$$P[U(n,2n)^c] \leq \sum_{x,y\in\Lambda_n} P_p[x \leftrightarrow \partial\Lambda_{2n}, y \leftrightarrow \partial\Lambda_{2n}, x \overset{\Lambda_{2n}}{\not\longleftrightarrow} y]$$



Two disjoint clusters crossing $\Lambda_{2n} \setminus \Lambda_n$ implies the existence of $x,y \in \Lambda_n$ s.t. $x \leftrightarrow \partial\Lambda_{2n}$, $y \leftrightarrow \partial\Lambda_{2n}$, $x \overset{\Lambda_{2n}}{\not\longleftrightarrow} y$.

Fix $x, y \in \Lambda_n$.

Let $A_i = \{x \leftrightarrow \partial\Lambda_{n+iL},\ y \leftrightarrow \partial\Lambda_{n+iL},\ x \overset{\Lambda_{n+iL}}{\not\leftrightarrow} y\}$.



For $\omega \in A_i$, we define $V_i(\omega), W_i(\omega) \in \partial\Lambda_{n+iL}$ that are resp. connected to x, y in Λ_{n+iL}.

Using $(*)$, we can prove (exercise) that

$$\mathbb{P}_p\left[V_i \overset{R_i}{\longleftrightarrow} W_i \mid A_i\right] \geq \delta^d,$$

where $R_i = (\Lambda_{n+(i+1)L} \setminus \Lambda_{n+iL}) \cup \partial\Lambda_{n+iL}$.

Finally, using that

$$\mathbb{P}_p[x \leftrightarrow \partial\Lambda_{2n}, y \leftrightarrow \partial\Lambda_{2n};\ x \overset{\Lambda_{2n}}{\not\leftrightarrow} y] = \mathbb{P}_p[A_k]$$

$$= \mathbb{P}_p[A_0] \cdot \prod_{0 \leq i < L} \underbrace{\mathbb{P}_p[A_{i+1} \mid A_i]}_{\substack{\leq \mathbb{P}_p[V_i \not\leftrightarrow W_i \mid A_i] \\ \leq 1-\delta^d}}$$

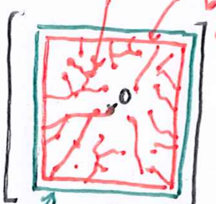
$$\leq (1-\delta^d)^k = (1-\delta^d)^{n/L}$$

■

4 VOLUME OF A FINITE CLUSTER.

First try (mimic the 2D-proof):

Lower bound

$$e^{-c_0 n^{\frac{d-1}{d}}} \leqslant \mathbb{P}_p\Big[$$ 

$$\Big] \leqslant \mathbb{P}\big[|C_0| \geqslant n, |C_0| < \infty\big).$$

upper bound.

$$\mathbb{P}_p\big[|C_0| \geqslant n, \ |C_0| < \infty\big) \leqslant \mathbb{P}_p\big[0 \leftrightarrow \partial\Lambda_{n^{\frac{1}{d}}}, 0 \not\leftrightarrow \infty\big] \leqslant e^{-c_1 n^{\frac{1}{d}}}.$$

→ when $d > 2$, the two bounds are not in the same order!

Thm:

Let $p_c < p < 1$. There exist $c_0, c_1 > 0$ s.t.

$$\forall n \geqslant 1 \quad e^{-c_0 n^{d/d-1}} \leqslant \mathbb{P}_p\big[|C_0| \geqslant n, \ |C_0| < \infty\big] \leqslant e^{-c_1 n^{d/d-1}}.$$

The lower bound follows from the same argument as in dimension $d=2$. In the rest of the section, we focus on the upper bound.

The proof is based on two arguments

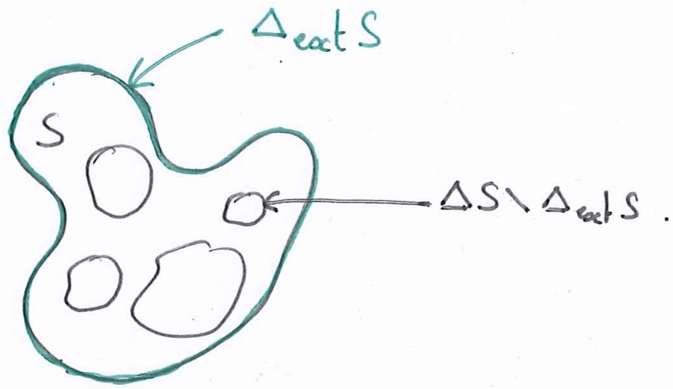
- a perturbative argument. (Peierls argument) that applies for p sufficiently close to one.
- a renormalisation argument: by rescaling the percolation process for $p > p_c$, we obtain a highly supercritical percolation

and the perturbative argument applies for this new process.

External ingredients. (admitted lemmas)

Def: Let $S \subset \mathbb{Z}^d$. Define

$$\Delta_{ext} S = \left\{ xy \in E : \begin{array}{l} x \in S,\ y \text{ belongs to an infinite} \\ \text{connected component of } \mathbb{Z}^d \setminus S \end{array} \right\}$$

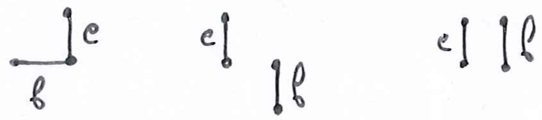


Lemma 1 (isoperimetry). There exists $c > 0$ s.t.

$$\boxed{\forall S \subset \mathbb{Z}^d \text{ finite} \quad |\Delta_{ext} S| \geq c\, |S|^{\frac{d-1}{d}}.}$$

Let G_E be the graph with vertex set E, and edge set given by $e \sim f \Leftrightarrow d_1(e,f) \leq 1$

L^1- distance



examples of neighbouring edges in G_E.

Lemma 2:

$$\boxed{\text{If } S \subset \mathbb{Z}^d \text{ finite, connected, then } \Delta_{ext} S \text{ is connected in } G_E.}$$

→ See Adam Timar "boundary connectivity via graph theory."

Lemma 3.

There exists a constant $C < \infty$ s.t., for every $e \in E$, $\left|\{A \subset E : e \in A, |A| = n, A \text{ connected in } G_E\}\right| \leq C^n$.

WARM-UPS

Warm-up 1. (perturbative argument)

There exist $p_0 < 1$ and $c_1 > 0$ s.t. for every $p \geq p_0$

$$\forall n \geq 1 \quad \mathbb{P}_p[|C_0| \geq n, |C_0| < \infty] \leq e^{-c_1 n^{\frac{d-1}{d}}}$$

Proof: For $k \geq 1$,

$$\mathbb{P}_p[|\Delta_{ext} C_0| = k] \overset{\text{Lemma 2}}{\leq} \sum_{\substack{e \in E: \\ d(e,0) \leq k}} \sum_{\substack{A \subset E \\ \text{connected in } G_E \\ |A| = k}} \underbrace{\mathbb{P}_p[\Delta_{ext} C_0 = A]}_{\leq (1-p)^k}$$

$$\overset{\text{Lemma 3}}{\leq} C' k^d \times C^k \times (1-p)^k$$

If $C(1-p) < 1$, then $\exists\, c_0 > 0$ s.t.

$$\forall k \geq 1 \quad \mathbb{P}_p[|\Delta_{ext} C_0| = k] \leq e^{-c_0 k}.$$

Therefore, if $C(1-p)<1$.

$$\mathbb{P}_p[|C_0|\geq n, |C_0|<\infty] \overset{\text{Lemma 1}}{\leq} \mathbb{P}_p\left[|\Delta_{ext} C_0| \geq c\, n^{\frac{d-1}{d}}, |C_0|<\infty\right]$$

$$\leq \sum_{k\geq c n^{\frac{d-1}{d}}} \mathbb{P}_p[|\Delta_{ext} C_0| = k]$$

$$\leq \frac{1}{1-e^{-c_0}} \times e^{-c_0 \cdot c n^{\frac{d-1}{d}}}.$$

$$\leq e^{-c_1 n^{\frac{d-1}{d}}} \quad \text{for } n \text{ large enough.} \qquad \blacksquare$$

<u>Warm up 2</u> (renormalization argument.)

There exists $\delta>0$ small enough s.t.

$$(p>p_c) \Longleftrightarrow \left(\exists k\geq 1 \text{ s.t. } \mathbb{P}_p\left[U(3k,6k), \Lambda_k \longleftrightarrow \partial\Lambda_{7k}\right] > 1-\delta\right)$$

↑

"finite size criterion for $p>p_c$"

<u>Proof</u>: $\boxed{\Rightarrow}$ true for every $\delta>0$ because $p>p_c \Rightarrow \begin{cases} \mathbb{P}_p[U(k,2k)] \xrightarrow[k\to\infty]{} 1, \\ \mathbb{P}_p[\Lambda_k \longleftrightarrow \infty] \xrightarrow[k\to\infty]{} 1. \end{cases}$

$\boxed{\Leftarrow}$ Let $\delta>0$ small (to be fixed later).

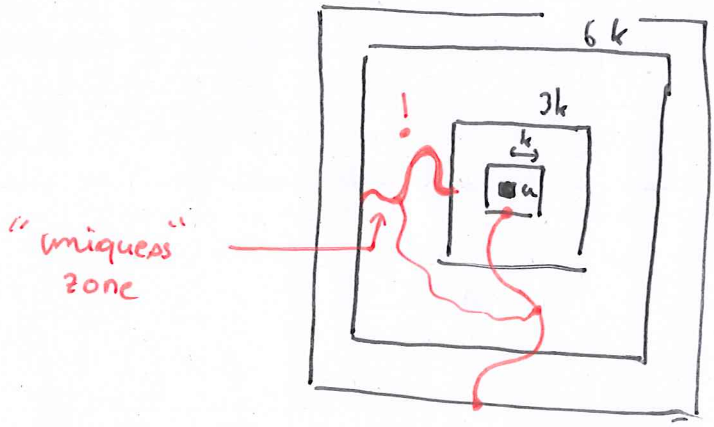
Let $k\geq 1$ s.t. $\mathbb{P}_p\left[U(3k,6k), \Lambda_k \longleftrightarrow \partial\Lambda_{7k}\right] > 1-\delta$.

Consider the graph $G_k = (V_k, E_k)$ $\quad V_k = 2k\,\mathbb{Z}^d$

$$E_k = \{\{2kx, 2ky\},\ x \sim y\}.$$

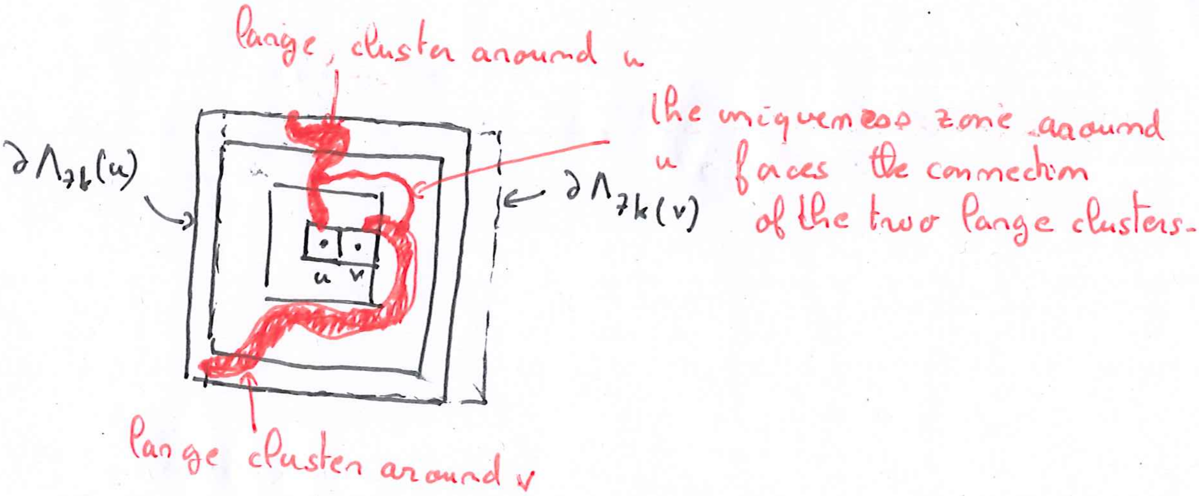
For $u \in V_k$, consider the event

$$A_k(u) = \begin{array}{l}\{\exists!\ \text{cluster in } \Lambda_{6k}(u) \text{ connecting } \Lambda_{3k}(u) \text{ to } \partial\Lambda_{6k}(u)\}\\ \cap\ \{\Lambda_k(u) \longleftrightarrow \partial\Lambda_{7k}(u)\}.\end{array}$$



Intuition: “locally around u”, there is a unique large cluster”

For $uv \in E_k$, set $\eta(uv) = \begin{cases} 1 & \text{if } A_k(u) \cap A_k(v) \text{ occurs,} \\ 0 & \text{otherwise.}\end{cases}$



Intuition: “If $\eta(uv) = 1$, the unique large cluster around u is connected to the unique large cluster around v.”

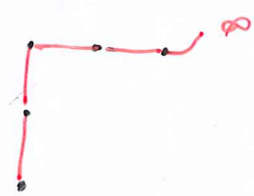
Notice that η is a bond percolation process on G_k that is 1S-independent (the configuration in $A \subset V_k$ is independent of the configuration of $B \subset V_k$, provided that A and B are sufficiently far apart).

- $\forall e \in E_k \quad \mathbb{P}_p[\eta(e)=1] \geq 1-2\delta.$

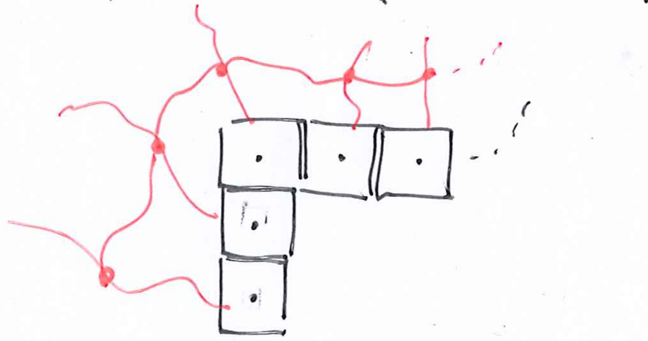
By exercise 2 in sheet 8, if $\delta > 0$ small enough (independently of k), we have

$$\mathbb{P}_p[0 \longleftrightarrow \infty \text{ in } \eta] > 0$$

Observe that $0 \longleftrightarrow \infty$ in η implies that $\Lambda_k \longleftrightarrow \infty$ in ω.



open path in η



$\Lambda_k \longleftrightarrow \infty$ in ω.

Hence $\mathbb{P}_p[\Lambda_k \longleftrightarrow \infty \text{ in } \omega] > 0$, which implies $p > p_c$.

Proof of the Thm (upper bound). Fix $p > p_c$.

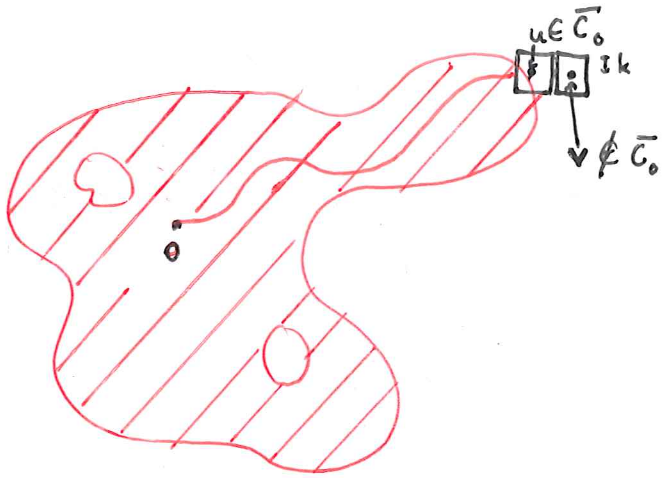
Fix $\delta > 0$ very small, and let $k \geq 1$ such that

$$P_p\left[U(3k, 6k) \cap \{ \Lambda_k \longleftrightarrow \partial \Lambda_{7k} \} \right] \geq 1 - \delta.$$

Consider $G_k = (V_k, E_k)$ and $\eta \in \{0,1\}^{E_k}$ as in warm-up 2.

(in particular $P_p[\eta(uv) = 1] \geq 1 - 2\delta$)

Let $\overline{C_0} = \{ u \in V_k : C_0 \cap \Lambda_k(u) \neq \emptyset \}$.



key observation: if $uv \in \Delta \overline{C_0}$ and $\operatorname{diam}(C_0) \gg k$

then $\eta(uv) = 0$.

Let n large $(n \gg |\Lambda_k|)$. Then

$$P_p[n \leq |C_0| < \infty] \leq P_p\left[\frac{n}{|\Lambda_k|} \leq |\overline{C_0}| < \infty \right.$$

$$\overset{\text{Lemma}}{\leq} P_p\left[c \left(\frac{n}{\Lambda_k} \right)^{\frac{d-1}{d}} \leq |\Delta_{ext} \overline{C_0}| < \infty \right]$$

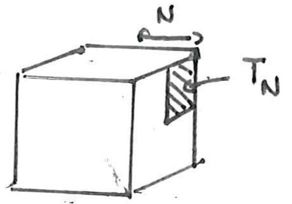
$$\leq e^{-c' \cdot \left(\frac{n}{\Lambda_k} \right)^{\frac{d-1}{d}}}$$

Same reasoning as in Warmup 2, provided δ small enough (use the key observation, and the fact that η is 14-independent.) ∎

5 PROOF OF GRIMMETT-MARSTRAND THEOREM.

Notation: • $\mathcal{S}_m = \{ S \subset \mathbb{Z}^d$ connected s.t. $0 \in S$, $|S| = m \}$

• $T_N = \{N\} \times \{0, \dots, N\}^{d-1}$



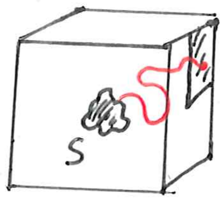
Grimmett-Marstrand theorem follows from the following two propositions.

Prop. 1

Let $\eta > 0$, $p \in [0,1]$. Then

$$\left(\theta(p) > 0 \right) \Rightarrow \left(\exists\, N \geq m \geq 1 \text{ s.t. } \underbrace{\forall S \in \mathcal{S}_m \quad P_p\left[S \overset{\Lambda_N}{\longleftrightarrow} T_N \right] > 1 - \eta}_{=: \, FC^{\eta}_{m,N}(p)} \right)$$

FC "finite criterion".



The event $S \overset{\Lambda_N}{\longleftrightarrow} T_N$

Prop. 2

Let $\eta > 0$, $p \in [0,1)$. Then $\forall m \geq 1$ $\forall N \geq 3m$

$$\left(FC^{\eta}_{m,N}(p) \right) \Rightarrow \left(P_{p+\delta(\eta)}\left[0 \overset{\mathbb{Z}^2 \times \{-2N, \dots, 2N\}^{d-2}}{\longleftrightarrow} \infty \right] > 0 \right)$$

where $\delta(\eta) \underset{\eta \to 0}{\longrightarrow} 0$

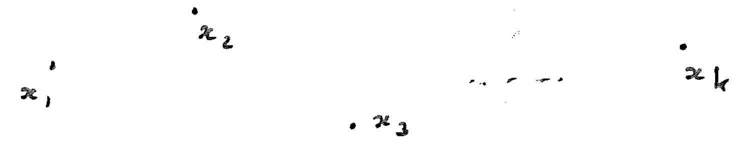
Lemma 1:

Let $p \in [0,1]$ s.t. $\theta(p) > 0$. Then

$$\min_{S \in \mathcal{S}_m} \mathbb{P}_p[S \longleftrightarrow \infty] \xrightarrow[m \to \infty]{} 1$$

Proof: idea: By the mixing property, if we have k points $x_1, \dots, x_k$ far apart, then $\mathbb{P}_p[\{x_1, \dots, x_k\} \not\longleftrightarrow \infty] \simeq (1-\theta)^k$.

(the events $x_i \longleftrightarrow \infty$ are "roughly independent")



Let $\varepsilon > 0$. Choose $k \geq 1$ s.t. $(1-\theta)^k \leq \frac{\varepsilon}{2}$,

$n \geq 1$ s.t. $k \, \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n, 0 \not\longleftrightarrow \infty] \leq \frac{\varepsilon}{2}$.

If $|S| \geq k|\Lambda_n|$, then we can find $x_1, \dots, x_k \in S$ s.t. for $i \neq j$ $\Lambda_n(x_i) \cap \Lambda_n(x_j) = \emptyset$.

$$\begin{aligned}
\mathbb{P}_p[S \not\longleftrightarrow \infty] &\leq \mathbb{P}_p[\forall i \ x_i \not\longleftrightarrow \infty] \\
&\leq \mathbb{P}_p[\forall i \ x_i \not\longleftrightarrow \partial\Lambda_n(x_i)] \\
&\qquad + \mathbb{P}_p[\exists i \ x_i \longleftrightarrow \partial\Lambda_n(x_i), x_i \not\longleftrightarrow \infty] \\
&\leq (1-\theta)^k + k \, \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n, 0 \not\longleftrightarrow \infty] \\
&\leq \varepsilon
\end{aligned}$$

■

Proof of prop. 1

Let $\eta > 0$, assume $\theta(p) > 0$.

By Lemma 1, we can pick $m \geq 1$ large enoug such that

$$\forall S \in \mathcal{S}_m \quad P_p[S \longleftrightarrow \infty] \geq 1 - \left(\frac{\eta}{3}\right)^{2d2^{d-1}}.$$

Choose L large enough s.t. $P_p[U(m,L)] \geq 1 - \frac{\eta}{3}$.

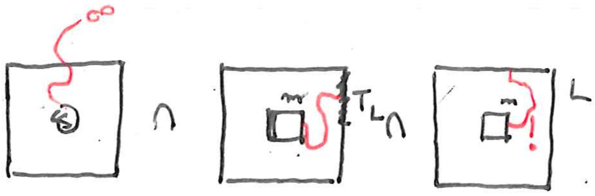
We have $P_p[\Lambda_m \longleftrightarrow \partial \Lambda_L] \geq 1 - \left(\frac{\eta}{3}\right)^{2d2^{d-1}}$.

By symmetry and the square root trick

$$P_p[\Lambda_m \overset{\Lambda_L}{\longleftrightarrow} T_L] \geq 1 - \frac{\eta}{3}.$$

Finally, for every $S \in \mathcal{S}_m$

$$P_p[S \overset{\Lambda_L}{\longleftrightarrow} T_L] \geq P_p[\{S \longleftrightarrow \infty\} \cap \{\Lambda_m \overset{\Lambda_L}{\longleftrightarrow} \partial\Lambda_L\} \cap U(m,L)]$$

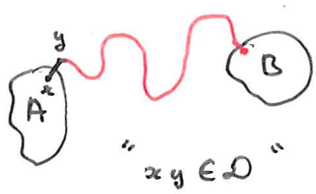


$$\geq 1 - \eta \quad \blacksquare$$

Lemma 2

Let $A, B \subset \mathbb{Z}^d$ s.t. $A \cap B = \emptyset$.

Write $\mathcal{D} = \{xy \in \Delta A \text{ s.t. } y \overset{\mathbb{Z}^d \setminus A}{\longleftrightarrow} B\}$



$\forall \eta > 0$

$$\left(P_p[A \longleftrightarrow B] \geq 1 - \eta\right) \Rightarrow P_p\left[|\mathcal{D}| \geq \frac{1}{\varepsilon(\eta)}\right] \geq 1 - \varepsilon(\eta).$$

where $\varepsilon(\eta) \underset{\eta \to 0}{\longrightarrow} 0$.

Proof: Let $\varepsilon > 0$

$$\eta \geq P_p[A \not\leftrightarrow D] \geq P_p\left[\text{all the edges of } \mathcal{D} \text{ are closed}, |\mathcal{D}| < \tfrac{1}{\varepsilon}\right]$$

$$\overset{\text{independence}}{\geq} (1-p)^{1/\varepsilon} \times P_p\left[|\mathcal{D}| < \tfrac{1}{\varepsilon}\right].$$

Therefore, $\forall \varepsilon > 0$

$$P_p\left[|\mathcal{D}| < \frac{1}{\varepsilon}\right] \leq \frac{\eta}{(1-p)^{1/\varepsilon}}.$$

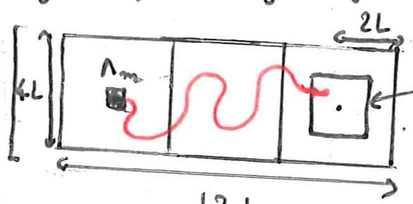
Choose $\varepsilon(\eta)$ defined by $\varepsilon(\eta) = \dfrac{\eta}{(1-p)^{1/\varepsilon(\eta)}}$ $\blacksquare$

Proof of prop 2 (sketch) fix $\eta > 0$.

Let $p > 0$ $m \geq 1$ $L \geq 3m$ s.t.

$$\forall S \in \mathcal{S}_m \quad P_p[S \overset{\Lambda_N}{\longleftrightarrow} T_N] \geq 1-\eta.$$

Warmup (stirring + sprinkling arguments)

$$P_{p+\delta(\eta)}\Big[$$  $$\longleftarrow (8L, 0, \dots, 0) + \Lambda_L \Big] \geq 1 - \varepsilon(\eta).$$

where $\delta(\eta) \searrow 0$. and $\varepsilon(\eta) \searrow 0$ as $\eta \searrow 0$.

Let $\omega \sim P_p$.

Step 1: explore the ω-cluster $C_0^{\Lambda_L}$ of Λ_m inside Λ_L

We know $P_p[C_0^{\Lambda_N} \cap T_N] > 1-\eta$.

Choose $X_1 = X_1(\omega) \in C_0^{\Lambda_N} \cap T_N$

Step 2: "skinning"

there exists $T_L^{(1)}$ translates of T_L in $\{2L\} \times \{-L, \dots, L\}^{d-1}$

s.t. $\quad P_p\left[S_1 \xleftrightarrow{x_1 + \Lambda_L} T_L^{(1)} \right] > 1 - \eta.$

~~Therefore~~ $\cancel{P\left[\Lambda_m \xleftrightarrow{\Lambda_L \cup x_1 + \Lambda_L} \{2L\} \times \{-L, \dots, L\}^{d-1} \right] > 1 - 2\eta}$

NO!

Step 3: "sprinkling argument"

By lemma 2,

$$P_p\left[\exists \text{ many edges } xy \text{ of } \Delta C_0^{\Lambda_L} \text{ s.t. } y \xleftrightarrow{\Lambda_L(x_1)} T_L^{(1)} \right] > 1 - \varepsilon(\eta)$$

Let $\xi \sim P_\delta$ ind. of ω. $\quad$ ($\delta > 0$ small).

with high probability, $\exists$ an edge of $\Delta C_0^{\Lambda_L}$ that is ξ-open and connected to T_L^1 in $\Lambda_L(x_1)$.

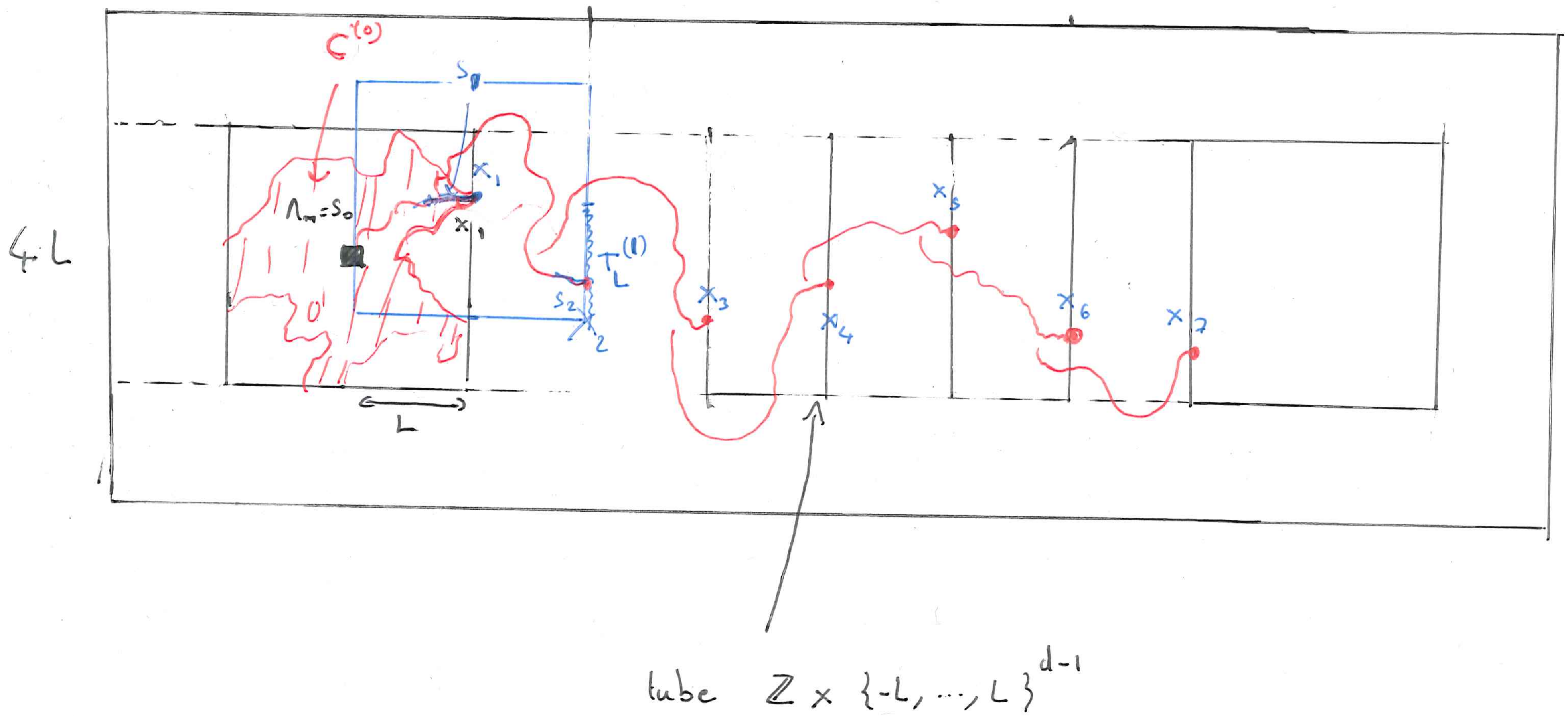
We can find $x_2 \in T_L^{(2)}$ s.t. Λ_m is connected to x_2 in $\Lambda_L \cup \Lambda_L(x_1)$ in ω + "sprinkling".

Step 4: iterate:

We can find $x_1, x_2, \dots, x_\eta$ as

$x_i \in \{iL\} \times \mathbb{Z}^d \qquad x_i \xleftrightarrow{\Lambda_L(x_i)} x_{i+1}$ in ω + "sprinkling"

This concludes the W.U.



4L
$C^{(0)}$
$\Lambda_m = S_0$
S_1
x_1
$T_L^{(1)}$
S_2
x_2
x_3
x_4
x_5
x_6
x_7
L
tube $\mathbb{Z} \times \{-L, \dots, L\}^{d-1}$