I. Making a vector-field conservative. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $\varphi(0) = -1$ and $X : \mathbb{R}^2 \to \mathbb{R}$ be the vector field such that for all $(x, y) \in \mathbb{R}^2$

$$X(x,y) = \left(\frac{2xy}{(1+x^2)^2}, \varphi(x)\right).$$

(1) Find a function φ satisfying the previous condition $\varphi(0) = -1$ such that X is conservative.

(2) Find a potential for the conservative vector field X found in question (1).

(3) Compute

$$\int_\gamma X(s) \cdot d\vec{s}$$

where γ is a direct parametrisation of the ellipse $5x^2 + 2y^2 = 7$.

Solution:

(1) As $\mathbb{R}^2$ is star-shaped, $X = (X_1, X_2)$ is exact if and only if $\partial_y X_1 = \partial_x X_2$, or

$$\varphi'(x) = \frac{2x}{(1+x)^2}$$

and as $\varphi(0) = -1$, we find

$$\varphi(x) = -\frac{1}{1+x^2},$$

so that

$$X(x,y) = \left(\frac{2xy}{(1+x)^2}, -\frac{1}{1+x^2}\right).$$

(2) By integrating $X_2 = \varphi$ in y, if f is a primitive of X, we find

$$f(x,y) = \frac{-y}{1+x^2} + g(x)$$

for some function $g : \mathbb{R} \to \mathbb{R}$. Now, we have

$$\partial_x f(x,y) = X_1(x,y) + g'(x)$$

so g must be constant and the primitives of X are

$$f(x,y) = \frac{-y}{1+x^2} + c, \quad c \in \mathbb{R}.$$

(3) As the curve γ is closed and X is conservative, the integral vanishes.

II. Fubini's theorem for explicit functions (1).

(1) Compute

$$\int_{[-1,1]\times[2,3]} \left(x^4 y - y^5 x + y^3\right) dxdy$$

(2) Let $D^2 = \mathbb{R}^2 \cap \left\{(x,y) : x^2 + y^2 \leq 1\right\}$ be the unit disk in the plan. Compute

$$\int_{D^2} x^2 y^2 dxdy$$

by following the following steps.

(a) Show that

$$\int_0^{\frac{\pi}{2}} \cos^4(\theta)\sin^2(\theta)d\theta = \frac{\pi}{32}$$

(b) Show that for all continuous function $f : D^2 \to \mathbb{R}$, we have

$$\int_{D^2} f(x,y)dxdy = \int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)dy\right)dx.$$

(c) Compute

$$\int_{D^2} x^2y^2dxdy,$$

by making the formula of question (2) and a (1-dimensional) change of variable using trigonometric functions and symmetry.

Solution:

(1) We have

$$\begin{aligned}\int_{[-1,1]\times[2,3]} \left(x^4y - y^5x + y^3\right)dxdy &= \int_2^3\left(\int_{-1}^1 x^4y - y^5x + y^3dx\right)dy\\ &= \int_2^3\left(\left[y\frac{x^5}{5} - y^5\frac{x^2}{2}\right]_{-1}^1 + 2y^3\right)dxdy\\ &= \int_2^3\left(\frac{2y}{5} + 2y^3\right)dy\\ &= \left[\frac{y^2}{5} + \frac{y^4}{2}\right]_2^3 = \frac{9-4}{5} + \frac{81-16}{2} = \frac{67}{2}.\end{aligned}$$

(2) (a) Using $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$, $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\cos^2 + \sin^2 = 1$, we obtain

$$\begin{aligned}\cos^4(\theta)\sin^2(\theta) &= \frac{1}{4}\cos^2(\theta)\sin^2(2\theta) = \frac{1}{4}\left(\frac{1+\cos(2\theta)}{2}\right)\left(1-\cos^2(2\theta)\right)\\ &= \frac{1}{4}\left(\frac{1+\cos(2\theta)}{2}\right)\left(\frac{1-\cos(4\theta)}{2}\right) = \frac{1}{16}\left(1+\cos(2\theta)-\cos(4\theta)-\cos(2\theta)\cos(4\theta)\right)\\ &= \frac{1}{32}\left(2+2\cos(2\theta)-2\cos(4\theta)-2\cos(2\theta)\cos(4\theta)\right)\\ &= \frac{1}{32}\left(2+2\cos(2\theta)-2\cos(4\theta)-\cos(2\theta)-\cos(6\theta)\right)\\ &= \frac{1}{32}\left(2+\cos(2\theta)-2\cos(4\theta)-\cos(6\theta)\right) \qquad (1)\end{aligned}$$

where we used

$$\cos(2\theta)\cos(4\theta) = \frac{1}{2}\left(\cos(2\theta)+\cos(6\theta)\right),$$

an identity which can be derived from the de Moivre's formula

$$\begin{aligned}\cos(2\theta)\cos(4\theta) &= \frac{\left(e^{2i\theta}+e^{-2i\theta}\right)}{2}\frac{\left(e^{4i\theta}+e^{-4i\theta}\right)}{2} = \frac{1}{4}\left(e^{2i\theta}+e^{-2i\theta}+e^{6i\theta}+e^{-6i\theta}\right)\\ &= \frac{1}{2}\left(\cos(2\theta)+\cos(2\theta)\right).\end{aligned}$$

Now, we have obviously for all integer $k \geq 1$

$$\int_0^{\frac{\pi}{2}} \cos(2k\theta)d\theta = \left[\frac{\sin(2k\theta)}{k}\right]_0^{\frac{\pi}{2}} = \sin(\pi k) = 0. \qquad (2)$$

Therefore, by (6), (1) and (2), we obtain

$$\int_0^{\frac{\pi}{2}} \cos^4(\theta)\sin^2(\theta)d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{32}\left(2+\cos(2\theta)-2\cos(4\theta)-\cos(6\theta)\right)d\theta = \frac{\pi}{32}.$$

Remark 1. One could also directly expand $\cos^4(\theta)\sin^2(\theta)$ with de Moivre's formula, but the computation would be slightly longer.

(b) We have for all $(x,y) \in D^2$ the inequality

$$x^2+y^2 \leq 1 \tag{3}$$

which implies that $-1 \leq x \leq 1$. Therefore, (3) holds if and only $-1 \leq x \leq 1$ and

$$y^2 \leq 1-x^2$$

which is equivalent to (notice that $1-x^2 \geq 0$) $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$. Finally, we have proved that

$$\begin{aligned} D^2 &= \mathbb{R}^2 \cap \left\{(x,y) : x^2+y^2 \leq 1\right\} \\ &= \mathbb{R}^2 \cap \left\{(x,y) : -1 \leq x \leq 1 \text{ and } -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\right\}. \end{aligned} \tag{4}$$

The integral formula is then a direct consequence of (4) and Fubini's theorem.

(c) Using the formula in (b), we find

$$\begin{aligned} \int_{D^2} x^2y^2dxdy &= \int_{-1}^1 x^2\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^2dy\right)dx = \int_{-1}^1 x^2\left[\frac{y^3}{3}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dx \\ &= \frac{2}{3}\int_{-1}^1 x^2\left(1-x^2\right)^{\frac{3}{2}}dx = \frac{4}{3}\int_0^1 x^2\left(1-x^2\right)^{\frac{3}{2}}dx \end{aligned} \tag{5}$$

where we used the symmetry of the integral in the last equality (formally, one can make a change of variable $t=-x$ in the integral $\int_{-1}^0$ to obtain the result). Now, we make the change of variable $x=\sin(\theta)$ to obtain (using $1-\sin^2=\cos^2$)

$$\int_0^1 x^2\left(1-x^2\right)^{\frac{3}{2}}dx = \int_0^{\frac{\pi}{2}} \sin^2(\theta)\left(1-\sin^2(\theta)\right)^{\frac{3}{2}}\cos(\theta)d\theta = \int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos^4(\theta)d\theta. \tag{6}$$

Therefore, thanks of the computation in (a), (5) and (6)

$$\int_{D^2} x^2y^2dxdy = \frac{4}{3}\int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos^4(\theta)d\theta = \frac{4}{3}\times\frac{\pi}{32} = \frac{\pi}{24}.$$

Remark 2. Once we know the change of variables, we can use polar coordinates to find

$$\begin{aligned} \int_{D^2} x^2y^2dxdy &= \int_0^1\int_0^{2\pi} r^5\cos^2(\theta)\sin^2(\theta)d\theta = \frac{1}{6}\int_0^{2\pi}\cos^2(\theta)\sin^2(\theta)d\theta \\ &= \frac{1}{24}\int_0^{2\pi}\sin^2(2\theta)d\theta = \frac{1}{24}\int_0^{2\pi}(1-\cos^2(2\theta))d\theta = \frac{1}{24}\int_0^{2\pi}\left(1-\frac{1+\cos(4\theta)}{2}\right)d\theta = \frac{\pi}{24}, \end{aligned}$$

where we used $\sin(2\theta)=\cos(\theta)\sin(\theta)$ and $\cos(2\theta)=2\cos^2(\theta)-1$.

III. Fubini's theorem for explicit functions (2).

Compute the following double integrals $\int_D f(x,y)dxdy$, where the continuous function $f : D \to \mathbb{R}$ and the domain D are given by

1. $f(x,y)=x$, and $D=\mathbb{R}^2 \cap \{(x,y) : y \geq 0, x-y+1 \geq 0, x+2y-4 \leq 0\}$.
2. $f(x,y)=\cos(xy)$, and $D=\mathbb{R}^2 \cap \left\{(x,y) : 1 \leq x \leq 2, 0 \leq xy \leq \frac{\pi}{2}\right\}$.

3. $f(x,y) = \frac{1}{(x+y)^3}$, and $D = \mathbb{R}^2 \cap \{(x,y) : 1 \leq x \leq 3, y \geq 2, x+y \leq 5\}$.

4. $f(x,y) = \frac{xy}{1+x^2+y^2}$, and $D = \mathbb{R}^2 \cap \left\{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x^2+y^2 \geq 1\right\}$.

Solution:

1. For all $(x,y) \in D$, we have $y \geq 0$, and $y-1 \leq x \leq 4-2y$, which is non-empty if and only $y-1 \leq 4-2y$, or $y \leq \dfrac{5}{3}$. Therefore, we have

$$\int_D f(x,y)dxdy = \int_0^{\frac{5}{3}} \int_{y-1}^{4-2y} xdx = \frac{1}{2}\int_0^{\frac{5}{3}} \left((4-2y)^2 - (y-1)^2\right) dy = \frac{275}{54}.$$

2. We have

$$\int_D f(x,y)dxdy = \int_1^2 \left(\int_0^{\frac{\pi}{2x}} \cos(xy)dy\right) dx = \int_1^2 \left[\frac{\sin(xy)}{x}\right]_0^{\frac{\pi}{2x}} dx = \int_1^2 \frac{dx}{x} = \log(2).$$

3. We have

$$\int_D f(x,y)dxdy = \int_1^3 \int_2^{5-x} \frac{dy}{(x+y)^3} = \int_1^3 -\frac{1}{2}\left(\frac{1}{25} - \frac{1}{(x+2)^2}\right) = \frac{2}{75}.$$

4. We have

$$\begin{aligned}
\int_D f(x,y)dxdy &= \int_0^1 \left(\int_{\sqrt{1-x^2}}^1 \frac{xy}{1+x^2+y^2}dy\right) dx \\
&= \int_0^1 \left[\frac{x}{2}\log\left(1+x^2+y^2\right)\right]_{\sqrt{1-x^2}}^1 dx \\
&= \int_0^1 \frac{x}{2}\left(\log(2+x^2) - \log(2)\right) dx \\
&= \frac{3}{4}\log\left(\frac{3}{2}\right) - \frac{1}{4}.
\end{aligned}$$

IV. Fubini's theorem for explicit functions (3).

Compute the area of the domain

$$D = \mathbb{R}^2 \cap \left\{(x,y) : -1 \leq x \leq 1, x^2 \leq y \leq 4 - x^3\right\}.$$

Solution: We have

$$\text{area}(D) = \int_D dxdy = \int_{-1}^1 \int_{x^2}^{4-x^3} dydx = \int_{-1}^1 \left(4 - x^3 - x^2\right) dx = \left[4x - \frac{x^4}{4} - \frac{x^3}{3}\right]_{-1}^1 = \frac{22}{3}.$$

Remark 3. Notice that for all $-1 \leq x \leq 1$, we have $4 - x^3 \geq 3 > x^2$, so the decomposition of the domain in the previous question is correct.