I. Volume of the 3-dimensional ball. Let $r > 0$ and $B_3(0,r) = \mathbb{R}^3 \cap \{(x,y,z) : x^2+y^2+z^2 \leq r^2\}$ be the open ball of radius $r > 0$. By using a change of coordinates $f : [0,r) \times [0,2\pi) \times [0,\pi) \to B_3(0,r)$ given as follows :

$$f(t,\theta,\varphi) = \begin{cases} t\cos(\theta)\sin(\varphi) \\ t\sin(\theta)\sin(\varphi) \\ t\cos(\varphi) \end{cases}$$

compute the volume of $B_3(0,r)$, defined by

$$\int_{B_3(0,r)} dxdydz.$$

Solution: One immediately checks that $f : [0,r) \times [0,2\pi) \times [0,\pi) \to B_3(0,r)$ is a diffeomorphism, and we compute

$$\nabla f(r,\theta,\varphi) = \begin{pmatrix} \cos(\theta)\sin(\varphi) & -t\sin(\theta)\sin(\varphi) & t\cos(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) & t\cos(\theta)\sin(\varphi) & t\sin(\theta)\cos(\varphi) \\ \cos(\varphi) & 0 & -t\sin(\varphi). \end{pmatrix}$$

By expanding the last line, we find

$$\begin{aligned} \det\nabla f(t,\theta,\varphi) &= t^2\cos(\varphi)\det\begin{pmatrix} -\sin(\theta)\sin(\varphi) & \cos(\theta)\cos(\varphi) \\ \cos(\theta)\sin(\varphi) & \sin(\theta)\cos(\varphi) \end{pmatrix} - t^2\sin^3(\varphi)\det\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= -t^2\cos(\varphi)\sin^2(\varphi) - t^2\sin^3(\varphi) \\ &= -t^2\cos(\varphi). \end{aligned}$$

Therefore, by the change of variable formula, we find (notice the absolute value)

$$\int_{B_3(0,r)} dxdydz = \int_0^r \int_0^{2\pi} \int_0^{\pi} t^2|\cos(\varphi)|dtd\theta d\varphi = 2\pi\left[\frac{t^3}{3}\right]_0^r \int_0^{\frac{\pi}{2}} 2\cos(\varphi)d\varphi = \frac{4\pi}{3}r^3.$$

where we have used

$$\int_0^{\pi} |\cos(\varphi)|d\varphi = 2\int_0^{\frac{\pi}{2}} \cos(\varphi)d\varphi$$

by obvious symmetry of the integrand.

II. n-dimensional volume of a ball. Let $n \geq 2$, $r > 0$ and

$$B_n(0,r) = \mathbb{R}^n \cap \{x = (x_1,\cdots,x_n) : x_1^2+\cdots+x_n^2 \leq r^2\}$$

be the closed ball of radius r. Let

$$V(n,r) = \int_{B_n(0,r)} dx_1\cdots dx_n$$

be the volume of the ball of radius 1 and write for simplicity $V(n) = V(n,1)$ for the volume of the unit ball.

1. Compute $V(1)$ and $V(2)$.
2. Show by a change of variable that for all $r > 0$, we have

$$V(n,r) = r^nV(n).$$

3. Show by Fubini's theorem that for all $n \geq 2$, we have

$$V(n) = V(n-2)\int_{B_2(0,1)} (1-x_{n-1}^2-x_n^2)^{\frac{n-2}{2}}dx_{n-1}dx_n$$

4. Deduce that for all $n \in \mathbb{N}$, we have

$$V(2n) = \frac{\pi^n}{n!} \quad V(2n+1) = \frac{2^{2n+1} n!}{(2n+1)!}\pi^n.$$

where we recall that $n! = 1 \times 2 \times \cdots (n-1) \times n$ for all $n \geq 1$.

Hint: Notice in 3. that

$$B_n(0,1) = \mathbb{R}^n \cap \left\{x = (x_1, \cdots, x_n) : x_{n-1}^2 + x_n^2 \leq 1,\ x_1^2 + \cdots + x_{n-2}^2 \leq 1 - x_{n-1}^2 - x_n^2\right\}$$

Solution:

1. We have $B_1(0,1) = [-1,1]$, so $V(1) = 2$, and $B_2(0,1)$ is the unit disk, which has area π, as one immediately checks by taking polar coordinates $x_1 = r\cos(\theta), x_2 = r\sin(\theta)$:

$$V(2) = \int_{B_2(0,1)} dx_1 dx_2 = \int_0^1 \int_0^{2\pi} r dr d\theta = 2\pi \left[\frac{r^2}{2}\right]_0^1 = \pi.$$

2. Making the change of variable $x = ry$, we notice that $dx_1 \cdots dx_n = r^n dy_1 \cdots dy_n$ (the Jacobian matrix is rI_n, where I_n is the identity matrix, and $\det(rA) = r^n \det(A)$ for all matrix $A \in M_n(\mathbb{R})$ as one checks directly from the definition), so that

$$V(n,r) = \int_{B(0,r)} dx_1 \cdots dx_n = \int_{B(0,1)} r^n dy_1 \cdots dy_n = r^n V(n)$$

3. We have

$$B_n(0,1) = \mathbb{R}^n \cap \left\{x = (x_1, \cdots, x_n) : x_{n-1}^2 + x_n^2 \leq 1,\ x_1^2 + \cdots + x_{n-2}^2 \leq 1 - x_{n-1}^2 - x_n^2\right\}$$

which implies by Fubini's theorem and 1. that

$$\begin{aligned}
V(n) &= \int_{\left\{(x_{n-1},x_n): x_{n-1}^2 + x_n^2 \leq 1\right\}} \left(\int_{\left\{(x_1,\cdots,x_{n-2}): x_1^2 + \cdots + x_{n-2}^2 \leq 1 - x_{n-1}^2 - x_n^2\right\}} dx_1 \cdots dx_{n-2}\right) dx_{n-1} dx_n \\
&= \int_{\left\{(x_{n-1},x_n): x_{n-1}^2 + x_n^2 \leq 1\right\}} V_{n-2}(\sqrt{1 - x_{n-1}^2 - x_n^2}) dx_{n-1} dx_n \\
&= V(n-2) \int_{B_2(0,1)} (1 - x_{n-1}^2 - x_n^2)^{\frac{n-2}{2}} dx_{n-1} dx_n.
\end{aligned}$$

Now, making a change of variables in polar coordinates, we obtain

$$\begin{aligned}
\int_{B_2(0,1)} (1 - x_{n-1}^2 - x_n^2)^{\frac{n-2}{2}} dx_{n-1} dx_n = 2\pi \int_0^1 (1-r^2)^{\frac{n-2}{2}} r dr = 2\pi \left[-\frac{1}{n}(1-r^2)^{\frac{n}{2}}\right]_0^1 \\
= \frac{2\pi}{n}.
\end{aligned}$$

Therefore, we obtain for all $n \geq 2$

$$V(n) = \frac{2\pi}{n} V(n-2).$$

4. Finally, by an immediate induction, we obtain

$$V(2n) = \frac{\pi}{n} V(2n-2) = \prod_{j=2}^{n} \frac{\pi}{j} V(2) = \frac{\pi^{n-1}}{n!} V(2) = \frac{\pi^n}{n!},$$

while

$$V(2n+1) = \frac{2\pi}{(2n+1)} V(2n-1) = \prod_{j=1}^{n} \frac{2\pi}{(2j+1)} V(1)$$

Now, noticing that

$$\prod_{j=1}^{n}(2j+1) = \left(\prod_{j=1}^{n} 2j\right)^{-1} \prod_{j=1}^{n} 2j(2j+1) = \left(\prod_{j=1}^{n} 2j\right)^{-1} (2n+1)! = \frac{(2n+1)!}{2^n n!}$$

we obtain (recalling that $V(1) = 2$)

$$V(2n+1) = \frac{2^{2n+1} n!}{(2n+1)!}\pi^n.$$

Remark 1. One may also use directly spherical coordinates in n dimension, but this leads to the computation of a large determinant.

III. Fubini's theorem for an explicit integral. By computing the integral (justify why it converges)

$$I = \int_{[0,\infty)\times[a,b]} e^{-xy} dxdy$$

in two different ways, where $0 < a < b$, compute

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$

Solution: As $0 < a \le y \le b$, and $e^{-xy} \ge 0$, we observe that for all $R > 0$

$$0 \le \int_{[0,R]\times[a,b]} e^{-xy} dxdy \le (b-a)\int_0^R e^{-ax} dx = \left(\frac{b}{a} - 1\right)(1 - e^{-aR}) \le 1.$$

so the integral converges. Therefore, we have

$$I = \lim_{R\to\infty} \int_{[0,R]\times[a,b]} e^{-xy} dxdy = \lim_{R\to\infty} I(R),$$

and we have by Fubini's theorem

$$I(R) = \int_a^b \left(\int_0^R e^{-xy} dx\right) dy = \int_a^b \left(\frac{1}{y} - \frac{e^{-Ry}}{y}\right) dy = \log\left(\frac{b}{a}\right) - \int_a^b \frac{e^{-Ry}}{y} dy$$

Furthermore, we have (as $a > 0$)

$$\left|\int_a^b \frac{e^{-Ry}}{y} dy\right| = \int_a^b \frac{e^{-Ry}}{y} dy \le \frac{e^{-Ra}}{b} \int_a^b dy = \left(1 - \frac{a}{b}\right) e^{-Ra} \underset{R\to\infty}{\longrightarrow} 0$$

so that

$$I = \log\left(\frac{b}{a}\right).$$

Now, by Fubini theorem, we also have (by definition of the 'improper' integral)

$$I(R) = \int_0^R \left(\int_a^b e^{-xy} dy\right) dx = \int_0^R \frac{e^{-ax} - e^{-bx}}{x} dx \underset{R\to\infty}{\longrightarrow} \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$

Finally, we deduce that

$$I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \log\left(\frac{b}{a}\right).$$

Remark 2. Notice that the integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$$

converges absolutely as $\frac{e^{-ax} - e^{-bx}}{x} = (b-a) + O(|x|)$ as $x \to 0$, and the convergence at infinity is trivial with the two exponential functions.