

I. Change of variables. Let $D = \mathbb{R}^2 \cap \{(x, y) : x^2 + y^2 - 2x \leq 0\}$. Compute

$$\int_D \sqrt{x^2 + y^2} dx dy$$

Hint: Notice that D is a disk.

Solution: First, we have

$$x^2 + y^2 - 2x = (x - 1)^2 + y^2 - 1,$$

so D is the disk as centre $(1, 0)$ and radius 1. By using polar coordinates, $x = r \cos(\theta)$, $y = r \sin(\theta)$, we find

$$(x - 1)^2 + y^2 = r^2 - 2r \cos(\theta) \leq 0 \Leftrightarrow 0 \leq r \leq 2 \cos(\theta),$$

which implies in particular that $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, by the change of variables formula, we have

$$\int_D \sqrt{x^2 + y^2} dx dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2 \cos(\theta)} r^2 dr d\theta = \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3(\theta) d\theta.$$

Now, we have as $\cos(2\theta) = 2 \cos^2(\theta) - 1$

$$\cos^3(\theta) = \frac{1}{2} \cos(\theta) (\cos(2\theta) + 1) = \frac{1}{4} (\cos(3\theta) + \cos(\theta)) + \frac{1}{2} \cos(\theta) = \frac{1}{4} \cos(3\theta) + \frac{3}{4} \cos(\theta).$$

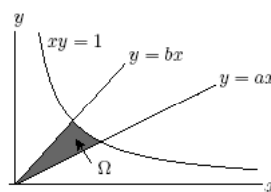
Furthermore, as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) d\theta = [\sin(\theta)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2, \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(3\theta) d\theta = \left[\frac{\sin(3\theta)}{3} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\frac{2}{3},$$

we obtain

$$\int_D \sqrt{x^2 + y^2} dx dy = \frac{8}{3} \left(-\frac{1}{4} \times \frac{2}{3} + \frac{3}{4} \times 2 \right) = \frac{32}{9}$$

II. Fubini's theorem. Let $0 < a < b < \infty$ and Ω the domain defined as below :



With formulas, this means that

$$\Omega = \mathbb{R}^2 \cap \{(x, y) : x \geq 0, y \geq 0, 0 \leq xy \leq 1, ax \leq y \leq bx\}$$

Show that

$$\int_{\Omega} xy dx dy = \frac{1}{4} \log \left(\frac{b}{a} \right).$$

Solution: The extremities are

$$xy = 1 \text{ und } y = bx \rightarrow x = \frac{1}{\sqrt{b}}$$

$$xy = 1 \text{ und } y = ax \rightarrow x = \frac{1}{\sqrt{a}}.$$

We split that domain in two parts $0 \leq x \leq \frac{1}{\sqrt{b}}$ and $\frac{1}{\sqrt{b}} \leq x \leq \frac{1}{\sqrt{a}}$ and we get

$$\int_{\Omega} xy \, dx dy = \int_0^{\frac{1}{\sqrt{b}}} \int_{ax}^{bx} xy \, dy dx + \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} \int_{ax}^{\frac{1}{x}} xy \, dy dx$$

By Fubini's theorem, one obtains

$$\begin{aligned} \int_{\Omega} xy \, dx dy &= \int_0^{\frac{1}{\sqrt{b}}} \int_{ax}^{bx} xy \, dy dx + \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} \int_{ax}^{\frac{1}{x}} xy \, dy dx \\ &= \int_0^{\frac{1}{\sqrt{b}}} x \left[\frac{y^2}{2} \right]_{ax}^{bx} dx + \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} x \left[\frac{y^2}{2} \right]_{ax}^{\frac{1}{x}} dx \\ &= \frac{1}{2} \int_0^{\frac{1}{\sqrt{b}}} (x(bx)^2 - x(ax)^2) dx + \frac{1}{2} \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} \left(x \left(\frac{1}{x} \right)^2 - x(ax)^2 \right) dx \\ &= \frac{b^2}{2} \int_0^{\frac{1}{\sqrt{b}}} x^3 dx + \frac{1}{2} \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} \frac{1}{x} dx - \frac{a^2}{2} \int_0^{\frac{1}{\sqrt{a}}} x^3 dx \\ &= \frac{b^2}{2} \left[\frac{x^4}{4} \right]_0^{\frac{1}{\sqrt{b}}} + \frac{1}{2} \left[\ln|x| \right]_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} - \frac{a^2}{2} \left[\frac{x^4}{4} \right]_0^{\frac{1}{\sqrt{a}}} \\ &= \frac{b^2}{2} \frac{1}{4b^2} + \frac{1}{2} \ln \left(\frac{1}{\sqrt{a}} \right) - \frac{1}{2} \ln \left(\frac{1}{\sqrt{b}} \right) - \frac{a^2}{2} \frac{1}{4a^2} \\ &= \frac{1}{8} + \frac{1}{2} \log \left(\frac{1}{\sqrt{a}} \right) - \frac{1}{2} \log \left(\frac{1}{\sqrt{b}} \right) - \frac{1}{8} \\ &= \frac{1}{2} \left(\log \left(\frac{1}{\sqrt{a}} \right) - \log \left(\frac{1}{\sqrt{b}} \right) \right) = \frac{1}{4} \log \left(\frac{b}{a} \right). \end{aligned}$$

III. Centre of mass. Compute the centre of mass of a half ball of radius $R > 0$

$$B_+^3(0, R) = \mathbb{R}^3 \cap \{(x, y, z) : x^2 + y^2 + z^2 \leq R^2, z \geq 0\},$$

defined by

$$\left(\frac{1}{\text{Vol}(B_+^3(0, R))} \int_{B_+^3(0, R)} x \, dx dy dz, \frac{1}{\text{Vol}(B_+^3(0, R))} \int_{B_+^3(0, R)} y \, dx dy dz, \frac{1}{\text{Vol}(B_+^3(0, R))} \int_{B_+^3(0, R)} z \, dx dy dz \right),$$

where $\text{vol}(B_+^3(0, R))$ is the volume of the half-ball (of radius $R > 0$) in 3-space (one can refer to Serie 11 for its value).

Solution: First, by obvious symmetry, we have

$$\frac{1}{\text{Vol}(B_+^3(0, R))} \int_{B_+^3} x \, dx dy dz = \frac{1}{\text{Vol}(B_+^3(0, R))} \int_{B_+^3} y \, dx dy dz = 0,$$

and as the ball of radius has volume $\frac{4\pi}{3}R^3$, the volume of the half-ball is $\frac{2\pi}{3}R^3$, so we just need to compute the following integral

$$\int_{B_+^3(0, R)} z \, dx dy dz.$$

Using spherical coordinates

$$\begin{cases} x = r \cos(\theta) \sin(\varphi) \\ y = r \sin(\theta) \sin(\varphi) \\ z = r \cos(\varphi) \end{cases}$$

where $0 \leq r \leq R$, $-\pi \leq \theta \leq \pi$ and $0 \leq \varphi \leq \frac{\pi}{2}$ (notice that the angle φ must be positive). Now, we have (using $2 \cos(\theta) \sin(\theta) = 2 \sin(2\theta)$)

$$\begin{aligned} \int_{B_3^+(0,R)} z \, dx dy dz &= \int_{-\pi}^{\pi} \int_0^{\frac{\pi}{2}} \int_0^R r^3 \cos(\varphi) \sin(\varphi) dr d\varphi d\theta \\ &= 2\pi \left(\int_0^R r^3 dr \right) \left(\int_0^{\frac{\pi}{2}} \frac{1}{2} \sin(2\varphi) d\varphi \right) \\ &= 2\pi \left[\frac{r^4}{4} \right]_0^R \left[-\frac{1}{4} \cos(2\varphi) \right]_0^{\frac{\pi}{2}} \\ &= 2\pi \times \frac{R^4}{4} \times \frac{1}{2} \\ &= \frac{\pi R^4}{4}. \end{aligned}$$

As the volume of the upper half-ball of radius $R > 0$ in \mathbb{R}^3 is $\frac{2\pi}{3}R^3$, we find

$$\frac{1}{\text{vol}(B_3^+(0,R))} \int_{B_3^+(0,R)} z \, dx dy dz = \frac{3}{2\pi R^3} \times \frac{\pi R^4}{4} = \frac{3R}{8} < R$$

as it should be. Therefore, the centre of gravity (or mass) of the upper-half ball of radius $R > 0$ in \mathbb{R}^3 is

$$\left(0, 0, \frac{3R}{8} \right).$$

IV. The Astroid Let $a > 0$. The Astroid $A(a) \subset \mathbb{R}^2$ is the geometric figure in the plane defined by

$$A(a) := \left\{ (x, y) \in \mathbb{R}^2 \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \right\}$$

The construction of an Astroid is very geometric (see link). Let $B(a)$ denote the set

$$B(a) = \{(rx, ry) \in \mathbb{R}^2 \mid r \in [0, 1], (x, y) \in A(a)\}.$$

Compute the area of $B(a)$ using the theorem of Green.

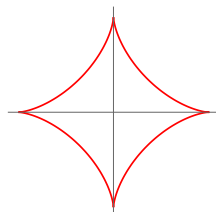


Abbildung 1: The red Astroid $A(a)$ is the boundary of $B(a)$.

Solution: A parametrisation of $A(a)$ is such that $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$,

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) = (a \cos^3(t), a \sin^3(t)).$$

Thanks of Green's theorem, we have

$$\begin{aligned} \text{Area}(B(a)) &= \iint_{B(a)} 1 \, dx dy = \int_{A(a)} (0, x) \cdot d\vec{s} \\ &= \int_0^{2\pi} \gamma_1(t) \dot{\gamma}_2(t) \, dt \\ &= 3a^2 \int_0^{2\pi} \cos^4(t) \sin^2(t) \, dt \end{aligned}$$

Now, recall from Lösung 10 that

$$\cos^4(\theta) \sin^2(\theta) = \frac{1}{32} (2 + \cos(2\theta) - 2 \cos(4\theta) - \cos(6\theta))$$

and as for all integer $k \geq 1$, we have

$$\int_0^{2\pi} \cos(k\theta) d\theta = 0,$$

we deduce that

$$\int_0^{2\pi} \cos^4(t) \sin^2(t) dt = \frac{\pi}{8},$$

so that

$$\text{Area}(B(a)) = \frac{3\pi a^2}{8}.$$