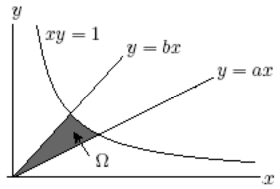
I. Change of variables. Let $D = \mathbb{R}^2 \cap \{(x,y) : x^2 + y^2 - 2x \leq 0\}$. Compute

$$\int_D \sqrt{x^2+y^2}dxdy$$

Hint: Notice that D is a disk.

II. Fubini's theorem. Let $0 < a < b < \infty$ and Ω the domain defined as below :



With formulas, this means that

$$\Omega = \mathbb{R}^2 \cap \{(x,y) : x \geq 0,\ \ y \geq 0,\ \ 0 \leq xy \leq 1,\ \ ax \leq y \leq bx\}$$

Show that

$$\int_\Omega xy\, dxdy = \frac{1}{4}\log\left(\frac{b}{a}\right).$$

III. Centre of mass. Compute the centre of mass of a half ball of radius $R > 0$

$$B^3_+(0,R) = \mathbb{R}^3 \cap \{(x,y,z) : x^2+y^2+z^2 \leq R^2, z \geq 0\},$$

defined by

$$\left(\frac{1}{\mathrm{Vol}(B^3_+(0,R))}\int_{B^3_+(0,R)} x\, dxdydz, \frac{1}{\mathrm{Vol}(B^3_+(0,R))}\int_{B^3_+(0,R)} y\, dxdydz, \frac{1}{\mathrm{Vol}(B^3_+(0,R))}\int_{B^3_+(0,R)} z\, dxdydz\right),$$

where $\mathrm{vol}(B^3_+(0,R))$ is the volume of the half-ball (of radius $R > 0$) in 3-space (one can refer to Serie 11 for its value).

IV. The Astroid Let $a > 0$. The Astroid $A(a) \subset \mathbb{R}^2$ is the geometric figure in the plane defined by

$$A(a) := \left\{(x,y) \in \mathbb{R}^2 \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\right\}$$

The construction of an Astroid is very geometric (see link). Let $B(a)$ denote the set

$$B(a) = \{(rx, ry) \subset \mathbb{R}^2 \mid r \in [0,1],\ (x,y) \in A(a)\}.$$

Compute the area of $B(a)$ using the theorem of Green.

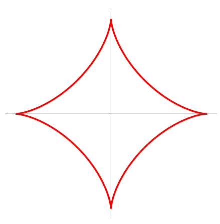


Figure 1: The red Astroid $A(a)$ is the boundary of $B(a)$.