Most important property of $\int_\gamma f(\vec{s}) \cdot d\vec{s}$:

independance of parameterization!

<u>**Def.**</u> (4.1.4)

$$\gamma : [a, b] \longrightarrow \mathbb{R}^n$$

An <u>oriented</u> reparameterization of γ is $\sigma : [c, d] \longrightarrow \mathbb{R}^n$

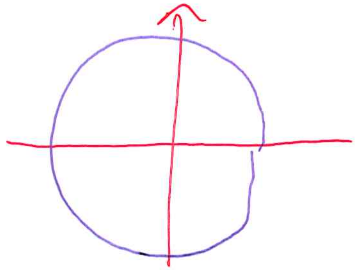
s.t. $\gamma(\cancel{t}) = \sigma(\varphi(u))$, $c \leq u \leq d$

where $\varphi : [c, d] \longrightarrow [a, b]$ is C^1 strictly increasing, $\varphi(c) = a$, $\varphi(d) = b$

Ex. $\gamma_n(t) = \left(\cos(2\pi t^n), \sin(2\pi t^n)\right)$

$n \geq 1$ integer, $0 \leq t \leq 1$

repar. of the circle γ_1



Prop. 4.1.5 – γ, σ oriented reparameteri-zation of γ

$f : X \longrightarrow \mathbb{R}^n$

$$\int_\gamma f(s) \cdot d\vec{s} = \int_\sigma f(s) \cdot d\vec{s}$$

Proof.

$$\sigma = \gamma \circ \varphi$$

$$\varphi : [c, d] \longrightarrow [a, b], \quad u \mapsto t$$

$$\int_\sigma f(s) \cdot d\vec{s} = \int_c^d f(\sigma(u)) \cdot \sigma'(u)\, du$$

$$= \int_c^d f(\gamma(\varphi(u))) \cdot \overbrace{\underbrace{\varphi'(u)}_{\mathbb{R}}\,\underbrace{\gamma'(\varphi(u))}_{\mathbb{R}^n}}^{\text{chain rule}}\, du$$

$$= \int_c^d \underbrace{f(\gamma(\varphi(u))) \cdot \gamma'(\varphi(u))}_{\mathbb{R}} \; \underbrace{\varphi'(u)}_{\mathbb{R}}\, du$$

Substitute

$\varphi(u) = t$

$\varphi'(u)\,du = dt$

$$= \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

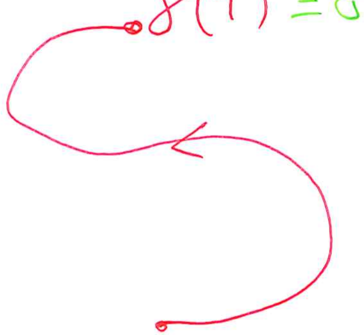
$$= \int_\gamma f(s) \cdot d\vec{s}$$

Remark: (1) oriented reparameterizations only!

For instance a non-oriented reparameterization of $\gamma : [0,1] \longrightarrow \mathbb{R}^n$ is

$$\sigma(u) = \gamma(1-u)$$

$$\sigma : [0,1] \longrightarrow \mathbb{R}$$

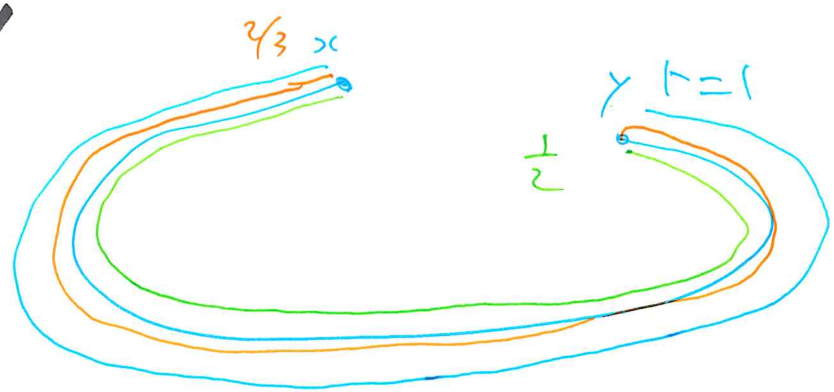


$$\left(\int_0^1 g = - \int_1^0 g \right)$$

Then

$$\int_\gamma f(s) \cdot d\vec{s} = - \int_\sigma f(s) \cdot d\vec{s}$$

(2) So $\int_\gamma f(s) \cdot d\vec{s}$ "only depends on the image of γ in $\mathbb{R}^n$"

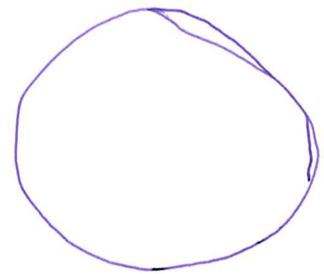


This is true, but only if one goes along the curve only once! If σ goes from x to y with $0 \le t \le 1/2$, then y to x with $1/2 \le t \le 2/3$, then to y, $\frac{2}{3} \le t \le 1$.

Then $\displaystyle\int_\sigma f(s)\cdot d\vec{s} \neq \int_\gamma f(s)\cdot d\vec{s}$

even if the image of σ is the same as that of γ.

<u>Ex.</u> "Compute the line integral of $f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$ along the circle $x^2+y^2=1$"



Then one has to find a parameterization going around the circle <u>only once</u>.

Conservative vector fields

Let $X \subset \mathbb{R}^n$ be open

Let $g : X \longrightarrow \mathbb{R}$ be C^1

Define $$f = \nabla g = \begin{pmatrix} \partial_{x_1} g \\ \vdots \\ \partial_{x_n} g \end{pmatrix}$$

This is a vector field on X ; it is continuous.

Let $\gamma : [a, b] \longrightarrow X$ be a parameterized curve.

$$\int_\gamma f(s) \cdot d\vec{s} = \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, dt$$

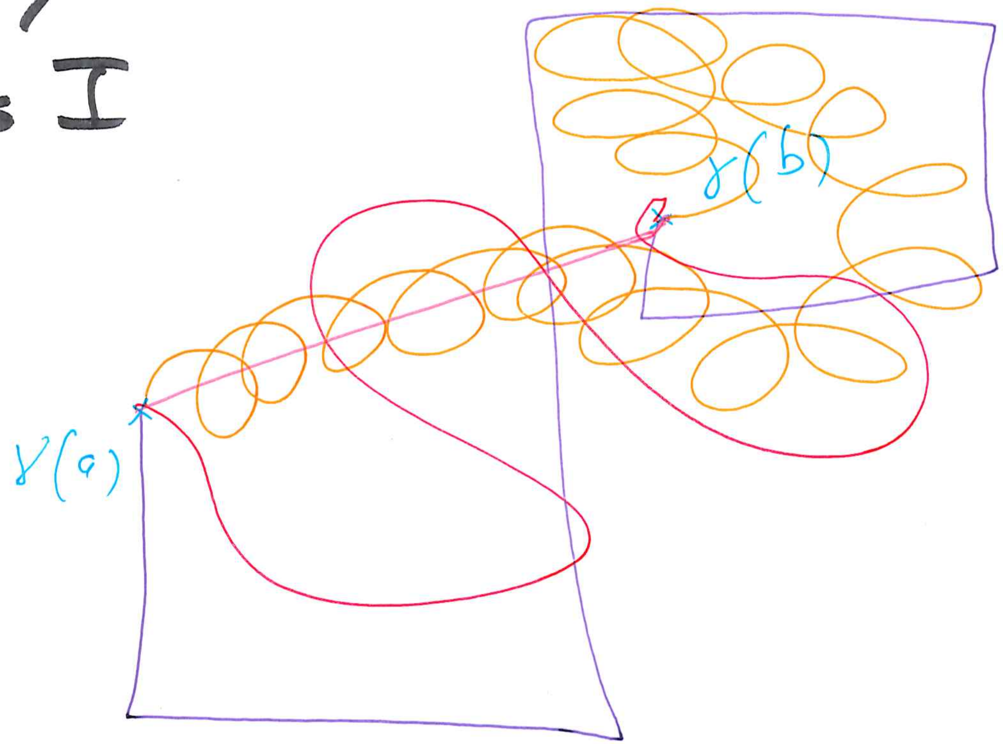
$$\left[\begin{array}{l} \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t)) \\ \gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t)) \end{array} \right]$$

$$= \int_a^b \sum_{i=1}^n \underbrace{\frac{\partial f}{\partial x_i}(\gamma(t))}_{\mathbb{R}} \underbrace{\gamma_i'(t)}_{\mathbb{R}} \, dt$$

$$\underset{\substack{\text{(Chain rule} \\ \text{in } \mathbb{R}^n)}}{=} \int_a^b \frac{d}{dt} g(\gamma(t)) \, dt$$

$$\underset{\text{Analysis I}}{=} g(\gamma(b)) - g(\gamma(a))$$

This only depends on the endpoints of γ!



Def. (4.18) A vector field
$$f : X \longrightarrow \mathbb{R}^n$$
is **conservative** if the line integral
$$\int f(s) \cdot d\vec{s}$$
~~only depends on the values o~~

is the same for all parameterized curves in X with the same starting point and endpoint.

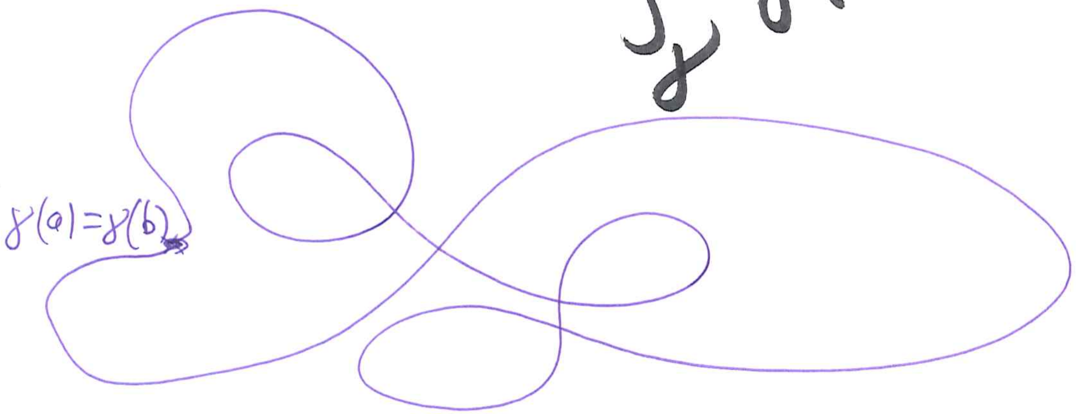
Remark: (4.1.9)

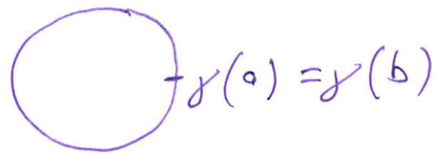
$f : X \longrightarrow \mathbb{R}^n$ conservative

$$\Updownarrow$$

for any closed curve $\gamma : [a,b] \longrightarrow X$ (that is, $\gamma(b) = \gamma(a)$), we have

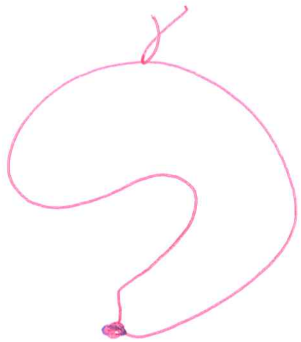
$$\int_\gamma f(s) \cdot d\vec{s} = 0$$





[f conservative;

γ closed;



$$\int_\gamma f(s)\cdot d\vec{s} = \int_\sigma f(s)\cdot d\vec{s}$$

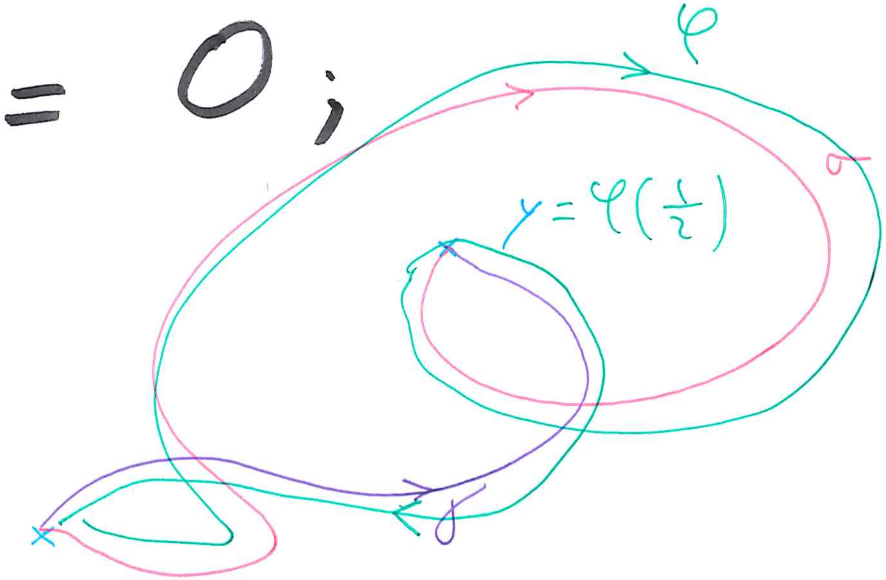
where $\sigma(t) = \gamma(a)$

$$= 0;$$

conversely:

$$\int_\varphi f(s)\cdot d\vec{s} = 0$$

$$\int_\sigma f(s)\cdot d\vec{s} - \int_\gamma f(s)\cdot d\vec{s}$$



]

Th. (4.1.10)

Let $f: X \longrightarrow \mathbb{R}^n$ be (continuous) conservative. Then there is

$$g: X \longrightarrow \mathbb{R}, \quad g \; C^1,$$

such that $f = \nabla g$.

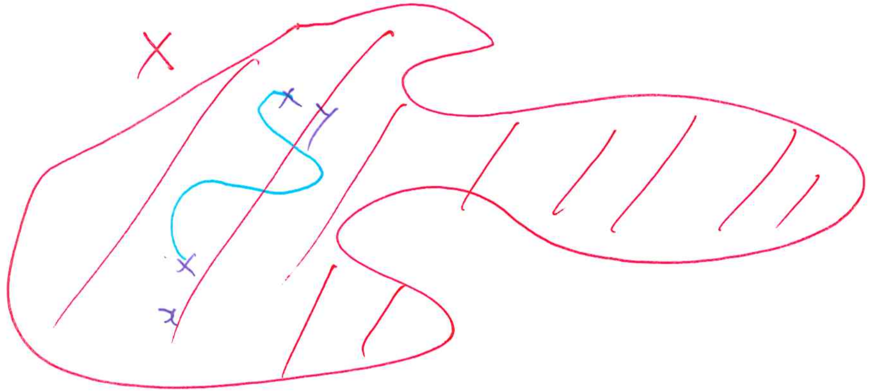
[**Def.** g is called a <u>potential</u> of the vector field f.]

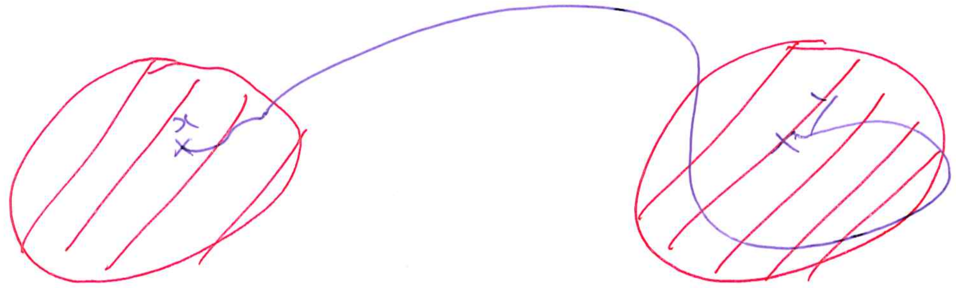
Remark: (1) the potential is not unique: one replace g by $g+a$, where $a \in \mathbb{R}$.

(2) if X is path-connected then two potentials g_1, g_2 of f satisfy

$$g_2 - g_1 = \text{constant on } X.$$

Def. X is path-connected $\Leftrightarrow$ for all x, y in X there is curve $\gamma : [a,b] \longrightarrow X$ s.t. $\gamma(a) = x$, $\gamma(b) = y$

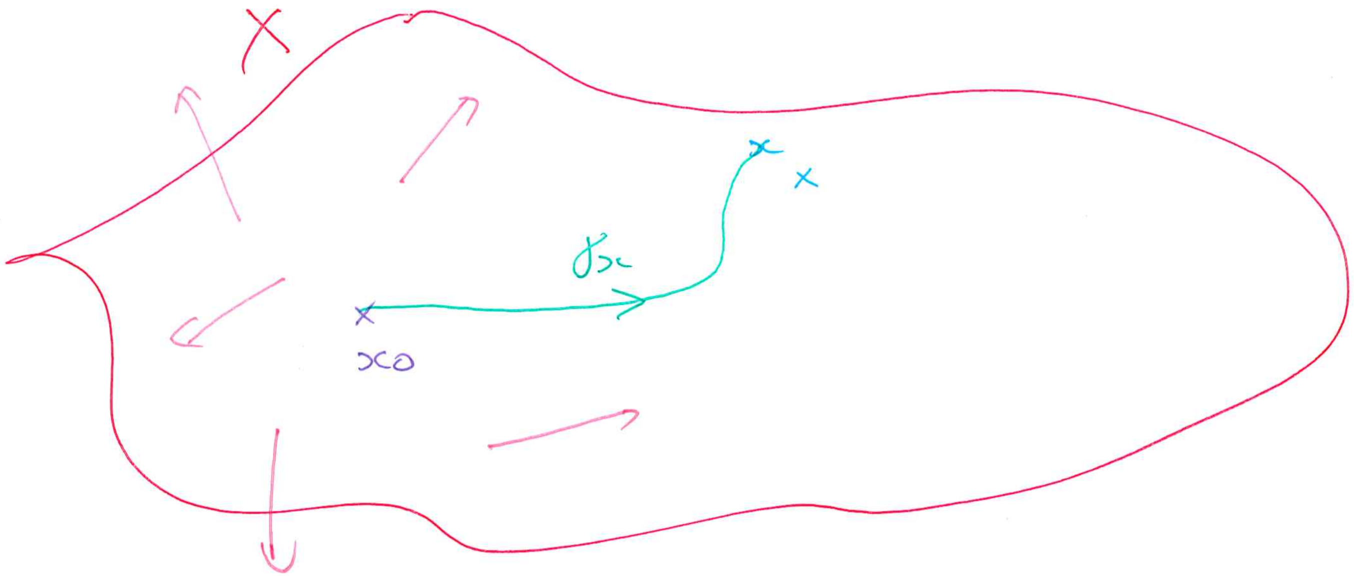




X = union of two disjoint discs is <u>not</u> path-connected

<u>Idea of proof.</u>

(X path connected)



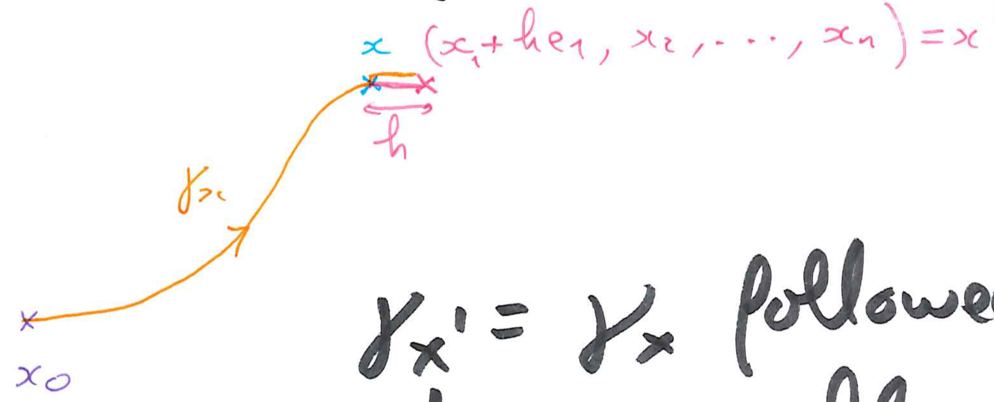
$$g(x) = \int_{\gamma_x} f(s) \cdot d\vec{s}$$

where γ_x goes from x_0 to x

f conservative $\Rightarrow$ $g(x)$ does not depend on the choice of γ_x.

To compute

$$\frac{\partial g}{\partial x_1}$$



$\gamma_{x'} = \gamma_x$ followed by a small horizontal segment

$\Rightarrow$ computations like fundamental th. of calculus.

Questions:

(1) How does one check that a "concrete" vector field is conservative or not?

(2) How does one compute a potential g of a conservative vector field f?

Example / answer for (2): $f = (f_1, \dots, f_n)$

Method: solve $\dfrac{\partial g}{\partial x_1} = f_1$ by primitive w.r.t x_1 $\Rightarrow$ $g = F_1 + g_1(x_2, \dots, x_n)$

solve $f_2 = \dfrac{\partial F_1}{\partial x_2} + \dfrac{\partial g_1}{\partial x_2}$

$\Rightarrow$ $g = F_1 + F_2 + g_2(x_3, \dots)$

$\boxed{4.1.12\ (2)}$

$$f(x, y, z) = \begin{pmatrix} 6x^2 \cos(yz) + z\sin(y) \\ -3x^3 \sin(yz) + xz\cos(y) + 2y \\ -3x^3 \sin(yz) + x\sin(y) + 2z \end{pmatrix}$$

$$\Rightarrow \quad g = 2x^3 \cos(yz) + xz \sin(y) + g_1(y, z)$$

$$\Rightarrow \quad -2x^3 z \sin(y) + xz \cos(y) + \frac{\partial g_1}{\partial y} = \ldots$$

$$\Rightarrow \quad g_1(y, z) = y^2 + k(z)$$

$$\vdots$$

$$\Rightarrow \quad g(x, y, z) = 3x^3 \cos(yz) + xz \sin(y) + y^2 + z^2$$

Q. How does one check that a vector field is conservative?

Necessary condition:

Prop. 4.1.13

$X \subset \mathbb{R}^n$ open

$f : X \longrightarrow \mathbb{R}^n$ $\underline{\underline{C^1}}$ vector field

$\underline{\underline{\text{If}}}$ f is conservative then

$$(*)_{i,j} \quad \frac{\partial^2 g}{\partial x_j \partial x_i} = \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} = \frac{\partial^2 g}{\partial x_i \partial x_j} \quad \text{for } 1 \leq i \neq j \leq n$$

Ex:

$$f(x,y,z) = \begin{pmatrix} y^2 \\ xz \\ 1 \end{pmatrix} \quad \text{in } \mathbb{R}^3$$

$$\frac{\partial f_1}{\partial y} = 2y \neq \frac{\partial f_2}{\partial x} = z$$

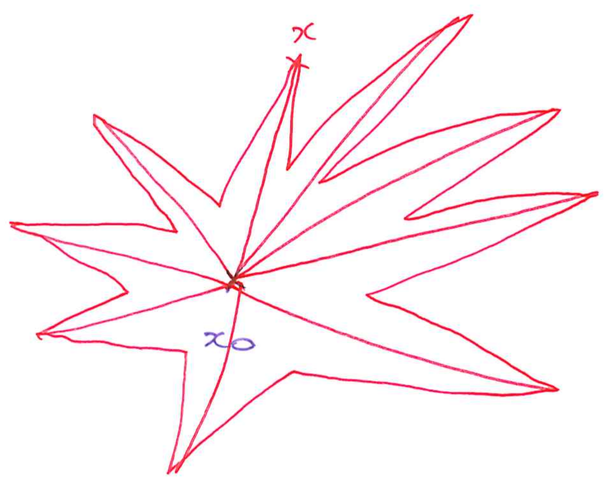
so f is <u>not</u> conservative

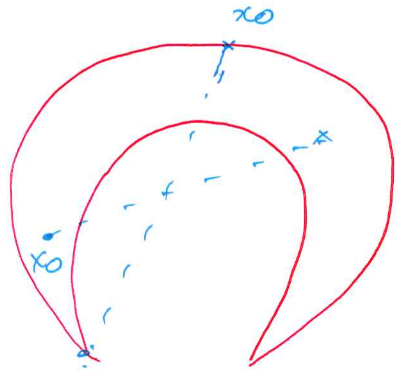
<u>Question:</u> if all equalities $(*)_{i,j}$ are true for f (C^1), is it conservative?

The answer depends on the <u>shape of X</u>.

<u>Def.</u> (4.1.15)

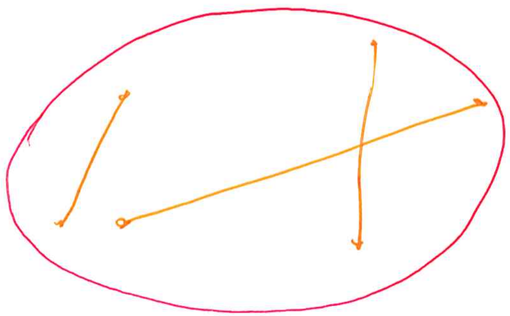
(1) $X \subset \mathbb{R}^n$ is called <u>star-shaped</u> around $x_0 \in X$ if for all $x \in X$, the segment from x_0 to x is contained in X.

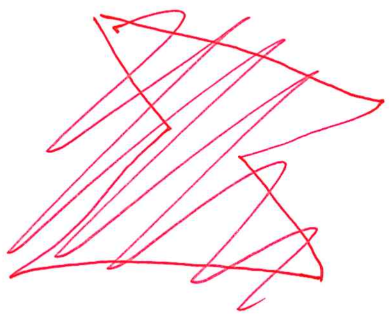


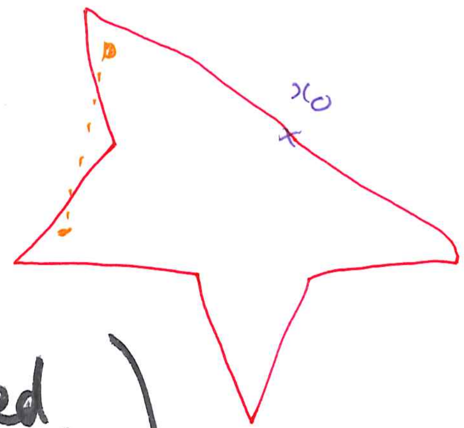
[

→ not star-shaped]

(2) $X \subset \mathbb{R}^n$ is <u>convex</u> if for all x, y in X, the segment joining them is contained in X.



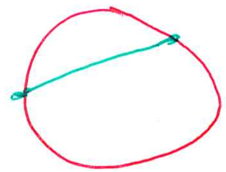




(star-shaped, not convex)

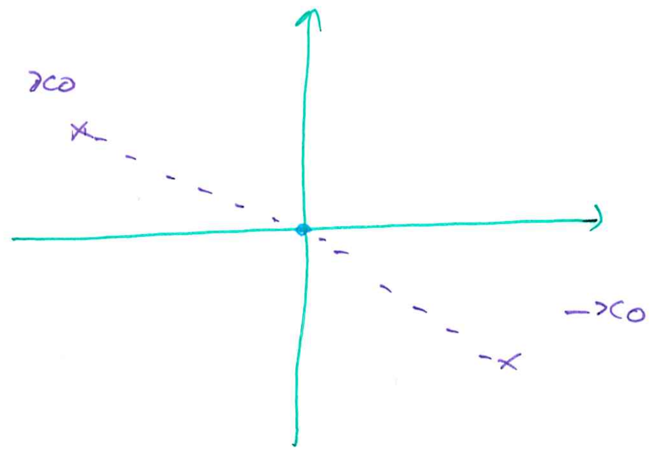
<u>Ex.</u> · any ball is convex $\{x \in \mathbb{R}^n \mid \|x - x_0\| \le r\}$

· a sphere is not convex $\{x \in \mathbb{R}^n \mid \|x - x_0\| = r\}$



· $\mathbb{R}^n$ is convex

· $\mathbb{R}^n - \{0\}$ is <u>not</u> star-shaped



(Ball of radius 1 around 0) $- \{0\}$ is not star-shaped

Th. (4.1.17)

$X \subset \mathbb{R}^n$ open, star-shaped

$f: X \longrightarrow \mathbb{R}^n$ C^1 vector field

If $$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad 1 \leq i \neq j \leq n$$

then f is conservative.

Ex. $(a, b, c) \in \mathbb{R}^3$

$$f(x,y) = \begin{pmatrix} ax^3y + bxy^3 \\ bx^4 + cx^2y^2 \end{pmatrix}$$

For which (a,b,c) is f conservative?

$X = \mathbb{R}^2$ is star-shaped, so this is

so $\Longleftrightarrow$

$$\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y}$$

$$4bx^3 + 2cxy^2 \overset{?}{=} ax^3 + 3bxy^2$$

$\Longleftrightarrow$ $4b = a$ and $2c = 3b$

so $(a, b, c) = \left(4b, b, \frac{3b}{2}\right)$.

[<u>Note:</u> if f_1, f_2 conservative, then $f = af_1 + bf_2$ is conservative for all a, b in $\mathbb{R}$]

$g = ag_1 + bg_2$