I. Differential equations.

Find the solutions of the following first order linear differential equations

1. $7y' + 2y = 2x^3 - 5x^2 + 4x - 1$.
2. $y' + y = xe^{-x}$.
3. $y' - 2y = \cos(x) + 2\sin(x)$.
4. $y'' - 2y' + y = x$, $y(0) = y'(0) = 0$.
5. $y'' + 9y = x + 1$, $y(0) = 0$.
6. $(x+1)y' + xy = x^2 - 2x + 1$, $y(1) = 1$, where $x > -1$.

II. Differential equations (2)**.**

Solve the following differential equations :

1. $(1 + e^x)y'' + 2e^x y' + (2e^x + 1)y = xe^x$, with the change of function $z = (1 + e^x)y$.
2. $y'' - y' - e^{2x}y = e^{3x}$, with the change of variable $t = e^x$.

III. Extrema (1)**.** Find the local and global extrema of the following functions.

1. $f(x, y) = x^2 + y^2 + xy + 1$.
2. $f(x, y) = x^2 + y^2 + 4xy - 2$.
3. $f(x, y) = y(x^2 + \log^2(y))$, with $y > 0$.

IV. Extrema (2)**.** Find the maximum of the following functions on the given compact sets $K \subset \mathbb{R}^2$:

1. $f(x, y) = xy(1 - x - y)$ on $K = \{(x, y) : x, y \geq 0,\ x + y \leq 1\}$.
2. $f(x, y) = x - y + x^3 + y^3$ on $K = [0, 1]^2$.
3. $f(x, y) = \sin(x)\sin(y)\sin(x + y)$ on $K = \left[0, \frac{\pi}{2}\right]^2$.

V. Implicit functions. Show that the relation

$$e^{xy} + y^2 - xy - 3y + 2x = -1$$

defines y as a function of x for x close to 0 and y close to 1. Compute $y'(0)$.

VI. Line integrals. Compute the line integrals

$$\int_\gamma f(s) \cdot d\vec{s},$$

in $\mathbb{R}^2$ where

1. $f(x, y) = (xy, x + y)$, where γ is the arc of parabola $y = x^2$, $-1 \leq x \leq 2$ in direct orientation.
2. $f(x, y) = (y\sin(x), x\cos(y))$, where γ is the line segment from $(0, 0)$ to $(1, 1)$.
3. $f(x, y) = (y, 2x)$ and γ is the boundary (with usual orientation) of the domain defined by the equations

$$\begin{cases} x^2 + y^2 - 2x \leq 0 \\ x^2 + y^2 - 2y \leq 0. \end{cases}$$

VII. Change of variable.

1. Compute

$$\int_{\Delta} \frac{dxdy}{1+x^2+y^2},$$

where $\Delta = \{(x,y) : x^2+y^2 \leq 1,\ 0 \leq x, y \leq 1\}$.

2.

$$\int_{B} \frac{dxdydz}{\sqrt{x^2+y^2+(z-a)^2}}.$$

where B is the unit ball in $\mathbb{R}^3$, and $a > 1$.

VIII. Fubini's theorem

By using Fubini's theorem to evaluate the following integral (one can admit that it converges) in two different ways

$$\int_{[0,\infty)\times[0,\infty)} \frac{dxdy}{(1+x^2y)(1+y)},$$

deduce the value of

$$\int_0^{\infty} \frac{\log(x)}{x^2-1} dx.$$

IX. Potential.

Is the vector-field $F(x,y,z) = (3x^2y+z^3, 3y^2z+x^3, 3xz^2+y^3)$ conservative on $\mathbb{R}^3$? If it is, then determine a potential for F.

X. Green's theorem. The Piriform curve C in $\mathbb{R}^2$ is the set

$$C = \{(x,y) \in \mathbb{R}^2 \mid y^2 = x^3(2-x)\}.$$

A parametrization of C is given by $\gamma : [-\frac{\pi}{2}, \frac{3\pi}{2}] \to \mathbb{R}^2$,

$$\gamma(t) = \begin{pmatrix} 1+\sin(t) \\ \cos(t)(1+\sin(t)) \end{pmatrix}$$

The Piriform curve is the boundary of the set

$$\Omega = \mathbb{R}^2 \left\{(x,y) : 0 \leq x \leq 2 \text{ and } -\sqrt{x^3(2-x)} \leq y \leq \sqrt{x^3(2-x)}\right\}.$$

Compute the area of Ω.

XI. Integration by substitution The cardioid C is the curve in $\mathbb{R}^2$ defined by

$$C = \{(x,y) \in \mathbb{R}^2 \mid (x^2+y^2-2x)^2 = 4(x^2+y^2)\}$$

C is the boundary of the set

$$\Omega := \{(tx,ty) \in \mathbb{R}^2 \mid t \in [0,1],\ (x,y) \in C\}.$$

Compute the area of Ω.

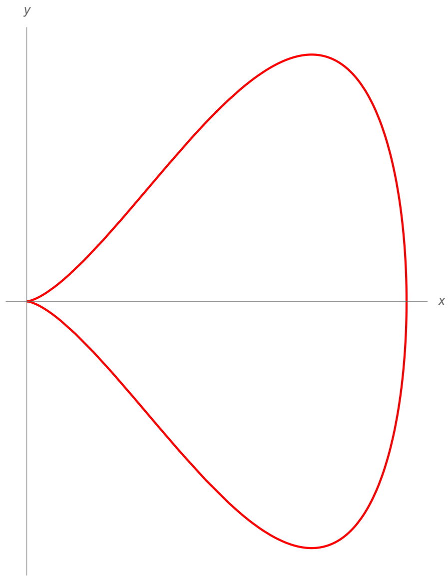

Figure 1: The Piriform curve

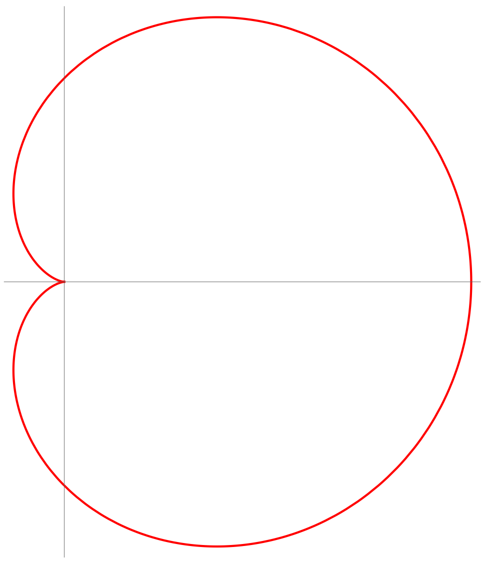

Figure 2: The cardioid