$\lambda_i \neq 0$

Replace v_i by

$$w_i = \begin{cases} \frac{1}{\sqrt{\lambda_i}} v_i, & \lambda_i > 0 \\ \frac{1}{\sqrt{-\lambda_i}} v_i, & \lambda_i < 0 \end{cases}$$

Then for

$$y = u_1 w_1 + \dots + u_n w_n$$

$(q = n-p)$

we get

$$y^t H y = u_1^2 + \dots + u_p^2 - u_{p+1}^2 - \dots - u_n^2$$

if $\lambda_1, \dots, \lambda_p > 0$, $\lambda_{p+1}, \dots, \lambda_n < 0$

Taylor approximation:

$$f(x) \approx f(x_0) + y^t H y$$

$$(y = x - x_0)$$

$$\approx f(x_0) + u_1^2 + \cdots + u_p^2 - u_{p+1}^2 - \cdots - u_n^2$$

If $p \geq 1$, going "in the direction of w_1"

$[x - x_0 = \underset{u_1}{\cancel{4}}\, w_1]$ gives $f(x) - f(x_0) > 0$

If $q \geq 1$, ______________ w_{p+1}

______________ $f(x) - f(x_0) < 0$

Cor. 3.8.7. f C^2

x_0 non-degenerate critical point

p nb. of >0 eigenvalues of $\mathrm{Hess}_f(x_0)$

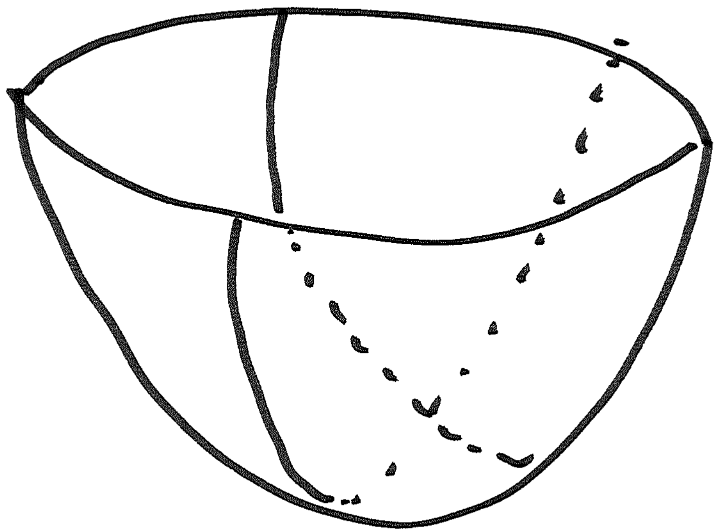
q —— <0 ——————

(1) If $p \geq 1$ and $q \geq 1$ [H indefinite] then x_0 is a saddle point.

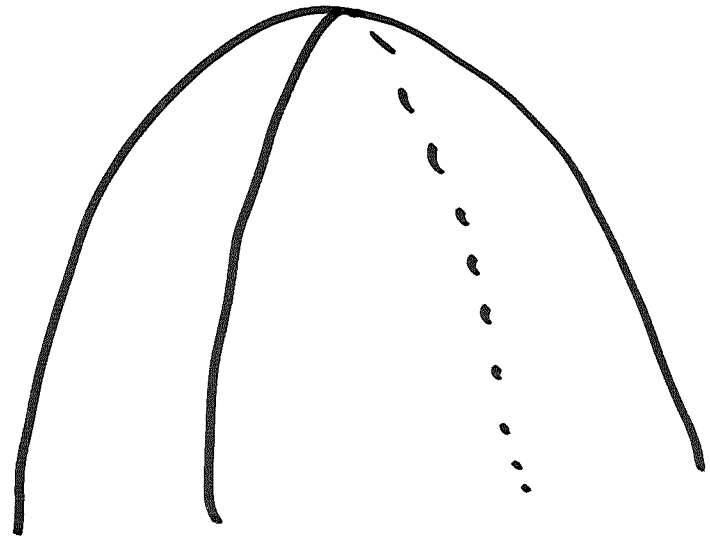
(2) If $p = n$ [H positive definite] then x_0 is a local minimum.

(3) If $q = n$ [H ~~neg~~ negative ——] then x_0 is a local maximum.

$n = 2$



local min.
$x^2 + y^2$



local max
$-x^2 - y^2$

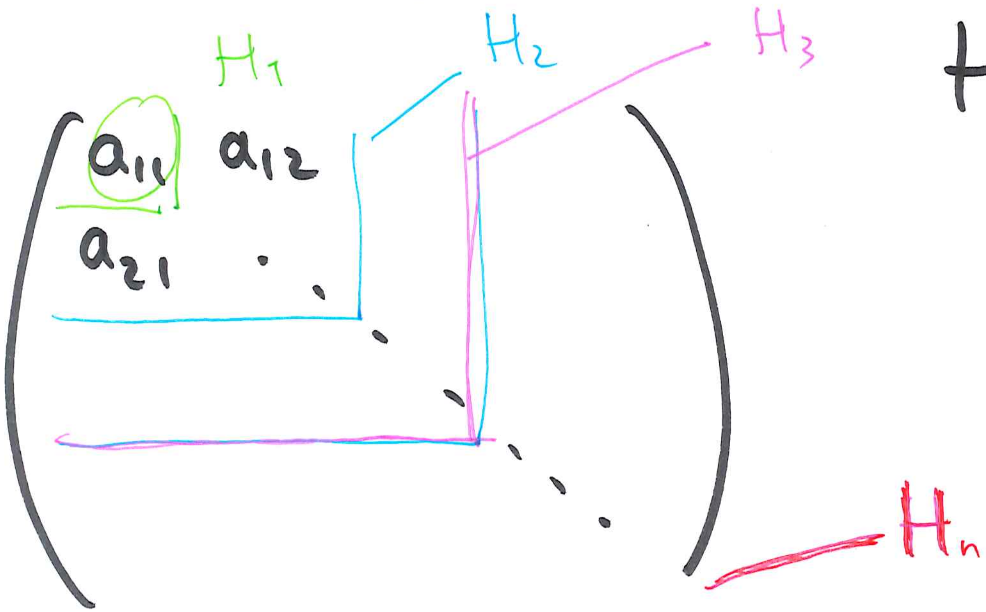
saddle
$x^2 - y^2$ or xy

In practice:

Criterion from linear algebra:

$H = (a_{ij})$ symmetric matrix

$H_k = (a_{ij})_{1 \le i,j \le k}$



H is $\boxed{\underline{\text{positive}}\ \underline{\text{definite}}}$

$\Updownarrow$

$\det(H_k) > 0$

for $1 \le k \le n$

H negative definite $\iff$ $-H$ is positive definite

[**Warning!** $\det(-H) = (-1)^n \det H$]

$$\iff \det H_k > 0, \quad k \text{ even}$$
$$\text{———} < 0, \quad k \text{ odd}$$

If $\det H_n \neq 0$, and none of these two holds, then H is indefinite.

<u>$n = 2$</u>

$$H = \begin{pmatrix} \partial^2_{x^2} f & \partial_x \partial_y f \\ \partial_y \partial_x f & \partial^2_{y^2} f \end{pmatrix}, \quad \det H \neq 0$$

local ~~max~~ min. $\Longleftrightarrow$ $\dfrac{\partial^2 f}{\partial x^2}(x_0) > 0$

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$$

local max $\Longleftrightarrow$ $\dfrac{\partial^2 f}{\partial x^2}(x_0) < 0$

$\det H > 0$

<u>else</u> saddle point

Ex. 3.8.9 (2)

$$f(x,y) = e^{\cos(x-y)} + x^2$$

with $(x,y) \in]-4,4[^2$

Critical points are

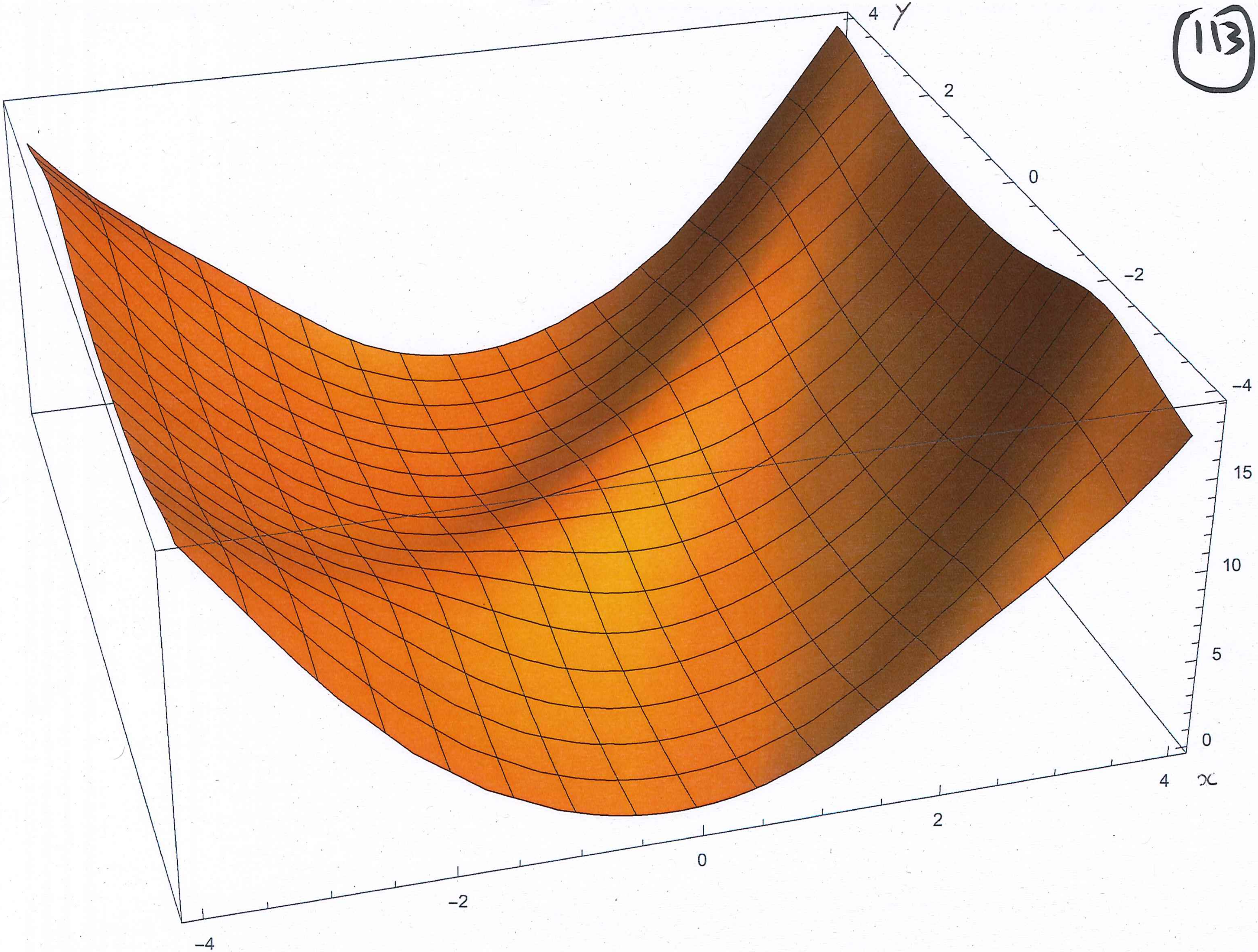
saddle point — $(0,0)$ — $H = \begin{pmatrix} 2-e & e \\ e & -e \end{pmatrix}$, $\det = -2e$

$(0,\pi)$, $(0,-\pi)$ — $H = \begin{pmatrix} 2+\frac{1}{e} & -\frac{1}{e} \\ -\frac{1}{e} & \frac{1}{e} \end{pmatrix}$

local minimum

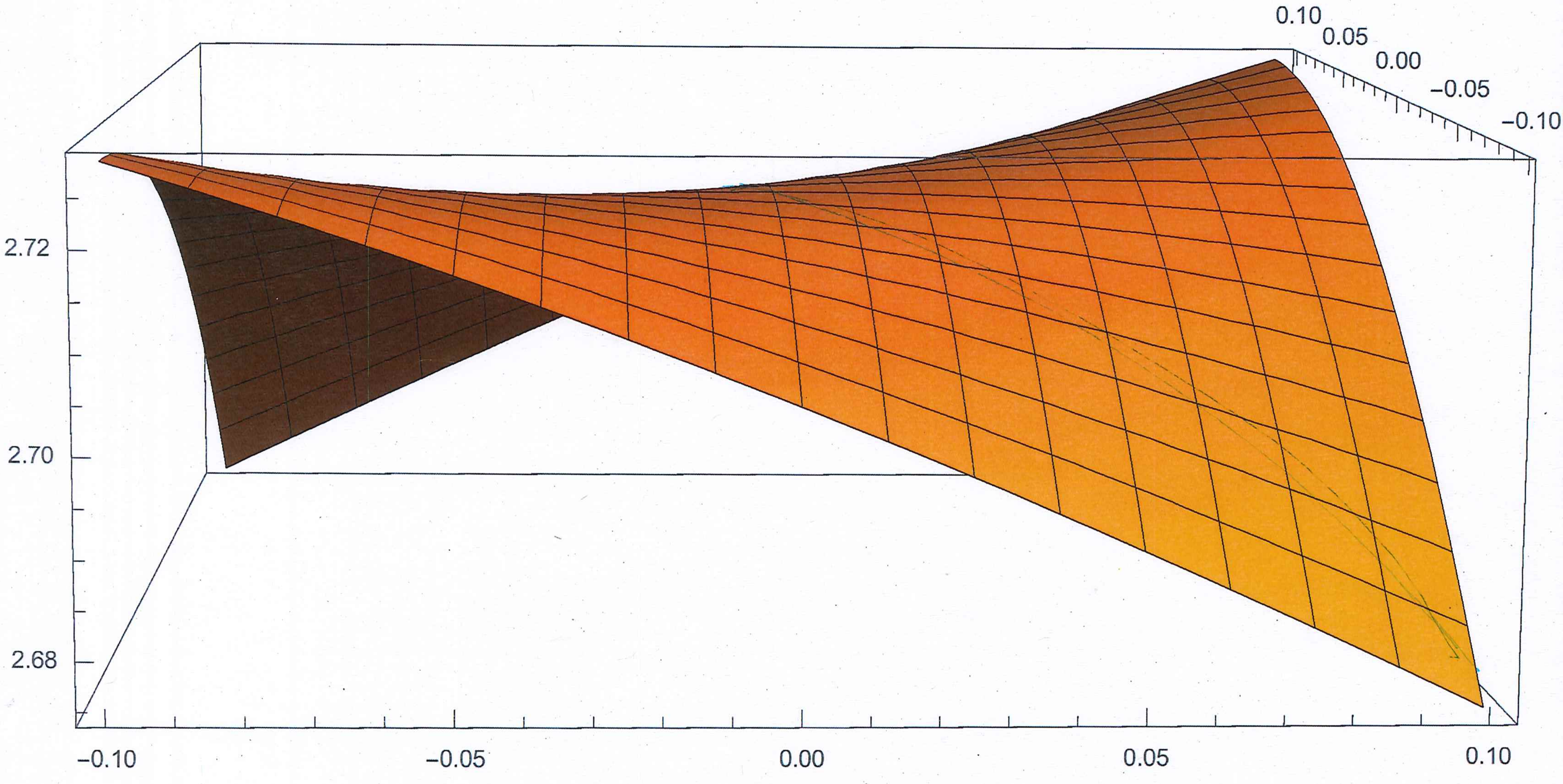
positive definite

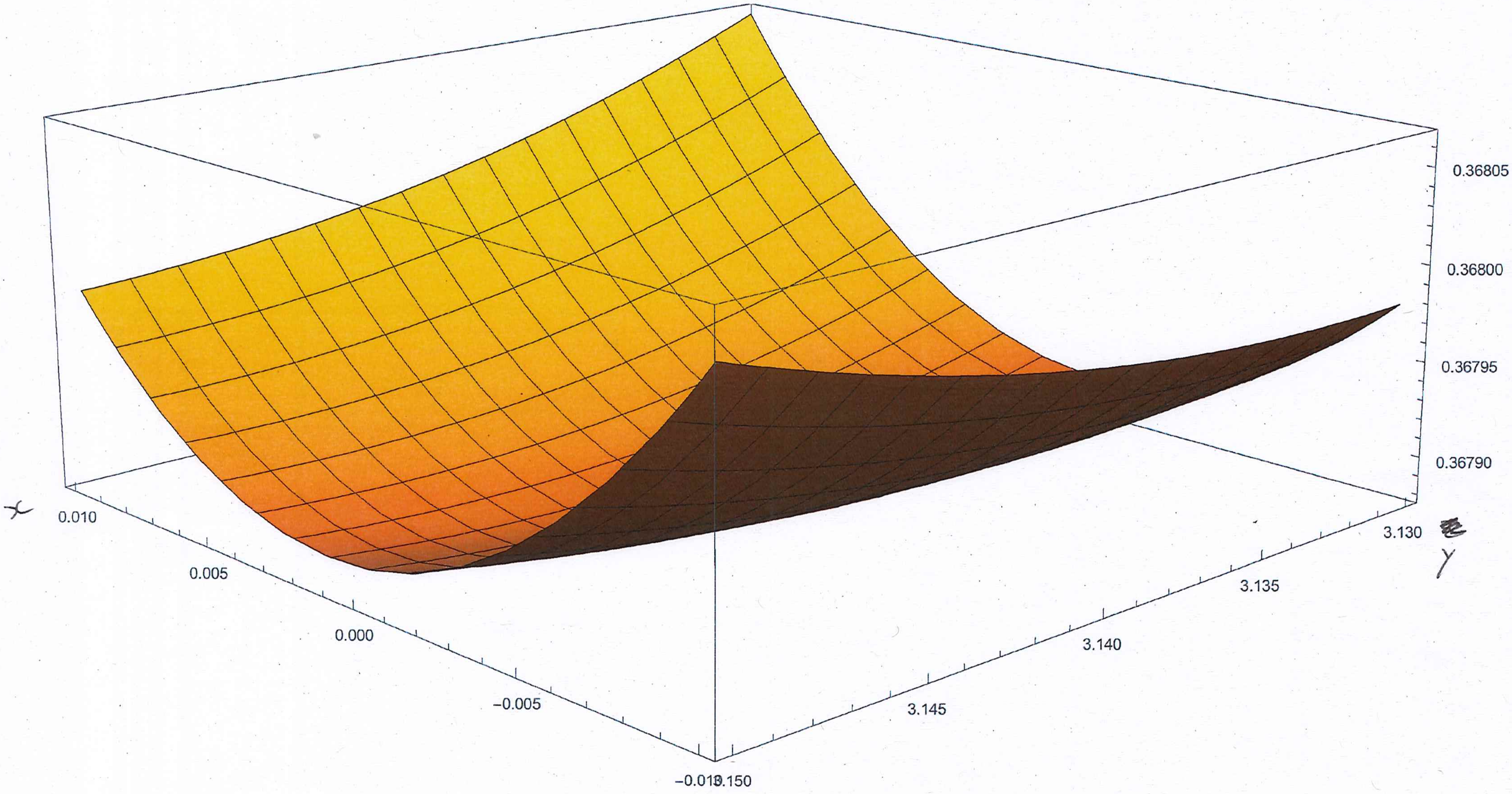
$\det = 2 + \frac{1}{e}$



4
y
2
0
−2
−4
15
10
5
0
4
x
2
0
−2
−4

114




x
0.010
0.005
0.000
−0.005
−0.010
3.150
3.145
3.140
3.135
3.130
y
z
0.36790
0.36795
0.36800
0.36805

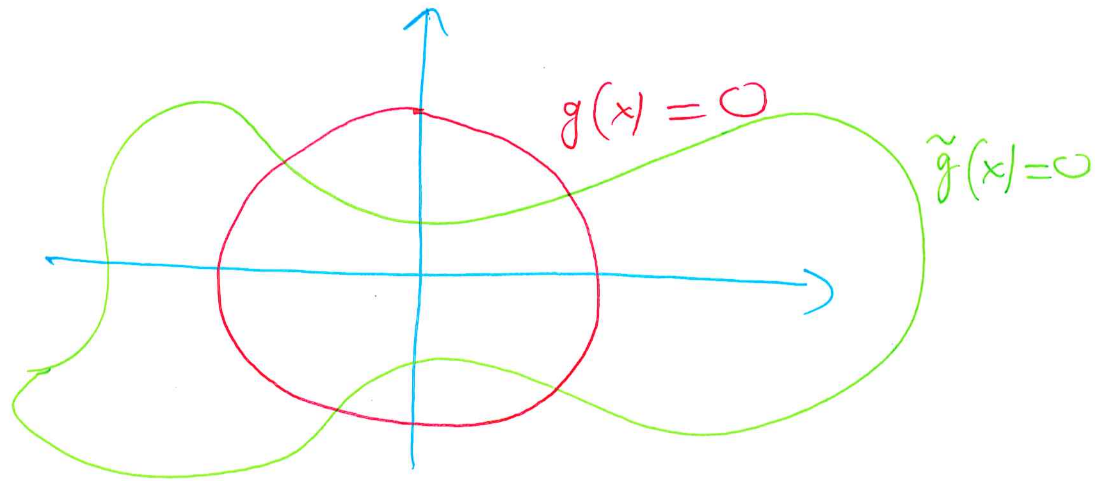
3.9. Lagrange multipliers

Maximizing / minimizing $f(x)$ subject to a "constraint" $g(x) = 0$

<u>Ex.</u> Maximize $f(x)$, $x \in \mathbb{R}^n$, only for x on the sphere

$$\|x\| = 1$$

$$\left(g(x) = \|x\| - 1 \right)$$



Approach 1: parameterize the set of x such that $g(x) = 0$ (say by $x = \varphi(t)$, t parameter) $\varphi : X \longrightarrow \mathbb{R}^n$

Then maximize / min. $f(\varphi(t))$ by previous methods (critical points, etc...)

Approach 2: Lagrange multipliers

Advantage of (2): keep track of possible symmetries of the problem

Prop. 3.9.2

$f : X \longrightarrow \mathbb{R}$ C^1 (X open)

$g : X \longrightarrow \mathbb{R}$ C^1

If $x_0 \in X$ is a local extremum of f restricted to

$$Y = \{ x \in X \mid g(x) = 0 \}$$

then either $\nabla g(x_0) = 0$ (and $x_0 \in Y$)

or $\exists\, \lambda \in \mathbb{R}$ s.t.

n+1 equations

$$\begin{cases} \nabla f(x_0) = \lambda \nabla g(x_0) \\ g(x_0) = 0 \end{cases}$$

Lagrange multiplier

<u>Note:</u> (1) if $\nabla f(x_0) = 0$, then

$$(x_0, \lambda_0 = 0)$$

is a solution. (Critical points of f are candidates for the restricted max/min problem).

(2) It is often the case that

$$Y = \{ x \in X \mid g(x) = 0 \}$$

is <u>compact</u> (= closed, bounded), so there <u>is</u> a global max/min.

Ex. 3.9.4 (3)

$n \geq 1$

Fix $y \in \mathbb{R}^n$, $y \neq 0$

Want to max/min

$$f(x_1, \dots, x_n) = x_1 y_1 + \dots + x_n y_n$$

with the constraint

$$g(x) = 0, \quad g(x) = x_1^2 + \dots + x_n^2 - 1$$

(could be done by Cauchy - Schwarz inequality).

$g(x) = 0$ defines a compact set, so there is a ~~local~~ min / max.

Possible $x \in Y$ where f is local extremum satisfy either

$$\begin{cases} g(x) = 0 \\ \nabla g(x) = 0 \iff (2x_1, \dots, 2x_n) = 0 \end{cases} \qquad g(0) = -1$$

No solutions

or

$$\begin{cases} \nabla f(x) = \lambda \nabla g(x) \\ g(x) = 0 \end{cases}$$

$$\boxed{f(x) = x_1 y_1 + \dots + x_n y_n}$$

$$\begin{cases} x_1^2 + \cdots + x_n^2 = 1 \\ y_1 = 2\lambda x_1 \\ \vdots \\ y_n = 2\lambda x_n \end{cases}$$

Observe that $\lambda = 0$ is impossible

[then $y_1 = \cdots = y_n = 0$, $y = 0$]

So $x_i = \dfrac{y_i}{2\lambda}$; then

$$f(x) = \frac{1}{2\lambda}(y_1^2 + \cdots + y_n^2)$$

First equation $\Longrightarrow$

$$\frac{1}{4\lambda^2}(y_1^2 + \cdots + y_n^2) = 1$$

$$\Rightarrow \lambda = \pm \frac{\|y\|}{2}$$

So the two candidates are

$$x = \frac{y}{2\lambda} = \pm \frac{y}{\|y\|}$$

and $f(x) = \pm \|y\|$.

So $\max_y f = \|y\|$

$\min_y f = -\|y\|$

$$-\|y\| \le x_1 y_1 + \cdots + x_n y_n \le \|y\|$$

if $\|x\| = 1$

$$\iff \quad |x_1 y_1 + \cdots + x_n y_n| \le \|y\|$$

if $\|x\| = 1$

3.10 – Inverse / implicit function Theorems

Inverse function Th:

when is $y = f(x)$, $f : \mathbb{R}^n \to \mathbb{R}^n$

a "change of variable"? [x determined by y]

Implicit function Th: $g : \underset{x}{\mathbb{R}^n} \times \underset{y}{\mathbb{R}} \to \mathbb{R}$

when can we represent the solutions of $g(x,y)=0$

by $y = f(x)$?

<u>Th.</u> (3.10.2)

$X \subset \mathbb{R}^n$ open, $f : X \longrightarrow \mathbb{R}^n$ f C^k

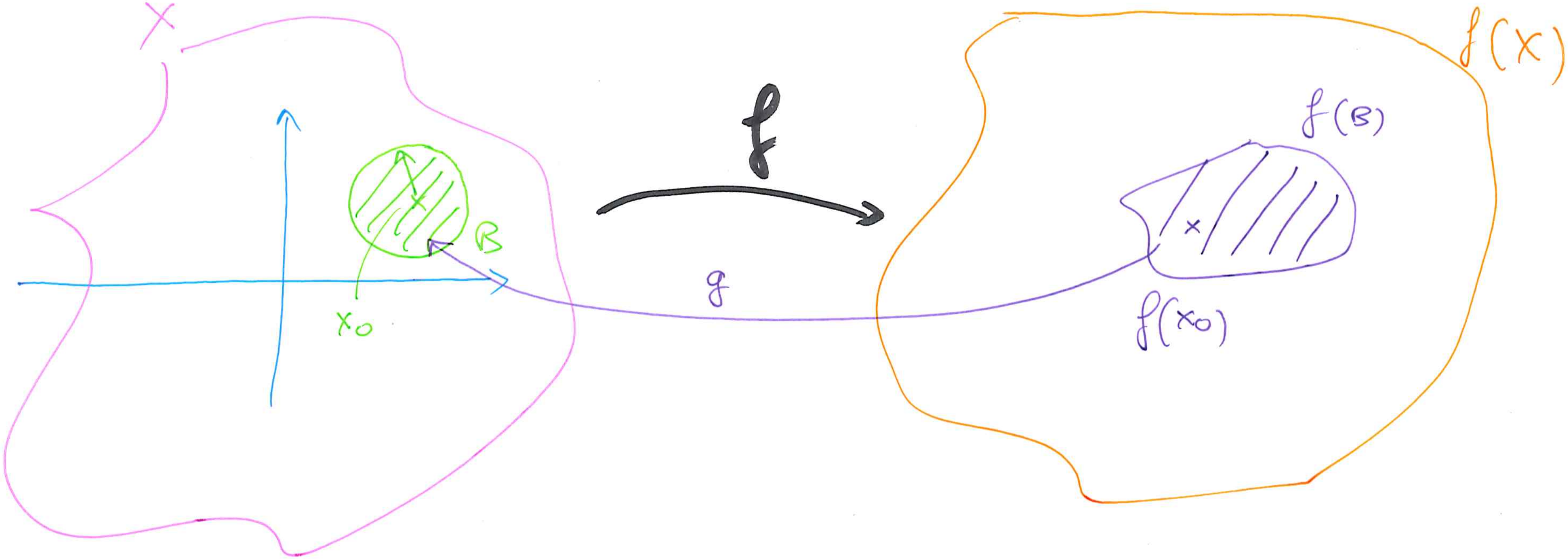
$x_0 \in X$

If $\det(J_f(x_0)) \neq 0$ then f is a change of variable close to x_0:

There is some ball B around x_0 of radius > 0 s.t.

$$f|B : B \underset{\xleftarrow{g}}{\longrightarrow} f(B)$$

is <u>bijective</u> and the inverse g is of class C^k, $J_g(f(x_0)) = J_f(x_0)^{-1}$



$\exists\!\!\!/\ g$ exists

$g \circ f = Id_B$

(Chain Rule) $\Rightarrow$ $J_g(f(x_0)) \cdot J_f(x_0) = Id_n$

Th. 3.10.4

$X \subset \mathbb{R}^{n+1}$ open

$g : X \longrightarrow \mathbb{R}$ C^k

$(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}$ s.t. $g(x_0, y_0) = 0$

If $(\partial_y g)(x_0, y_0) \neq 0$ Then : there exist

$U \subset \mathbb{R}^n$ open, $x_0 \in U$

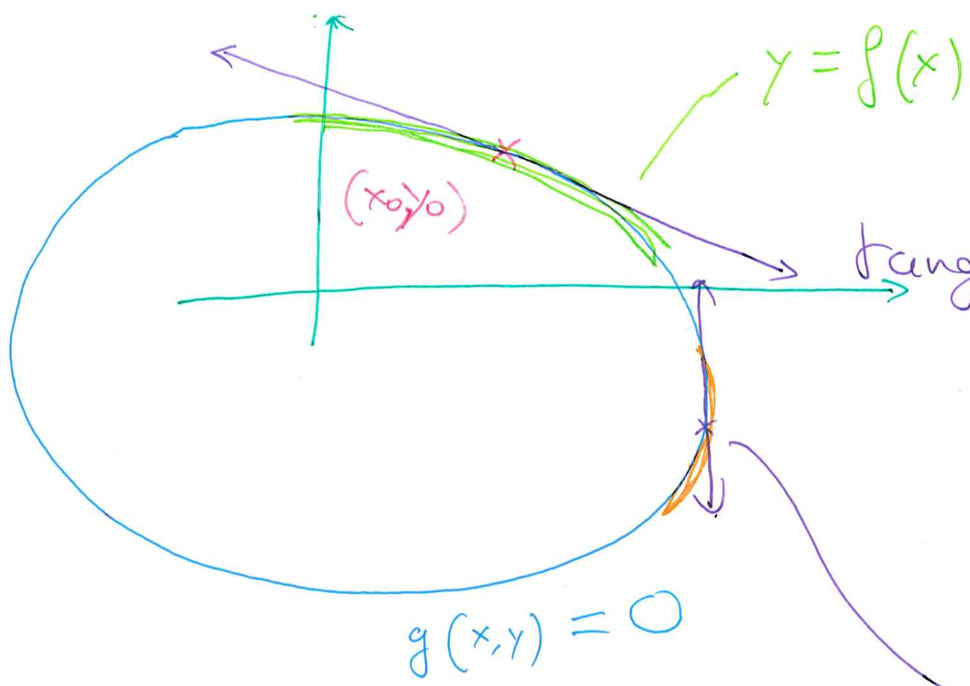
$I \subset \mathbb{R}$ open, $y_0 \in I$

$f : U \longrightarrow \mathbb{R}$, C^k s.t.

$y = f(x) \iff g(x, y) = 0$

for $x \in U$, $y \in I$.

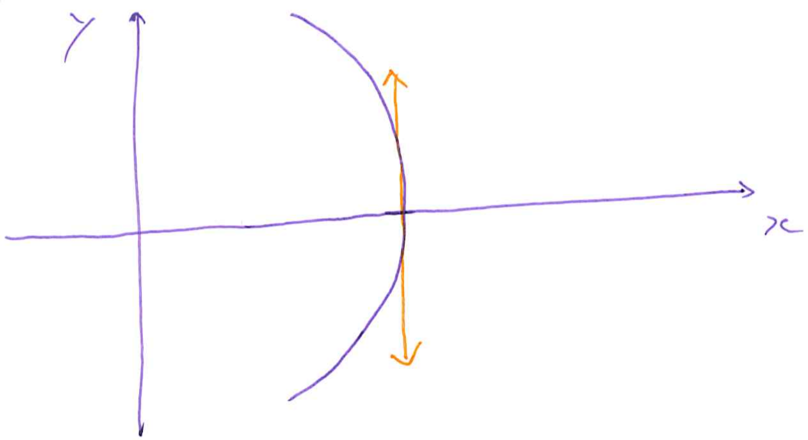
$n=2$



tangent not vertical $\Updownarrow$ $\partial_y g(x_0,y_0) \neq 0$

doesn't work

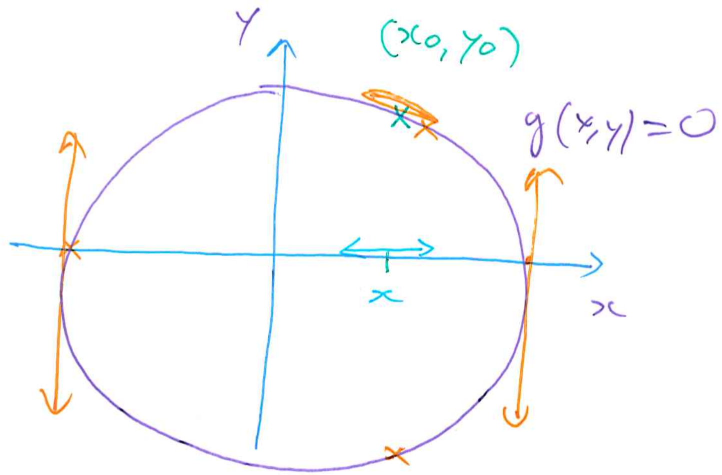
If the tangent is vertical, the picture is:



Ex. 3.10.5 (1) $n = 2$

$$g(x, y) = x^2 + y^2 - 1$$

$$\nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$



so $\partial_y g(x_0, y_0) = 0$

$$\Updownarrow$$

$$y_0 = 0$$

$$\Updownarrow$$

$(1, 0)$, $(-1, 0)$

If $x_0 \neq 0$, we have $\boxed{x \text{ close to } x_0}$, $\boxed{y \text{ close to } y_0}$

$g(x, y) = 0$

$$\Updownarrow$$

$$y = \sqrt{1 - x^2}$$