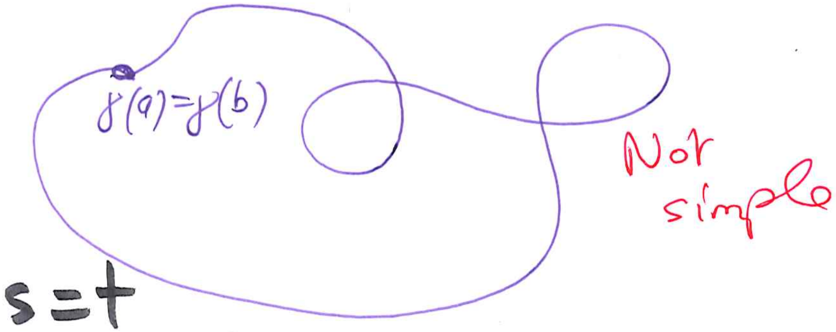
<u>**Def.**</u> (4.6.1) $\gamma : [a,b] \longrightarrow \mathbb{R}^n$

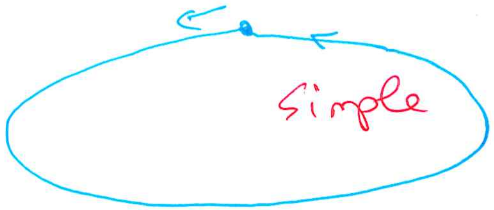
a parameterized closed curve

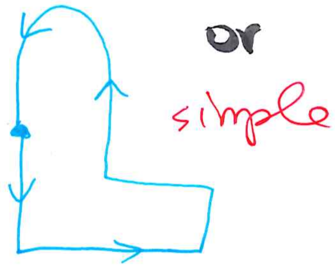
$(\gamma(a) = \gamma(b))$ is

<u>**simple**</u> if:

$$\left(\gamma(s) = \gamma(t) \implies \text{either } s = t \text{ or } s = a,\ t = b \text{ or } s = b,\ t = a \right)$$

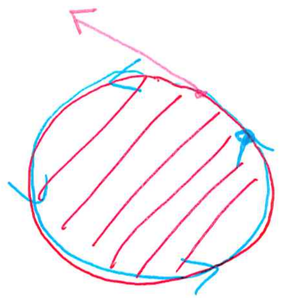






Ex.

(1)



$\gamma(t) = (r\cos t, r\sin t)$

is "positively oriented"

$f = (f_1, f_2)$

Green formula :

$$\int_{(\text{disc})} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy$$

$$= \int_{(\text{circle})} f \cdot d\vec{s}$$

$$= \int_0^{2\pi} \big(f_1(r\cos t, r\sin t) \cdot (-r\sin t) + f_2(r\cos t, r\sin t) \cdot r\cos t \big)\, dt$$

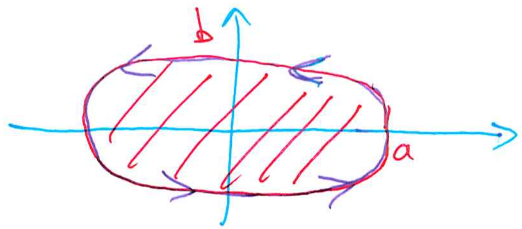
(2) Suppose f is conservative (and one boundary component):

$$0 = \int_X \underbrace{\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)}_{=0} dx\,dy$$

$$= \int_\gamma f \cdot d\vec{s} = 0 \quad (\gamma \text{ is a closed curve})$$

so we prove some cases of the formula.

(3) (Area computation)



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

Area of the interior of the ellipse?

$\gamma(t) = (a\cos t, b \sin t)$, $0 \le t \le 2\pi$

is a positively-oriented C^1 parameterization of ∂X (= ellipse) as a simple closed curve

$\gamma'(t) = (-a\sin t, b\cos t)$ is always $\neq 0$

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy = \int_\gamma f \cdot d\vec{s}$$

To compute area of X, we need to find $f = (f_1, f_2)$ $[C^1]$ such that

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 1 \text{ everywhere}$$

Take $f_1 = 0$; then the condition is

$$\frac{\partial f_2}{\partial x} = 1$$

e.g. $f_2(x,y) = x$

So we take $f(x,y) = (0, x)$ as vector field.

We get

$$\text{Area} = \int_X dx\, dy = \int_\gamma (0,x) \cdot d\vec{s}$$

$$= \int_0^{2\pi} a\cos t \,.\, b\cos t \, dt = ab\int_0^{2\pi} \frac{1}{2}(1+\cos 2t)\, dt$$

$$= \pi ab.$$

<u>Remark</u>: there is no simple formula for the perimeter of an ellipse! ($\simeq$ 1700-1750 "elliptic integrals")

In fact: for any $X \subset \mathbb{R}^2$ with $\partial X = \gamma$ simple closed curve, we get

$$\text{Area}(X) = \int_\gamma (0, x) \cdot d\vec{s}.$$

(4) <u>Generalization</u>:

$$\int_X g(x,y)\,dx\,dy \overset{?}{=} \int f \cdot d\vec{s}$$

g continuous on X

One can <u>always</u> find f so that $\int f \cdot d\vec{s}$ computes a given 2-dimensional integral:

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = g(x,y)$$

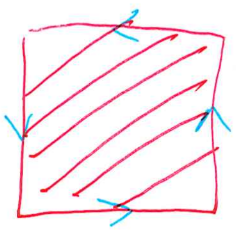
Take $f_1 = 0$, solve $\frac{\partial f_2}{\partial x} = g(x,y)$

For simple enough $g(x,y)$, this gives a simple one-variable integral to compute the 2-dimensional $\int_X g \, dx \, dy$.

For instance, $g(x,y)$ = polynomial in x,y
[Ex. Script 4.6.4 (3), $g(x,y) = x^2 y^2$
X = interior of ellipse]

15) [4.6.4 (4)]

$X = [0,1]^2$ $\qquad f(x,y) = (xy,\, x^2 - y^2)$



$$\int f \cdot d\vec{s} = \int_0^1 \int_0^1 (2x - x)\, dx\, dy$$

Green

$$= 1 \cdot \frac{1}{2}$$

[no need to even write a parameterization of the curve !]

(6) "Divergence form of the Green formula"

$f = (f_1, f_2)$ C^1 vector field

X

$\partial X = \underbrace{\text{simple closed curve}}_{}$ γ

We want to compute

$$\int_X \underbrace{\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} \right)}_{\mathrm{div}(f)} dx\,dy = \, ?$$

"divergence of the vector field"

Put $\quad \tilde{f}(x,y) = (-f_2(x,y), f_1(x,y))$.

Then
$$\frac{\partial \tilde{f}_2}{\partial x} - \frac{\partial \tilde{f}_1}{\partial y} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = \operatorname{div}(f)$$

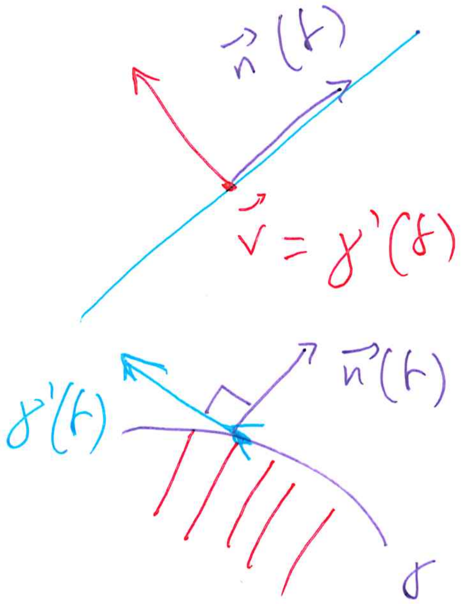
So Green's formula gives

$$\int_X \operatorname{div}(f)\, dx\, dy = \int_\gamma \tilde{f} \cdot d\vec{s}$$

$$= \int_a^b \left(-f_2(\gamma(t)) \cdot \gamma_1'(t) + f_1(\gamma(t)) \cdot \gamma_2'(t) \right) dt$$

$$= \int_a^b f(\gamma(t)) \cdot \vec{n}(t)\, dt$$

where $\vec{n}(t) = (\gamma_2'(t), -\gamma_1'(t))$.



$\vec{n}(t)$ is obtained from $\gamma'(t)$ by a clockwise quarter-turn

$\vec{n}(t)$ is perpendicular to ∂X, and pointing outside of X, and the length of $\vec{n}(t)$ is the same as that of $\gamma(t)$

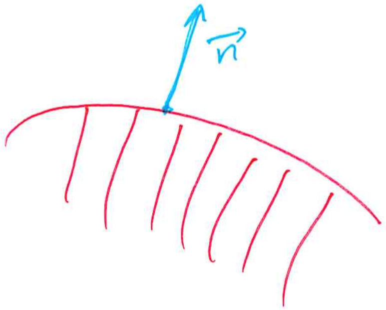
<u>Notation:</u>

$$\int_\gamma f \cdot \vec{n}\, ds = \int_\gamma f \cdot d\vec{n}$$

Divergence form of Green's Theorem:

X

$\partial X = \gamma$ simple closed curve



$\vec{n}$ = "exterior" normal vector along γ

f vector field, C^1, on X

$$\int_X \operatorname{div}(f)\, dx\, dy = \int_\gamma f \cdot d\vec{n}$$

END OF EXAM PROGRAM!

4.7 – Gauss – Ostrogradski formula

Case $n = 3$ of the general Stokes formula:

$$\text{LHS} = \int_X \operatorname{div}(f)\, dx\, dy\, dz$$

X — 3-dim compact set in $\mathbb{R}^3$

$f : X \to \mathbb{R}^3$

C^1

$$\operatorname{div}(f) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

RHS $=$ 2-dim. integral of f along the boundary ∂X of X, positively oriented

What needs to be done to state this formula?

(1) define the boundary as a surface

(2) ________ the orientation

(3) define $\int_{\partial X} g(x,y,z)$ [generalize line integral]

<u>Def.</u> (4.7.1)

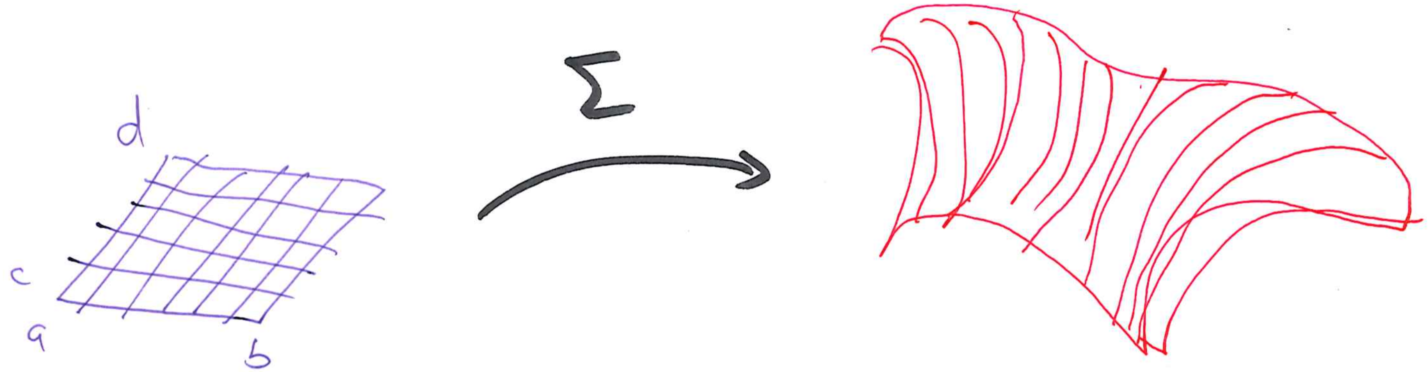
A parameterized surface Σ is

$$\Sigma : \underset{(s}{[a,b]} \times \underset{t)}{[c,d]} \longrightarrow \mathbb{R}^3$$

s.t. Σ is C^1 on $]a,b[\times]c,d[$,

and for all $(s,t) \in]a,b[\times]c,d[$,

the Jacobian $J_\Sigma(s,t)$ <u>has rank 2</u>



The rank condition ensures that the image of Σ in $\mathbb{R}^3$ is really two-dimensional.

<u>Ex.</u>

(1) let $g : [a,b] \times [c,d] \longrightarrow \mathbb{R}$
C^1 on $]a,b[\times]c,d[$

and $\Sigma(s,t) = \begin{pmatrix} s \\ t \\ g(s,t) \end{pmatrix}$. The image of Σ is the graph of Σ in $\mathbb{R}^2 \times \mathbb{R}$.

Rank condition:

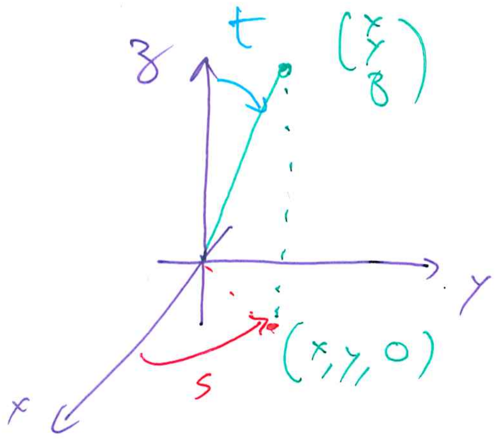
$$J_\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_s g & \partial_t g \end{pmatrix}$$ has rank 2 since the first two rows are linearly independent

(2) Sphere of radius $r > 0$ centered at (x_0, y_0, z_0) as a parameterized surface:

$$\Sigma(s,t) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} r\cos(s)\sin(t) \\ r\sin(s)\sin(t) \\ r\cos(t) \end{pmatrix},$$

$0 \leq s \leq 2\pi$

$0 \leq t \leq \pi$



$$\begin{pmatrix} 0 < s < 2\pi \\ 0 < t < \pi \end{pmatrix}$$

Rank condition:

$$J_\Sigma = \begin{pmatrix} -r\sin(s)\sin(t) & r\cos(s)\cos(t) \\ r\cos(s)\sin(t) & r\sin(s)\cos(t) \\ 0 & -r\sin(t) \end{pmatrix}$$

Last two rows are independent unless $\cos(s) = 0 \iff s = \frac{\pi}{2}$ or $\frac{3\pi}{2}$;

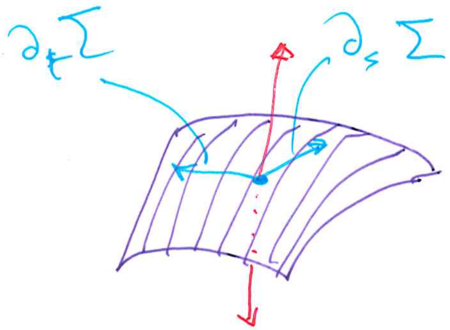
Then

$$J_\Sigma = \begin{pmatrix} \mp\sin(t) & 0 \\ 0 & \pm r\cos(t) \\ 0 & -r\sin(t) \end{pmatrix}$$

so 1^{st} and 3^{rd} rows are independent.

Next : orientation of the surface

<u>Pb.</u> : some surfaces cannot be oriented ! (Möbius strip for instance)



From a point $\Sigma(s,t)$, we get two vectors $(\partial_s \Sigma, \partial_t \Sigma)$; we "orient" Σ at that point by saying that the perpendicular vectors $\vec{n}$ s.t. $\det(\partial_s \Sigma, \partial_t \Sigma, \vec{n}) > 0$ are pointing "outside".

More precisely: we use the cross-product

Def. x, y in $\mathbb{R}^3$, linearly independent

$x \times y$ is the unique vector z perpendicular to x, y s.t.

(1) $\det(x, y, z) > 0$

(2) $\|z\| = \|x\| \, \|y\| \, \sin(\theta)$

θ = angle between x and y