Formula:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

$$= \det \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Ex.

(i) $e_1 \times e_2 = e_3$

(ii) $e_2 \times e_3 = e_1$

(iii) $e_3 \times e_1 = e_2$

\+ bilinearity

$(ax+by) \times z$

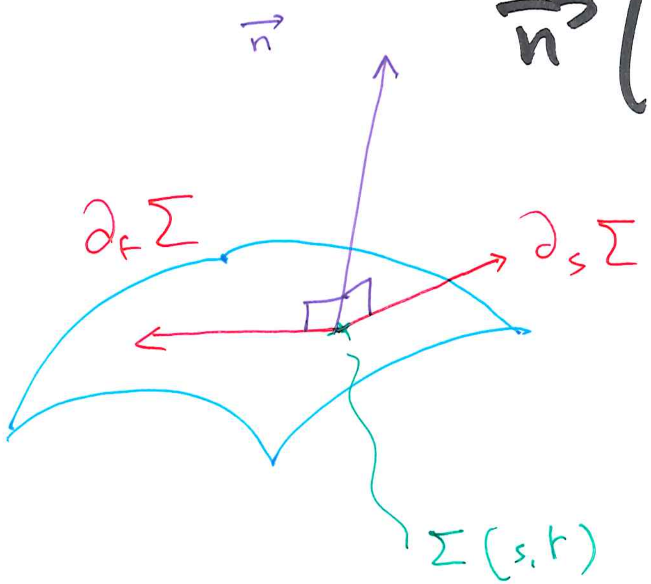
$= ax \times z + by \times z$

etc...

Let $\Sigma : [a,b] \times [c,d] \longrightarrow \mathbb{R}^3$ be a parameterized surface.

For s,t given, let

$$\vec{n}(s,t) = \partial_s \Sigma \times \partial_t \Sigma \in \mathbb{R}^3$$



This is a vector perpendicular to Σ, s.t.

$$\det(\partial_s \Sigma, \partial_t \Sigma, \vec{n}) > 0$$

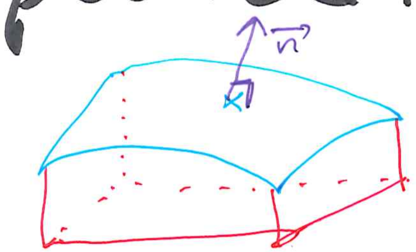
("positive orientation")

Let $X \subset \mathbb{R}^3$ 3-dim. compact

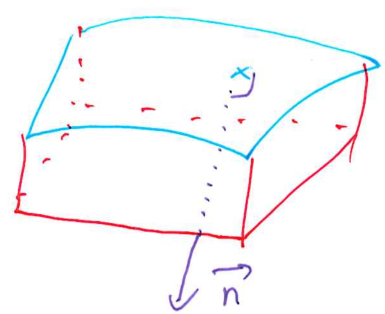
have boundary Σ (i.e. the image de Σ)

[ex. X = ball of radius r centered at 0
Σ = sphere ———————— ———————— 0]

At every (s,t), two pictures are possible:



"exterior normal vector"



"interior normal vector"

Theorem (4.7.6)

$X \subset \mathbb{R}^3$ 3-dim $\underline{a}$ compact

$\partial X = \Sigma$ param. surface

$\underline{\underline{\text{s.t}}}$ $\vec{n}$ is an exterior normal vector

Let $\vec{u} = \frac{\vec{n}}{\|\vec{n}\|}$ (unit normal vector)

Let $f = (f_1, f_2, f_3)$ be a C^1 vector field

Then

$$\int_X \operatorname{div}(f)\, dx\, dy\, dz = \int_\Sigma (f \cdot \vec{u})\, d\sigma$$

$\operatorname{div}(f) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

where the RHS is the surface integral of $f \cdot \vec{u}$ along Σ

Def. (Surface integral)

$\Sigma : [a,b] \times [c,d] \longrightarrow \mathbb{R}^3$ param. surface

$\vec{n}$ = normal vector

g: ~~a~~ continuous function on $\mathbb{R}^3$

$$\int_{\Sigma} g \, d\sigma \overset{\text{def}}{=} \int_a^b \int_c^d g(\Sigma(s,t)) \, \underline{\sigma(s,t)} \, ds \, dt$$

where $\sigma(s,t) = \| \partial_s \Sigma \times \partial_t \Sigma \| = \| \vec{n} \|$

<u>Key property:</u>

$$\int_\Sigma g \, d\sigma$$

is <u>independent</u> of the parameterization of Σ (<u>positively oriented</u>)

[if $\varphi : [a,b] \times [c,d] \longrightarrow [a',b'] \times [c',d']$ with coordinates u, v

is a change of variable and $\Sigma' \circ \varphi = \Sigma$

then $\int_{\Sigma'} g \, d\sigma = \int_\Sigma g \circ \varphi \, d\sigma$

Gauss - Ostrogradski:

$$\int_X \mathrm{div}(f)\,dx\,dy\,dz = \int_\Sigma (f\cdot\vec{u})\,d\sigma$$

"flux of f through Σ"

where $(f\cdot\vec{u})(\Sigma(s,t)) = f(\Sigma(s,t))\cdot\vec{u}(\Sigma(s,t))$

is a continuous function.

Examples 4.7.7

(1) Surface area for the graph of $f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$

so

$$\Sigma(s,t) = \begin{pmatrix} s \\ t \\ f(s,t) \end{pmatrix} \in \mathbb{R}^3$$

If we take $g = 1$ then

$$\int_\Sigma 1 . d\sigma = \text{Area of } \Sigma$$

Indeed

$$\partial_s \Sigma = \begin{pmatrix} 1 \\ 0 \\ \partial_s f \end{pmatrix}, \quad \partial_t \Sigma = \begin{pmatrix} 0 \\ 1 \\ \partial_t f \end{pmatrix}$$

$$\Rightarrow \quad \underbrace{\partial_s \Sigma \times \partial_t \Sigma}_{= \vec{n}} = \begin{pmatrix} -\partial_s f \\ -\partial_t f \\ 1 \end{pmatrix}$$

so $\|\vec{n}\| = \sqrt{1 + (\partial_s f)^2 + (\partial_t f)^2} = \sigma(s,t)$

and $\displaystyle\int_\Sigma d\sigma = \int_a^b \int_c^d \sqrt{1 + (\partial_s f)^2 + (\partial_t f)^2}\, ds\, dt$

= surface area of the graph of f

(2) $X =$ ball of radius $r > 0$ centered at 0

$\partial X = \Sigma$ sphere of radius $r > 0$

$$\int_X \operatorname{div}(f)\, dx\, dy\, dz = \int_\Sigma (f \cdot \vec{u})\, d\sigma$$

<u>Take</u> $f(x, y, z) = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ so $\operatorname{div}(f) = 1$.

Then

$$\operatorname{Vol}(X) = \int_\Sigma (f \cdot \vec{u})\, d\sigma$$

$$\Sigma(s,t) = \begin{pmatrix} r\cos s \sin t \\ r\sin s \sin t \\ r\cos t \end{pmatrix}, \quad \begin{array}{l} 0 \le s \le 2\pi \\ 0 \le t \le \pi \end{array}$$

We compute

$$\partial_s \Sigma \times \partial_t \Sigma = -r^2 \sin t \begin{pmatrix} \cos s \sin t \\ \sin s \sin t \\ \cos t \end{pmatrix}$$

<u>Check:</u> $\partial_s \Sigma \times \partial_t \Sigma$ is an <u>interior normal vector</u>

so $$\vec{u} = -\frac{\partial_s \Sigma \times \partial_t \Sigma}{\|\partial_s \Sigma \times \partial_t \Sigma\|}$$

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$$\int_{\Sigma} (f \cdot \vec{u}) \, d\sigma$$

$$= \int_0^{2\pi} \int_0^{\pi} (r \cos s \sin t) \cdot \frac{r^2 \sin^2 t \cos s}{r^2 \sin t} \, ds\,dt$$

? $\sigma(s,t)$

$\|\vec{n}\|$

$$= \left(r \int_0^{2\pi} \cos^2 s \, ds \right) \left(r^2 \int_0^{\pi} \sin^3(t) \, dt \right)$$

$$= \frac{4\pi r^3}{3}$$

$$\left(\sigma(s,t) = \|\vec{n}\| = r^2 \sin t \right)$$