I. Green's formula.

1. If $\gamma \subset \mathbb{R}^2$ is the oriented boundary (positively oriented) of $\Omega = \mathbb{R}^2 \cap \{x : 0 \leq x \leq 1,\ x^2 \leq y \leq \sqrt{x}\}$, and $F(x,y) = ((2xy - x^2), (x + y^2))$. Compute

$$\int_\gamma F \cdot d\vec{s}.$$

2. Let $0 < a, b < \infty$ and $E_{a,b} = \mathbb{R}^2 \cap \left\{(x,y) :\ x \geq 0,\ y \geq 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$ be the quarter of ellipse. Compute

$$\int_{E_{a,b}} (2x^3 - y) dx dy.$$

Solution:

1. Let $P(x,y) = 2xy - x^2$ and $Q(x,y) = x + y^2$. Then

$$\frac{\partial P}{\partial y}(x,y) = 2x, \quad \frac{\partial Q}{\partial x}(x,y) = 1$$

by Green's formula, we have

$$\begin{aligned}\int_\gamma F \cdot d\vec{s} &= \int_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy = \int_0^1 \int_{x^2}^{\sqrt{x}} (1-2x) dy dx = \int_0^1 (1-2x)(\sqrt{x} - x^2) dx \\ &= \int_0^1 \left(\sqrt{x} - 2x^{\frac{3}{2}} - x^2 + 2x^3\right) dx \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}} - \frac{1}{3}x^3 + \frac{1}{2}x^4\right]_0^1 \\ &= \frac{1}{30}.\end{aligned}$$

2. To use the Green's formula, let P, Q be such that

$$\frac{\partial P}{\partial y}(x,y) = y \quad , \frac{\partial Q}{\partial x}(x,y) = 2x^3$$

which can be solved (for example) with $P(x,y) = \frac{y^2}{2}$ and $Q(x,y) = \frac{x^4}{2}$. Now, the boundary of the quarter of ellipse D can be parametrised by

$$\begin{cases} \gamma_1(t) = (a\cos(t), b\sin(t)) & 0 \leq t \leq \frac{\pi}{2} \\ \gamma_2(t) = (0, b(1-t)) & 0 \leq t \leq 1 \\ \gamma_3(t) = (at, 0) & 0 \leq t \leq 1. \end{cases}$$

Now, the Green's formula implies that (if $F = (P,Q)$)

$$\int_{E_{a,b}} (2x^3 - y) dx dy = \int_{\partial E_{a,b}} F \cdot d\vec{s} = \sum_{i=1}^3 \int_{\gamma_i} F \cdot d\vec{s}.$$

Now, we have

$$\begin{aligned}\int_{\gamma_2} F \cdot d\vec{s} &= \int_0^1 \left\langle \left(\frac{1}{2}(b(1-t))^2, 0\right), (0,b) \right\rangle dt = 0 \\ \int_{\gamma_3} F \cdot d\vec{s} &= \int_0^1 \left\langle \left(0, \frac{1}{2}(at)^4\right), (a,0) \right\rangle dt = 0.\end{aligned}$$

Finally, we have

$$\begin{aligned}\int_{\gamma_1} F\cdot d\vec{s} &= \int_0^{\frac{\pi}{2}} \left\langle \left(\frac{1}{2}b^2\sin^2(t), \frac{1}{2}a^4\cos^4(t)\right), (-a\sin(t), a\cos(t)) \right\rangle dt \\ &= \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2}ab^2\sin^3(t) + \frac{1}{2}a^4 b\cos^5(t)\right) dt\end{aligned}$$

Now, we have for all $a, b \in \mathbb{R}$, $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b)$ (which implies in particular that $\cos(2t) = 2\cos^2(t) - 1$), so that

$$\begin{aligned}&\cos^5(t) = \cos(t)\frac{1}{4}\left(\cos(2t)+1\right)^2 = \frac{1}{4}\cos(t)\left(\cos^2(2t) + 2\cos(2t) + 1\right) \\ &= \frac{1}{4}\cos(t)\left(\frac{1}{2}(\cos(4t)+1) + 2\cos(2t) + 1\right) \\ &= \frac{1}{8}\cos(t)\cos(4t) + \frac{1}{8}\cos(t) + \frac{1}{2}\cos(t)\cos(2t) + \frac{1}{4}\cos(t) \\ &= \frac{1}{16}\left(\cos(5t) + \cos(3t)\right) + \frac{3}{8}\cos(t) + \frac{1}{4}\left(\cos(3t) + \cos(t)\right) \\ &= \frac{1}{16}\left(\cos(5t) + 5\cos(3t) + 10\cos(t)\right).\end{aligned}$$

Likewise, we have

$$\sin^3(t) = \sin(t)\sin^2(t) = \sin(t)\frac{1}{2}\left(1 - \cos(2t)\right) = \frac{1}{2}\sin(t) - \frac{1}{2}\cos(2t)\sin(t)$$

Now, we have

$$\begin{aligned}\cos(a)\sin(b) &= \frac{1}{4i}(e^{ia} + e^{-ia})(e^{ib} - e^{-ib}) = \frac{1}{4i}\left(e^{i(a+b)} - e^{i(a-b)} - e^{i(a-b)} + e^{-i(a-b)}\right) \\ &= \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b),\end{aligned}$$

which implies that

$$\cos(2t)\sin(t) = \frac{1}{2}\sin(3t) - \frac{1}{2}\sin(t)$$

and

$$\sin^3(t) = \frac{1}{2}\sin(t) - \frac{1}{4}\sin(3t) + \frac{1}{4}\sin(t) = \frac{1}{4}\left(3\sin(t) - \sin(3t)\right).$$

Therefore, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \cos^5(t)dt = \frac{1}{16}\left[\frac{1}{5}\sin(5t) + \frac{5}{3}\sin(3t) + 10\sin(t)\right]_0^{\frac{\pi}{2}} = \frac{1}{16}\left(\frac{1}{5} - \frac{5}{3} + 10\right) = \frac{8}{15} \\ &\int_0^{\frac{\pi}{2}} \sin^3(t)dt = \frac{1}{4}\left[-3\cos(t) + \frac{1}{3}\cos(3t)\right]_0^{\frac{\pi}{2}} = \frac{1}{4}\left(3 - \frac{1}{3}\right) = \frac{2}{3}.\end{aligned}$$

Finally, we find

$$\int_0^{\frac{\pi}{2}} \left(-\frac{1}{2}ab^2\sin^3(t) + \frac{1}{2}a^4 b\cos^5(t)\right) dt = \frac{4}{15}a^4 b - \frac{1}{3}ab^2.$$

II. Area computation.

Let $a > 0$ and Ω_a be the compact domain delimited by the arcs $y = 0$, and

$$(x(t), y(t)) = (a(t - \sin(t)), a(1 - \cos(t))), \quad t \in [0, 2\pi].$$

Compute the area of Ω_a.

Solution: Let γ the boundary of the compact domain Ω_a. If $F(x,y) = (-y, 0)$, we have

$$\text{Area}(\Omega_a) = \int_\gamma F \cdot d\vec{s},$$

and as the area on the axis $y = 0$ is trivially zero, we find by Green's formula (as $\int_0^{2\pi} \cos(nt)dt = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$)

$$\begin{aligned}\text{Area}(\Omega_a) &= \int_0^{2\pi} a(1-\cos(t)) \times a(1-\cos(t))dt = a^2 \int_0^{2\pi} (1 - 2\cos(t) + \cos^2(t))dt \\ &= a^2 \int_0^{2\pi} \left(1 + \frac{1}{2}(\cos(2t)+1)\right) dt = a^2 \times 2\pi \times \frac{3}{2} = 3\pi a^2.\end{aligned}$$

III. Two methods.

Let $K = \mathbb{R}^2 \cap \left\{(x,y) : x \geq 0,\ y \geq 0 \text{ and } x^2 + y^2 \leq 1\right\}$, and γ its oriented boundary, and $F : \mathbb{R}^2 \to \mathbb{R}^2$ the vector field such that for all $x, y \in \mathbb{R}$, we have $F(x,y) = (xy^2, 2xy)$. Compute $\int_\gamma F \cdot d\vec{s}$

1. By using a parametrisation of γ.
2. By using the Green's formula.

Solution.

1. We parametrise γ by

$$\begin{cases} \gamma_1(t) = (t, 0) & 0 \leq t \leq 1 \\ \gamma_2(t) = (\cos(t), \sin(t)) & 0 \leq t \leq \dfrac{\pi}{2} \\ \gamma_3(t) = (0, t) & 0 \leq t \leq 1. \end{cases}$$

As x or y vanishes identically on γ_1 and γ_2, the integrals on γ_1 and γ_2 vanish (F vanishes identically on these domains). Therefore, we have

$$\int_\gamma F \cdot d\vec{s} = \int_0^{\frac{\pi}{2}} \cos(t)\sin^2(t) \times (-\sin(t)) + 2\cos^2(t)\sin(t)dt = \int_0^{\frac{\pi}{2}} -\cos(t)\sin^3(t) + 2\cos^2(t)\sin(t)dt$$
$$= \left[-\frac{1}{4}\sin^4(t) - \frac{2}{3}\cos^3(t)\right]_0^{\frac{\pi}{2}} = -\frac{1}{4} + \frac{2}{3} = \frac{5}{12}.$$

(We can also linearise as previously).

2. We have by Green's formula

$$\int_\gamma F d\vec{s} = \int_K (2y - 2xy)dxdy.$$

Using polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, with $0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$, we find

$$\begin{aligned}\int_K (2y-2xy)dxdy &= \int_0^1 \int_0^{\frac{\pi}{2}} (2r\sin(\theta) - 2r^2\cos(\theta)\sin(\theta))rdrd\theta \\ &= \int_0^1 2r^2 dr \int_0^{\frac{\pi}{2}} \sin(\theta)d\theta - \int_0^1 r^3 dr \int_0^{\frac{\pi}{2}} 2\cos(\theta)\sin(\theta)d\theta \\ &= \left[\frac{2}{3}r^3\right]_0^1 \left[-\cos(\theta)\right]_0^{\frac{\pi}{2}} - \left[\frac{r^4}{4}\right]_0^1 \left[\sin^2(\theta)\right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}.\end{aligned}$$