

**I. Volume of the 3-dimensional ball.** Let  $r > 0$  and  $B_3(0, r) = \mathbb{R}^3 \cap \{(x, y, z) : x^2 + y^2 + z^2 \leq r^2\}$  be the open ball of radius  $r > 0$ . By using a change of coordinates  $f : [0, r] \times [0, 2\pi] \times [0, \pi] \rightarrow B_3(0, r)$  given as follows :

$$f(t, \theta, \varphi) = \begin{cases} t \cos(\theta) \sin(\varphi) \\ t \sin(\theta) \sin(\varphi) \\ t \cos(\varphi) \end{cases}$$

compute the volume of  $B_3(0, r)$ , defined by

$$\int_{B_3(0, r)} dx dy dz.$$

**II.  $n$ -dimensional volume of a ball.** Let  $n \geq 2$ ,  $r > 0$  and

$$B_n(0, r) = \mathbb{R}^n \cap \{x = (x_1, \dots, x_n) : x_1^2 + \dots + x_n^2 \leq r^2\}$$

be the closed ball of radius  $r$ . Let

$$V(n, r) = \int_{B_n(0, r)} dx_1 \cdots dx_n$$

be the volume of the ball of radius 1 and write for simplicity  $V(n) = V(n, 1)$  for the volume of the unit ball.

1. Compute  $V(1)$  and  $V(2)$ .
2. Show by a change of variable that for all  $r > 0$ , we have

$$V(n, r) = r^n V(n).$$

3. Show by Fubini's theorem that for all  $n \geq 2$ , we have

$$V(n) = V(n-2) \int_{B_2(0, 1)} (1 - x_{n-1}^2 - x_n^2)^{\frac{n-2}{2}} dx_{n-1} dx_n$$

4. Deduce that for all  $n \in \mathbb{N}$ , we have

$$V(2n) = \frac{\pi^n}{n!} \quad V(2n+1) = \frac{2^{2n+1} n!}{(2n+1)!} \pi^n.$$

where we recall that  $n! = 1 \times 2 \times \dots \times (n-1) \times n$  for all  $n \geq 1$ .

**Hint:** Notice in 3. that

$$B_n(0, 1) = \mathbb{R}^n \cap \{x = (x_1, \dots, x_n) : x_{n-1}^2 + x_n^2 \leq 1, x_1^2 + \dots + x_{n-2}^2 \leq 1 - x_{n-1}^2 - x_n^2\}$$

**III. Fubini's theorem for an explicit integral.** By computing the integral (justify why it converges)

$$I = \int_{[0, \infty) \times [a, b]} e^{-xy} dx dy$$

in two different ways, where  $0 < a < b$ , show that

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \log\left(\frac{b}{a}\right).$$