I. Conservative vector-fields

For which of the following vector-fields v does there exist a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $v = \nabla f$?

1. $v(x, y) = \begin{pmatrix} x - y \\ x - y \end{pmatrix}$
2. $v(x, y) = \begin{pmatrix} x^2 - y \\ x^3 + 2xy \end{pmatrix}$
3. $v(x, y) = \begin{pmatrix} x^3 + 2xy \\ x^2 - y \end{pmatrix}$
4. $v(x, y) = \begin{pmatrix} x^3 - xy^2 \\ x^2 y - y^5 \end{pmatrix}$

Solution : As $\mathbb{R}^2$ is star connected, it suffices to check the condition $\partial_y v_1 = \partial_x v_2$, where $v = (v_1, v_2)$ to determine if v is conservative or not.

1. We have $\partial_y v_1 = -1 \neq 1 = \partial_x v_2$, so v is not conservative.
2. We have $\partial_y v_1(x, y) = -1$, while $\partial_x v_2(x, y) = 3x^2$, so we have in particular $\partial_y v_1(0, 0) = -1 \neq 0 = \partial_x v_2(0, 0) = 0$. Therefore, v is not conservative. (One can also observe that $\partial_y v_1 = -1 < 0 \leq \partial_x v_2$.)
3. We have $\partial_y v_1(x, y) = 2x = \partial_x v_2(x, y)$, so v is conservative.
4. We have $\partial_y v(x, y) = -2xy$ while $\partial_x v(x, y) = 2xy$, so $\partial_y v(1, 1) = -2 \neq 2 = \partial_x v_2(1, 1)$. Therefore, v is not conservative.

II. A counter-example Let $V : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^2$ be defined by

$$V(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$$

1. Show that V satisfies the necessary condition to be conservative.
2. Compute the pathintegral of V around the curve $\gamma(t) = (\sin(t), -\cos(t))$, $0 \leq t \leq 2\pi$.
3. Is V conservative?

Solution:

1. Writing $V = (V_1, V_2)$, we have
$$\partial_y V_1(x, y) = \partial_x V_2(x, y) = -\frac{2xy}{x^2 + y^2}.$$
2. We have
$$\int_0^{2\pi} V(s) \cdot d\vec{s} = \int_0^{2\pi} (\sin(t) \cos(t) - \cos(t) \sin(t))\, dt = 0$$
3. As the integral vanishes, V is conservative. Indeed, we have
$$V(x, y) = \nabla \log \sqrt{x^2 + y^2}$$

III. Pathintegrals Compute in the following exercises the pathintegral of the vectorfield v along the path.

1. $v(x, y) = \begin{pmatrix} x^2 - 2xy \\ y^2 - 2xy \end{pmatrix}$, from $(-1, 1)$ to $(1, 1)$ along the path $y = x^2$.
2. $v(x, y) = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$, from $(0, 0)$ to $(2, 0)$ along the path $y = 1 - |1 - x|$.

3. $v(x,y,z) = \begin{pmatrix} x \\ y \\ xz - y \end{pmatrix}$, along the path $\gamma(t) = (t^2, 2t, 4t^3)$, $t \in [0,1]$.

4. $v(x,y) = \begin{pmatrix} 2a - y \\ x \end{pmatrix}$, along the path $\gamma(t) = (a(t - \sin(t)), a(1 - \cos(t)))$, $t \in [0, 2\pi]$, where $a \in \mathbb{R}$ is a constant.

5. $v(x,y) = \left(\frac{x}{x^2 + y^2 + 1}, \frac{y}{x^2 + y^2 + 1} \right)$, along the circle $x^2 + y^2 - 2x = 1$.

6. $v(x,y,z) = \begin{pmatrix} 2xy^2 z \\ 2x^2 yz \\ x^2 y^2 - 2z \end{pmatrix}$, along the path $\gamma(t) = \left(\cos(t), \frac{\sqrt{3}}{2} \sin(t), \frac{1}{2} \sin(t) \right)$, $0 \le t \le 2\pi$.

Solution:

1. We parametrisation the curve by $\gamma : [-1,1] \to \mathbb{R}^2$, where

$$\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \Rightarrow \gamma'(t) = \begin{pmatrix} 1 \\ 2t. \end{pmatrix}$$

Then we have

$$v(\gamma(t)) = \begin{pmatrix} t^2 - 2t^3 \\ t^4 - 2t^3. \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = t^2 - 2t^3 + 2t^5 - 4t^4.$$

Finally, we obtain

$$\begin{aligned} \int_\gamma v(s) \cdot d\vec{s} &= \int_{-1}^{1} t^2 - 2t^3 + 2t^5 - 4t^4 \, dt = \left[\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4t^5}{5} \right]_{t=-1}^{1} \\ &= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{4}{5} - \left(\frac{-1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{4}{5} \right) \\ &= \frac{1}{3} - \frac{4}{5} + \frac{1}{3} - \frac{4}{5} = \frac{2}{3} - \frac{8}{5} = -\frac{14}{15}. \end{aligned}$$

2. We parametrisation the curve by $\gamma(t) = (t, \gamma_2(t))$, $t \in [0,2]$, where

$$\gamma_2(t) = \begin{cases} t, & t \in [0,1] \\ 2 - t, & t \in [1,2]. \end{cases}$$

Then we have for all $t \in [0,1]$

$$v(\gamma(t)) = \begin{pmatrix} 2t^2 \\ 0 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = 2t^2$$

and for all $t \in [1,2]$

$$v(\gamma(t)) = \begin{pmatrix} t^2 + (2-t)^2 \\ t^2 - (2-t)^2 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = 2(2-t)^2.$$

Then, we obtain

$$\begin{aligned} \int_\gamma v(s) \, d\vec{s} &= \int_0^1 2t^2 \, dt + \int_1^2 2(2-t)^2 dt = \left[\frac{2t^3}{3} \right]_{t=0}^{1} + 2 \left[\frac{-(2-t)^3}{3} \right]_{t=1}^{2} \\ &= \frac{2}{3} + 2 \left(\frac{1}{3} \right) = \frac{4}{3}. \end{aligned}$$

3. We have

$$v(\gamma(t)) = \begin{pmatrix} t^2 \\ 2t \\ 4t^5 - 2t \end{pmatrix}, \quad \gamma'(t) = \begin{pmatrix} 2t \\ 2 \\ 12t^2 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = 2t^3 + 4t + 12t^2(4t^5 - 2t),$$

which implies that

$$\int_\gamma v(s) \cdot d\vec{s} = \int_0^1 2t^3 + 4t + 12t^2(4t^5 - 2t)\ dt = \int_0^1 4t + 48t^7 - 22t^3\ dt$$
$$= \left[2t^2 + 6t^8 - \frac{22t^4}{4}\right]_{t=0}^1 = 2 + 6 - \frac{11}{2} = \frac{4 + 12 - 11}{2} = \frac{5}{2}.$$

4. We have

$$v(\gamma(t)) = \begin{pmatrix} 2a - a(1 - \cos(t)) \\ a(t - \sin(t)) \end{pmatrix}, \quad \gamma'(t) = \begin{pmatrix} a(1 - \cos(t)) \\ a \sin(t) \end{pmatrix}$$

and this implies that

$$v(\gamma(t)) \cdot \gamma'(t) = 2a^2 - 2a^2\cos(t) - a^2 t + a^2\cos(t) + a^2\cos(t) - a^2\cos^2(t) + a^2\sin(t)t - a^2\sin^2(t).$$

Now, we have

$$\int_\gamma v(s)\ d\vec{s} = \int_0^{2\pi} 2a^2 - 2a^2\cos(t) - a^2 t + a^2\cos(t) + a^2\cos(t) - a^2\cos^2(t) + a^2\sin(t)t - a^2\sin^2(t)\ dt$$
$$= \int_0^{2\pi} 2a^2 - a^2 t + -a^2\cos^2(t) + a^2\sin(t)t - a^2\sin^2(t)\ dt = -2\pi^2 a^2.$$

5. We see by a direct computation that the gradient of v is

$$\nabla v(x,y) = \begin{pmatrix} \frac{1}{x^2+y^2+1} - \frac{2x^2}{(x^2+y^2+1)^2} & -\frac{2xy}{(x^2+y^2+1)^2} \\ \frac{-2xy}{(x^2+y^2+1)^2} & \frac{1}{x^2+y^2+1} - \frac{2y^2}{(x^2+y^2+1)^2}. \end{pmatrix}$$

If we write $v = (v_1, v_2)$, we obtain the condition $\partial_y v_1 = \partial_x v_2$, and as $\mathbb{R}^2$ is simply connected, v is conservative so the integral is equal to 0. This can be seen directly as

$$v(x,y) = \nabla \log(\sqrt{1 + x^2 + y^2}).$$

6. Here, one checks directly the condition on the first derivatives, but this is easier to see that

$$v(x,y,z) = \nabla(x^2y^2z - z^2).$$

Therefore, v is also conservative and the integral vanishes.