I. Differential equations.

Find the solutions of the following first order linear differential equations

1. $7y' + 2y = 2x^3 - 5x^2 + 4x - 1$.
2. $y' + y = xe^{-x}$.
3. $y' - 2y = \cos(x) + 2\sin(x)$.
4. $y'' - 2y' + y = x$, $y(0) = y'(0) = 0$.
5. $y'' + 9y = x + 1$, $y(0) = 0$.
6. $(x + 1)y' + xy = x^2 - 2x + 1$, $y(1) = 1$, where $x > -1$.

Solution

1. First, the solutions of the homogeneous equation $7y'+2y = 0$ are $y(x) = \lambda e^{-\frac{2x}{7}}$, where $\lambda \in \mathbb{R}$. Now, we look for a polynomial solution (necessarily of degree 3). We see that $P(x) = ax^3 + bx^2 + cx + d$ is a solution of the differential equation if and only if
$$7(3ax^2 + bx + c) + 2(ax^3 + bx^2 + cx + d) = 2x^3 - 5x^2 + 4x - 1$$
which gives the solution $P(x) = x^3 - 13x^2 + 93x - 326$. Therefore, the general solution is the family
$$y_\lambda(x) = \lambda e^{-\frac{2x}{7}} + x^3 - 13x^2 + 93x - 326, \quad \lambda \in \mathbb{R}$$

2. Here, $y(x) = \lambda e^{-x}$ $(\lambda \in \mathbb{R})$ are the solutions of the homogeneous equation. Let $y_0(x) = \lambda(x)e^{-x}$ be a solution of the (non-homogeneous) differential equation. Then we get
$$y_0'(x) + y_0(x) = \lambda'(x)e^{-x} = xe^{-x}$$
if and only if $\lambda'(x) = x$, so that $\lambda(x) = \dfrac{x^2}{2}$ is a solution, and $y_0(x) = \frac{x^2}{2}e^{-x}$ is a solution of the equation, and the general solution is the family
$$y_\lambda(x) = \left(\frac{x^2}{2} + \lambda\right) e^{-x}, \quad \lambda \in \mathbb{R}.$$

3. As previously, we find
$$y_\lambda(x) = \lambda e^{2x} - \frac{4}{5}\cos(x) - \frac{3}{5}\sin(x), \quad \lambda \in \mathbb{R}.$$

4. The solution of the homogeneous equation is $y(x) = \lambda_1 e^x + \lambda_2 xe^x$ $(\lambda_1, \lambda_2 \in \mathbb{R})$, and the linear solution $x \mapsto x + 2$ a solution of the equation. Therefore, we find that the general solution is
$$y(x) = \lambda_1 e^x + \lambda_2 xe^x + (x + 2),$$
and the conditions $y(0) = y'(0) = 0$ give
$$\lambda_1 + 2 = 0, \quad \lambda_1 + \lambda_2 + 1 = 0$$
so that $\lambda_1 = -2$ and $\lambda_2 = 1$. Finally, we deduce that the solution is
$$y(x) = (x - 2)e^x + (x + 2).$$

5. The solutions are
$$y_\lambda(x) = -\frac{1}{9}\cos(3x) + \lambda\sin(x) + \frac{x + 1}{9}, \quad \lambda \in \mathbb{R}.$$

6. Using separation of variables, y is a solution of the homogeneous equation $(x+1)y' + xy = 0$ on $(-1, \infty)$ if and only if

$$y' = -\frac{x}{x+1} y = \left(-1 + \frac{1}{1+x}\right) y$$

which holds if and only if

$$y(x) = \lambda e^{-x+\log(1+x)} = \lambda(x+1)e^{-x}, \quad \text{for some } \lambda \in \mathbb{R}.$$

Now, looking for a polynomial solution of degree 1 of the differential equation, one easily infers that $y_0(x) = x - 3$ is such solution. Therefore, we have

$$y(x) = \lambda\,(1+x)\,e^{-x} + x - 3.$$

Now, the condition $y(1) = 1$ yields $2\lambda e^{-1} - 2 = 1$, or $\lambda = \frac{3e}{2}$. Finally, the unique solution is

$$y(x) = \frac{3}{2}(1+x)e^{1-x} + x - 3$$

II. Differential equations (2)**.**

Solve the following differential equations :

1. $(1+e^x)y'' + 2e^x y' + (2e^x + 1)y = xe^x$, with the change of function $z = (1+e^x)y$.
2. $y'' - y' - e^{2x}y = e^{3x}$, with the change of variable $t = e^x$.

Solution:

1. The function z is smooth and y is a solution of the differential equation if and only if

$$z'' + z = xe^x,$$

which implies that

$$z(x) = \lambda_1 \cos(x) + \lambda_2 \sin(x) + \frac{x-1}{2}e^x,$$

and

$$y(x) = \frac{1}{1+e^x}\left(\lambda_1 \cos(x) + \lambda_2 \sin(x) + \frac{x-1}{2}e^x\right) \quad \text{for some } \lambda_1, \lambda_2 \in \mathbb{R}.$$

2. y is a solution of the equation if and only

$$z'' - z = t,$$

and we find the solutions of this equation to be

$$z(t) = \lambda_1 e^t + \lambda_2 e^{-t} - t.$$

Finally, we have

$$y(x) = \lambda_1 e^{e^x} + \lambda_2 e^{-e^x} - e^x, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

III. Extrema (1)**.** Find the local and global extrema of the following functions.

1. $f(x,y) = x^2 + y^2 + xy + 1$.
2. $f(x,y) = x^2 + y^2 + 4xy - 2$.
3. $f(x,y) = y(x^2 + \log^2(y))$, with $y > 0$.

Solution:

1. As $\nabla f(x,y) = (2x+y, x+2y)$, so $(0,0)$ is the only critical point of f. Furthermore, as
$$f(x,y) = \left(x + \frac{y}{2}\right)^2 + \frac{3y^2}{4} + 1 \geq 1 = f(0,0),$$
so $(0,0)$ is the unique global minimum of f.

2. We have
$$f(0,0) = -2$$
while
$$f(x,0) = 5x^2 - 2 > f(0,0),$$
while $f(x,-x) = -1 = 2x^2 - 2 = f(0,0)$.

3. We have $\nabla f(x,y) = (2xy, x^2 + \log^2(y) + 2\log(y))$. As $y > 0$, we obtain $x = 0$, $\log^2(y) + 2\log(y) = 0$, which holds $\log(y)\,(\log(y)+2) = 0$, so that $y = 1$ or $y = e^{-2}$.

IV. Extrema (2)**.** Find the maximum of the following functions on the given compact sets $K \subset \mathbb{R}^2$:

1. $f(x,y) = xy(1-x-y)$ on $K = \{(x,y) : x, y \geq 0,\ x + y \leq 1\}$.
2. $f(x,y) = x - y + x^3 + y^3$ on $K = [0,1]^2$.
3. $f(x,y) = \sin(x)\sin(y)\sin(x+y)$ on $K = \left[0, \frac{\pi}{2}\right]^2$.

Solution:

1. On the boundary ∂K, $x = 0$, $y = 0$, so that $f = 0$ on ∂K. As $f(1/4, 1/4) > 0$ the maximum of f is attained in an interior point of K. As
$$\nabla f(x,y) = (y(1-2x-y), x(1-x-2y))$$
the only critical points on K is
$$x = y = \frac{1}{3}.$$
Therefore, the only local maximum of f is $(1/3, 1/3)$ and the maximum is equal to $1/27$.

2. As $\partial_x f(x,y) = 1 + 3x^2 > 0$, f has no critical point in the interior of K. Therefore, the maximum of f must be attained on the boundary of K. We have
$$f(x,0) = x + x^3,$$
which has its maximum at $(1,0)$. As $f(0,y) = -y + y^3 \leq 2$ if $y \in [0,1]$, and $f(1,y) = 2 - y + y^3 \leq 2 - y + y = 2$, this implies that the maximum of f on K is equal to 3, attained uniquely at $(1,0) \in \partial K$.

3. We first study f in the interior of K. By the addition formula, we have
$$\nabla f(x,y) = (\sin(y)\sin(2x+y), \sin(x)\sin(x+2y)),$$
so that (x,y) is a critical point of f if and only if
$$\begin{cases} 2x + y = \pi \\ x + 2y = \pi. \end{cases}$$
Therefore, the only critical point of f is $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, and $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$.

Now, we study f on ∂K. As $f(0,t) = f(t,0) = 0$, and by symmetry, we deduce that we only need to study $t \mapsto f(\pi/2, t)$ on $\left[0, \frac{\pi}{2}\right]$. As

$$f\left(\frac{\pi}{2}, t\right) = \sin(t)\sin\left(t + \frac{\pi}{2}\right) = \sin(t)\cos(t) = \frac{1}{2}\sin(2t),$$

this implies that the maximum of $t \mapsto f(\pi/2, t)$ on $\left[0, \frac{\pi}{2}\right]$ is equal to $\frac{1}{2}$. As $\frac{1}{2} \leq \frac{3\sqrt{3}}{2}$, we deduce that $\max_K f = \frac{3\sqrt{3}}{2}$, attained uniquely in $(\pi/3, \pi/3)$.

V. Implicit functions. Show that the relation

$$e^{xy} + y^2 - xy - 3y + 2x = -1$$

defines y as a function of x for x close to 0 and y close to 1. Compute $y'(0)$. **Solution:** Let $f(x,y) = e^{xy} + y^2 - xy + 2x + 1$. We have

$$\partial_y f(x,y) = xe^{xy} + 2y - x - 3 \implies \partial_y f(0,1) = -1 \neq 0$$

As we also have $f(0,1)$, we deduce by the theorem of implicit functions that there exists open intervals $I, J \subset \mathbb{R}$ such that $0 \in I$, $j \in J$, and a smooth (as f is smooth) function $g : I \to J$ such that

$$\forall (x,y) \in I \times J, f(x,y) = 0 \Leftrightarrow y = g(x).$$

Now, we have $g(0) = 1$, and for all $x \in I$, we have

$$e^{xg(x)} + g(x)^2 - xg(x) - 3g(x) + 2x + 1 = 0.$$

By differentiating this relation, we get

$$(g(x) + xg'(x))e^{xg(x)} + 2g'(x)g(x) - g(x) - xg'(x) - 3g'(x) + 2 = 0,$$

and by evaluating at $x = 0$, and recalling that $g(0) = 1$, we find

$$1 + 2g'(0) - 1 - 3g'(0) + 2 = 0 \Leftrightarrow g'(0) = 2.$$

VI. Line integrals. Compute the line integrals

$$\int_\gamma f(s) \cdot d\vec{s},$$

in $\mathbb{R}^2$ where

1. $f(x,y) = (xy, x+y)$, where γ is the arc of parabola $y = x^2$, $-1 \leq x \leq 2$ in direct orientation.
2. $f(x,y) = (y\sin(x), x\cos(y))$, where γ is the line segment from $(0,0)$ to $(1,1)$.
3. $f(x,y) = (y, 2x)$ and γ is the boundary (with usual orientation) of the domain defined by the equations

$$\begin{cases} x^2 + y^2 - 2x \leq 0 \\ x^2 + y^2 - 2y \leq 0. \end{cases}$$

Solution:

1. We have

$$\int_\gamma f(s) \cdot d\vec{s} = \int_{-1}^{2} (x^3 + (x + x^2) \times 2x)dx = \frac{69}{4}.$$

2. We have

$$\begin{aligned}
\int_\gamma f(s)\cdot d\vec{s} &= \int_0^1 x(\cos(x)+\sin(x))dx \\
&= [x(\sin(x)-\cos(x))]_0^1 - \int_0^1 (\sin(x)-\cos(x))dx \\
&= \sin(1)-\cos(1)+[\cos(x)+\sin(x)]_0^1 = \sin(1)-\cos(1)+(\cos(1)+\sin(1))-1 = 2\sin(1)-1.
\end{aligned}$$

3. The domain is a reunion of two quarters of disks, that we parametrise by polar coordinates, as follows

$$\begin{aligned}
\gamma_1(t) &= (\cos(t), 1+\sin(t)), \quad -\frac{\pi}{2} \le t \le 0 \\
\gamma_2(t) &= (1+\cos(t), \sin(t)), \quad \frac{\pi}{2} \le t \le \pi.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_\gamma f(s)\cdot d\vec{s} &= \int_{\gamma_1} f(s)\cdot d\vec{s} + \int_{\gamma_2} f(s)\cdots d\vec{s} \\
&= \int_{-\frac{\pi}{2}}^0 \big(-(1+\sin(t))\sin(t)+2\cos^2(t)\big)\,dt + \int_{\frac{\pi}{2}}^{\pi} \big(-\sin^2(t)+2(1+\cos(t))\cos(t)\big)\,dt \\
&= \left(\frac{\pi}{4}+1\right)+\left(\frac{\pi}{4}-2\right) \\
&= \frac{\pi}{2}-1
\end{aligned}$$

where we used the duplication formula.

VII. Change of variable.

1. Compute

$$\int_\Delta \frac{dxdy}{1+x^2+y^2},$$

where $\Delta = \big\{(x,y) : x^2+y^2 \le 1,\ 0 \le x,y \le 1\big\}$.

2.

$$\int_B \frac{dxdydz}{\sqrt{x^2+y^2+(z-a)^2}}.$$

where B is the unit ball in $\mathbb{R}^3$, and $a > 1$. **Solution:**

1. Taking polar coordinates, we have

$$\int_\Delta \frac{dxdy}{1+x^2+y^2} = \frac{1}{4}\times 2\pi \int_0^1 \frac{r}{1+r^2} = \frac{\pi}{2}\left[\frac{1}{2}\log(1+r^2)\right] = \frac{\pi}{4}\log(2).$$

2. Taking spherical coordinates

$$\begin{cases} x = r\sin(\theta)\sin(\varphi) \\ y = r\cos(\theta)\sin(\varphi) \\ z = r\cos(\varphi) \end{cases}$$

we find

$$\begin{aligned}
\int_B \frac{dxdydz}{\sqrt{x^2+y^2+(z-a)^2}} &= \int_0^1\int_{-\pi}^{\pi}\int_0^{\pi} \frac{r^2\sin(\varphi)}{\sqrt{r^2+a^2-2ar\cos(\varphi)}}d\theta d\varphi dr \\
&= 2\pi\int_0^1 r^2\left(\int_0^\pi \frac{\sin(\varphi)d\varphi}{\sqrt{r^2+a^2-2ar\cos(\varphi)}}\right)dr.
\end{aligned}$$

Now, making the change of variable $t = r^2 + a^2 - 2ar\cos(\varphi)$, we obtain (as $dt = 2ar\sin(\varphi)d\varphi$)

$$\int_B \frac{dxdydz}{\sqrt{x^2+y^2+(z-a)^2}} = 2\pi \int_0^1 r^2 \left(\int_{(r-a)^2}^{(r+a)^2} \frac{dt}{2ar\sqrt{t}} \right) dr$$
$$= \frac{4\pi}{a} \int_0^1 r^2 dr = \frac{4\pi}{3a}.$$

VIII. Fubini's theorem

By using Fubini's theorem to evaluate the following integral (one can admit that it converges) in two different ways

$$\int_{[0,\infty)\times[0,\infty)} \frac{dxdy}{(1+x^2y)(1+y)},$$

deduce the value of

$$\int_0^\infty \frac{\log(x)}{x^2-1} dx.$$

Solution: As the function integrated is positive and locally bounded, we can use directly Fubini's theorem on bounded domains. For all $y > 0$, by classical growth comparison theorem, the integral

$$\int_0^\infty \frac{dx}{(1+x^2y)(1+y)}$$

converges and we compute directly

$$\int_0^\infty \frac{dx}{(1+x^2y)(1+y)} = \frac{1}{1+y}\left[\frac{1}{\sqrt{y}}\arctan(x\sqrt{y})\right]_0^\infty = \frac{\pi}{2\sqrt{y}(1+y)}$$

as $\arctan(x\sqrt{y}) \underset{x\to\infty}{\longrightarrow} \frac{\pi}{2}$ (recall that $y > 0$) and $\arctan(0) = 0$. Now, as

$$\frac{1}{\sqrt{y}(1+y)} \sim \frac{1}{y^{\frac{3}{2}}} \quad \text{as } y \to \infty.$$

the following integral converges

$$\int_0^\infty \frac{dy}{\sqrt{y}(1+y)}.$$

Now, the change of variable $t = \sqrt{y}$ yields

$$\int_0^\infty \frac{dy}{\sqrt{y}(1+y)} = 2\int_0^\infty \frac{dt}{1+t^2} = 2\left[\arctan(t)\right]_0^\infty = \pi.$$

Therefore, we have

$$\int_0^\infty \int_0^\infty \frac{dxdy}{(1+x^2y)(1+y)} = \int_0^\infty \left(\int_0^\infty \frac{dx}{(1+x^2y)(1+y)} \right) dy = \frac{\pi^2}{2}. \tag{1}$$

Now, we fix $x > 0$ and $x \neq 1$ and we make the decomposition

$$\frac{1}{(1+x^2y)(1+y)} = \frac{1}{1-x^2}\left(-\frac{x^2}{1+x^2y} + \frac{1}{1+y} \right)$$

which implies that we have for all $R > 0$

$$\int_0^R \frac{dy}{(1+x^2y)(1+y)} = \frac{1}{1-x^2}\int_0^R \left(-\frac{x^2}{1+x^2y} + \frac{1}{1+y} \right) dy = \frac{1}{1-x^2}\left[-\log(1+x^2y) + \log(1+y)\right]_0^R$$
$$= \frac{1}{1-x^2}\log\left(\frac{1+R}{1+Rx^2}\right) \underset{R\to\infty}{\longrightarrow} \frac{1}{1-x^2}\log\left(\frac{1}{x^2}\right) = \frac{2\log(x)}{x^2-1},$$

where we used

$$\frac{1+R}{1+Rx^2} = \frac{1+\frac{1}{R}}{x^2+\frac{1}{R}} \underset{R\to\infty}{\longrightarrow} \frac{1}{x^2}.$$

Finally, we have

$$\int_0^\infty \left(\int_0^\infty \frac{dy}{(1+x^2)(1+y)} \right) dx = 2\int_0^\infty \frac{\log(x)}{x^2-1} dx$$

and by Fubini and (1), we obtain

$$\int_0^\infty \frac{\log(x)}{x^2-1} dx = \frac{\pi^2}{4}.$$

Remark 1. One can see directly that this integral converges as follows. First, at infinity, we have

$$\frac{\log(x)}{x^2-1} \sim \frac{\log(x)}{x^2} = O\left(\frac{1}{|x|^{2-\varepsilon}}\right), \quad \text{for all } \varepsilon > 0$$

so the integral converges at infinity (by a standard comparison argument). Now, one also needs to analyse the behaviour as $x \to 1$ and $x \to 0$. As $x^2-1 = (x+1)(x-1)$, one needs to check that

$$\int_0^1 \log(x) dx, \quad \text{and} \quad \int_{\frac{1}{2}}^2 \frac{\log(x)}{x-1} dx \tag{2}$$

converge. For the first integral, the change of variable $y = -\log(x)$ yields

$$\int_0^1 |\log(x)| dx = -\int_0^1 \log(x) dx = \int_0^\infty y e^{-y} dy < \infty$$

which converges as $e^{-y} \leq \dfrac{1}{1+|y|^3}$ for y large enough (for example). Now, as $\log(x) \simeq x-1$ when $x \to 1$, the function

$$\frac{\log(x)}{x-1}$$

is bounded in $[\frac{1}{2}, 2]$, so the second integral in (2) converges.

IX. Potential.

Is the vector-field $F(x,y,z) = (3x^2y + z^3, 3y^2z + x^3, 3xz^2 + y^3)$ conservative on $\mathbb{R}^3$? If it is, then determine a potential for F.

Solution: We check directly that F is exact by an explicit computation. As $\mathbb{R}^2$ is a starred domain, this implies that F derives from a potential, and by integrating with respect to the different variables, we find that

$$F = \nabla f,$$

where $f : \mathbb{R}^3 \to \mathbb{R}, (x,y,z) \mapsto x^3y + xz^3 + y^3z$.

X. Green's theorem. The Piriform curve C in $\mathbb{R}^2$ is the set

$$C = \{(x,y) \in \mathbb{R}^2 \mid y^2 = x^3(2-x)\}.$$

A parametrization of C is given by $\gamma : [-\frac{\pi}{2}, \frac{3\pi}{2}] \to \mathbb{R}^2$,

$$\gamma(t) = \begin{pmatrix} 1+\sin(t) \\ \cos(t)(1+\sin(t)) \end{pmatrix}$$

The Piriform curve is the boundary of the set

$$\Omega = \mathbb{R}^2 \left\{ (x,y) : 0 \leq x \leq 2 \text{ and } -\sqrt{x^3(2-x)} \leq y \leq \sqrt{x^3(2-x)} \right\}.$$

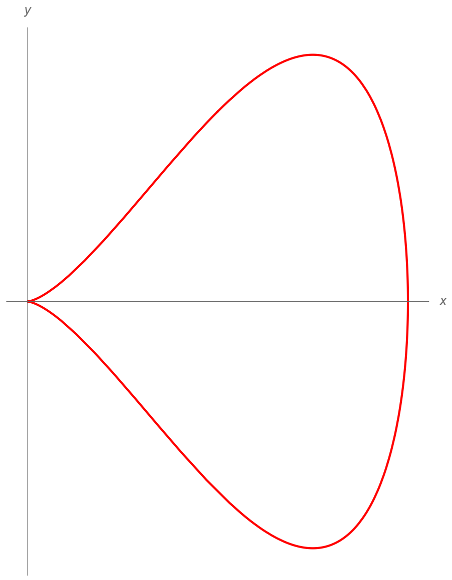

Figure 1: The Piriform curve

Compute the area of Ω.

Solution: As this curve is clockwise parametrised, we choose the parametrisation of $-\gamma$ to get the area, so that by Green's theorem

$$\begin{aligned}
\mathrm{Area}(\Omega) &= -\int_{-\gamma} y\ dx = \int_{\gamma} y\ dx = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2(t)(1+\sin(t))dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2(t)\ dt + \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2(t)\sin(t)\ dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1+\cos(2t)}{2}\ dt = \pi.
\end{aligned}$$

Here, we used the formula

$$\int_{-\pi/2}^{3\pi/2} \cos^2(t)\sin(t)\ dt = \left[-\frac{1}{3}\cos^3(t)\right]_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} = 0$$

XI. Integration by substitution The cardioid C is the curve in $\mathbb{R}^2$ defined by

$$C = \{(x,y) \in \mathbb{R}^2 \mid (x^2+y^2-2x)^2 = 4(x^2+y^2)\}$$

C is the boundary of the set

$$\Omega := \{(tx,ty) \in \mathbb{R}^2 \mid t \in [0,1],\ (x,y) \in C\}.$$

Compute the area of Ω.

Solution Using polar coordinates $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$, we have for all $(x,y) \neq (0,0)$:

$$\begin{aligned}
4(x^2+y^2) = (x^2+y^2-2x)^2 &\Leftrightarrow 4r^2 = (r^2 - 2r\cos(\varphi))^2 \\
&\Leftrightarrow 2r = |r^2 - 2r\cos(\varphi)| = r|r - 2\cos(\varphi)| \\
&\Leftrightarrow 2 = |r - 2\cos(\varphi)|.
\end{aligned}$$

Therefore, the function

$$C\backslash\{(0,0)\} \ni (r\cos(\varphi), r\sin(\varphi)) \mapsto r - 2\cos(\varphi)$$

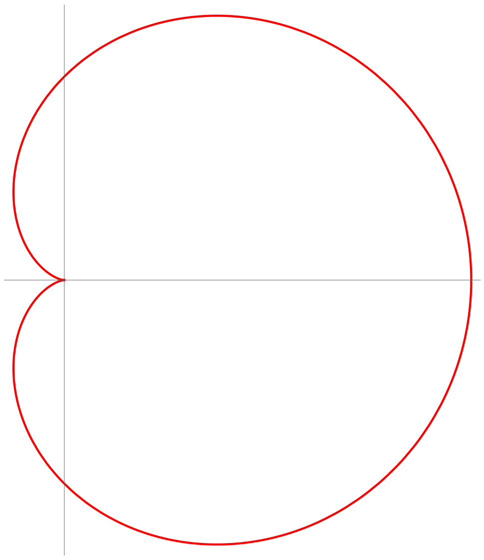

Figure 2: The cardioid

is pointwise equal to -2 and 2. As $C\backslash\{(0,0)\}$ is connected, this function is constant. As $(4,0) = (4\cos(0), 4\sin(0)) \in C\backslash\{(0,0)\}$ and $4 - 2\cos(0) = 2$ we deduce that for all $(x,y) \neq (0,0)$

$$\begin{aligned} 4(x^2+y^2) = (x^2+y^2-2x)^2 &\Leftrightarrow 2 = r - 2\cos(\varphi) \\ &\Leftrightarrow r = 2(1+\cos(\varphi)) \end{aligned}$$

Therefore, we have

$$\begin{aligned} \text{Area}(\Omega) &= \int_\Omega 1 \; dxdy = \int_0^{2\pi} \int_0^{2(1+\cos(\varphi))} r \; dr d\varphi \\ &= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{2(1+\cos(\varphi))} d\varphi \\ &= \int_0^{2\pi} 2(1+\cos(\varphi))^2 \; d\varphi \\ &= 2\int_0^{2\pi} 1 + \cos^2(\varphi) + 2\cos(\varphi) \; d\varphi \\ &= 2\left(2\pi + \int_0^{2\pi} \frac{1+\cos(2\varphi)}{2} \; d\varphi\right) \\ &= 2\,(2\pi + \pi) = 6\pi. \end{aligned}$$