I. Making a vector-field conservative. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $\varphi(0) = -1$ and $X : \mathbb{R}^2 \to \mathbb{R}$ be the vector field such that for all $(x, y) \in \mathbb{R}^2$

$$X(x, y) = \left(\frac{2xy}{(1 + x^2)^2}, \varphi(x) \right).$$

(1) Find a function φ satisfying the previous condition $\varphi(0) = -1$ such that X is conservative.

(2) Find a potential for the conservative vector field X found in question (1).

(3) Compute

$$\int_\gamma X(s) \cdot d\vec{s}$$

where γ is a direct parametrisation of the ellipse $5x^2 + 2y^2 = 7$.

II. Fubini's theorem for explicit functions (1).

(1) Compute

$$\int_{[-1,1]\times[2,3]} \left(x^4 y - y^5 x + y^3 \right) dxdy$$

(2) Let $D^2 = \mathbb{R}^2 \cap \left\{(x, y) : x^2 + y^2 \leq 1\right\}$ be the unit disk in the plan. Compute

$$\int_{D^2} x^2 y^2 dxdy$$

by following the following steps.

(a) Show that

$$\int_0^{\frac{\pi}{2}} \cos^4(\theta) \sin^2(\theta) d\theta = \frac{\pi}{32}$$

(b) Show that for all continuous function $f : D^2 \to \mathbb{R}$, we have

$$\int_{D^2} f(x, y)dxdy = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y)dy \right) dx.$$

(c) Compute

$$\int_{D^2} x^2 y^2 dxdy,$$

by making the formula of question (2) and a (1-dimensional) change of variable using trigonometric functions and symmetry.

III. Fubini's theorem for explicit functions (2).

Compute the following double integrals $\int_D f(x, y)dxdy$, where the continuous function $f : D \to \mathbb{R}$ and the domain D are given by

1. $f(x, y) = x$, and $D = \mathbb{R}^2 \cap \{(x, y) : y \geq 0, x - y + 1 \geq 0, x + 2y - 4 \leq 0\}$.
2. $f(x, y) = \cos(xy)$, and $D = \mathbb{R}^2 \cap \left\{(x, y) : 1 \leq x \leq 2, 0 \leq xy \leq \frac{\pi}{2}\right\}$.
3. $f(x, y) = \frac{1}{(x+y)^3}$, and $D = \mathbb{R}^2 \cap \{(x, y) : 1 \leq x \leq 3, y \geq 2, x + y \leq 5\}$.
4. $f(x, y) = \frac{xy}{1+x^2+y^2}$, and $D = \mathbb{R}^2 \cap \left\{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x^2 + y^2 \geq 1\right\}$.

IV. Fubini's theorem for explicit functions (3).

Compute the area of the domain

$$D = \mathbb{R}^2 \cap \left\{(x, y) : -1 \leq x \leq 1, x^2 \leq y \leq 4 - x^3\right\}.$$