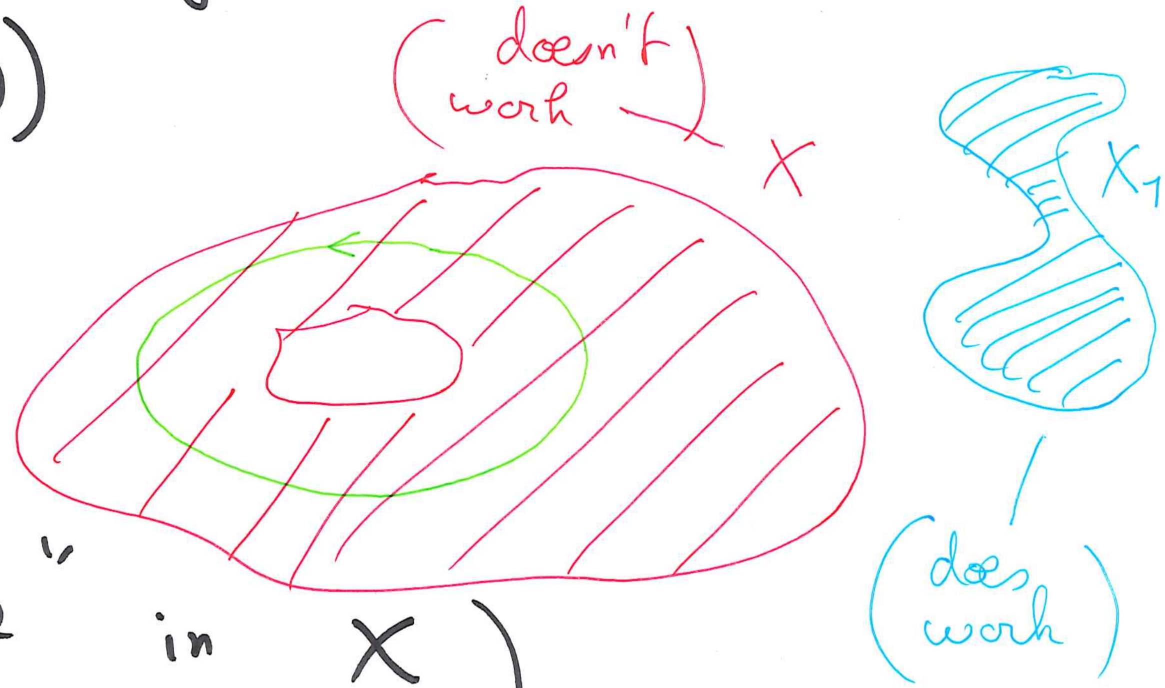
<u>Remark</u>: (1) There exist open sets X for which the necessary conditions

$$(*)_{i,j} \quad \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

are <u>not</u> sufficient for f to be conservative.

(Ex. 4.1.19 (1))

In $\mathbb{R}^2$: the best condition on X is that there is no "hole" in X).



Can find f on X s.t. (*) holds

but $\oint f \cdot d\vec{s} \neq 0$.

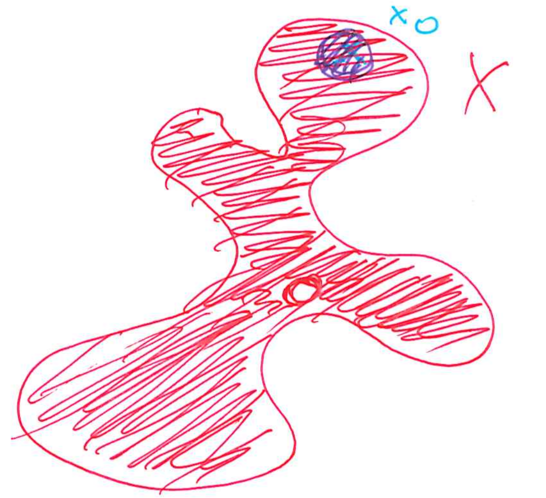
One can find extra conditions (in addition to (*)) that imply that f is conservative.

(2) X open

$\Longrightarrow$ for all $x_0 \in X$

we can find $r > 0$ s.t.

$B = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\} \subset X$



and since B is star-shaped there is

$g : B \longrightarrow \mathbb{R}$ s.t. $f(x) = \nabla g(x)$ for $x \in B$.

This local condition may be enough for applications.

<u>Remark</u>. $n = 3$ variables x, y, z

(*) $\longrightarrow$ 3 conditions (*)

$$\partial_y f_3 = \partial_z f_2$$

$$\partial_x f_3 = \partial_z f_1 , \qquad \partial_x f_2 = \partial_y f_1$$

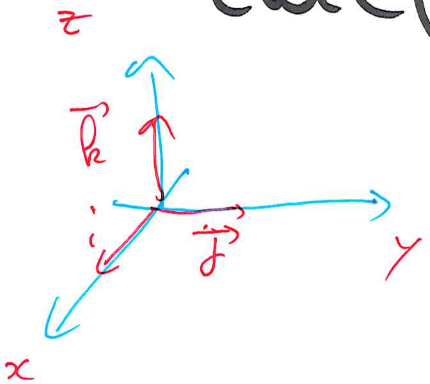
$$\Longleftrightarrow \quad \mathrm{curl}(f) = 0$$

where for any $f : \begin{matrix} X \longrightarrow \mathbb{R}^3 \\ \cap \\ \mathbb{R}^3 \end{matrix}$

$$\operatorname{curl}(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

Remembering this definition:

$$\operatorname{curl}(f) = \det \begin{array}{c} \begin{matrix} \vec{\imath} & \vec{\jmath} & \vec{k} \end{matrix} \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix} \end{array}$$



$$= \underbrace{e_1 \partial_y f_3}_{\partial_y f_3 \cdot e_1} + e_2 \partial_z f_1 + e_3 \partial_x f_2$$

$$- \underbrace{f_1 \partial_y e_3}_{(\partial_y f_1) \cdot e_3} - \ldots - \ldots$$

$$= \left(-\partial_y f_1 + \partial_x f_2\right) e_3 + \ldots$$

4.2 – (Riemann) integral in $\mathbb{R}^n$

Goals:

(1) ~~define~~ / understand / compute integrals in $\mathbb{R}^n$, $n \geq 2$

$\hookrightarrow$ compute volumes, areas, higher-dim. versions

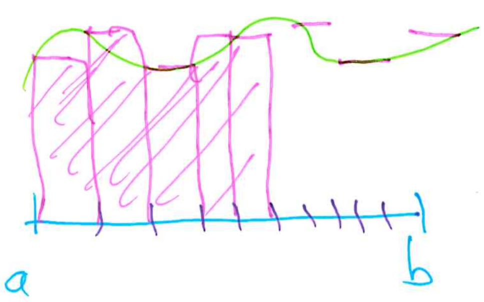
(2) study relations between certain integrals with n and $n-1$ variables (simplest ex.: partial integration)

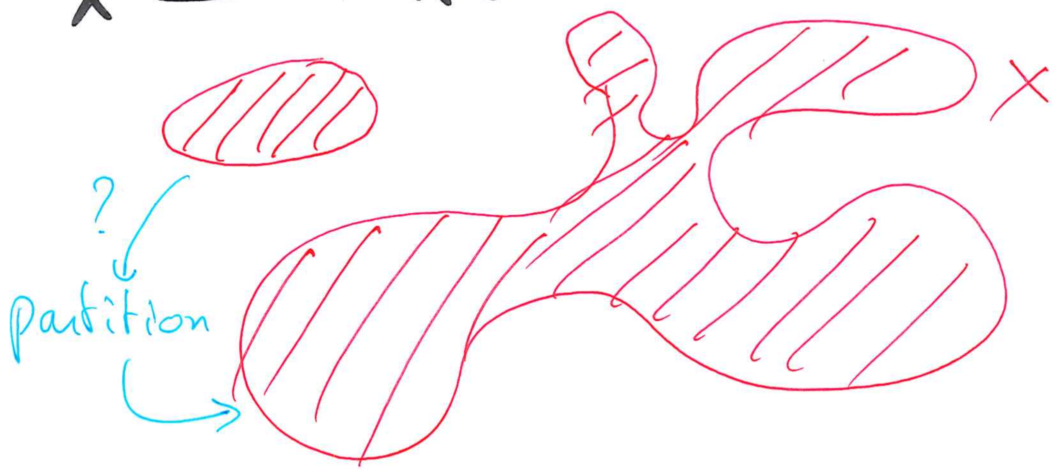
Why is it hard to define

$$\int_X f(x)\,dx$$

where $X \subset \mathbb{R}^n$ closed, bounded?

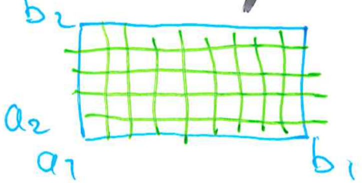
$f : X \longrightarrow \mathbb{R}$ continuous.





No obvious way to approximate X by smaller shapes if X is not

$$X = [a_1, b_1] \times \cdots \times [a_n, b_n]$$



Fact: we can define a real number

$$\int_X f(x_1, \dots, x_n)\, dx_1 \dots dx_n$$
$$\parallel$$
$$\int_X f(x)\, dx$$

("integral of f over X") for

- $X \subset \mathbb{R}^n$ closed, bounded ("compact")
- $f : X \longrightarrow \mathbb{R}$ continuous (then f is bounded on X)

satisfying conditions (1) to (6) below.

(1) (Compatibility)

For $n=1$, $X=[a,b]$,

$$\int_X f(x)\,dx = \int_a^b f(x)\,dx$$

where RHS is the Riemann integral of Analysis I

(2) (Linearity)

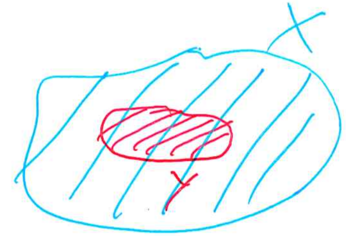
$$\int_X (af_1 + bf_2)\,dx = a\int_X f_1\,dx + b\int_X f_2\,dx$$

for all f_1, f_2 continuous, a, b in $\mathbb{R}$.

(3) (Positivity)

If $f \leq g$ on X then

$$\int_X f(x)\,dx \leq \int_X g(x)\,dx$$



and if $Y \subset X$ is compact

and $f \geq 0$ then

$$\int_Y f(x)\,dx \leq \int_X f(x)\,dx$$

Especially: if $f \geq 0$ then

$$\int_X f(x)\,dx \geq 0.$$

(4) (Triangle inequality)

$$\left| \int_X f(x)\,dx \right| \leq \int_X |f(x)|\,dx$$

more generally

$$\left| \int_X (f(x)+g(x))\,dx \right|$$

$$\leq \int_X |f(x)+g(x)|\,dx \leq \int |f(x)|\,dx + \int |g(x)|\,dx$$

(5) (Volume)

If $f = 1$, the integral $\int_X dx$ is the volume (in $\mathbb{R}^n$) of X.
[area if $n=2$]

↪ if $X = [a_1, b_1] \times \dots \times [a_n, b_n]$

Then $\int_X dx = (b_1 - a_1) \cdots (b_n - a_n)$
(for instance)

↪ in general, we define the volume to be the integral $\int_X dx$.

<u>Notation:</u> $\mathrm{Vol}(X)$ or $\mathrm{Vol}_n(X)$.

(6) (Fubini's Theorem or "integration by slices")

$n = n_1 + n_2, \quad n_1 \geq 1, \quad n_2 \geq 1$

$X_1 = \{x_1 \in \mathbb{R}^{n_1} \mid \exists x_2, (x_1, x_2) \in X\}$ (compact)

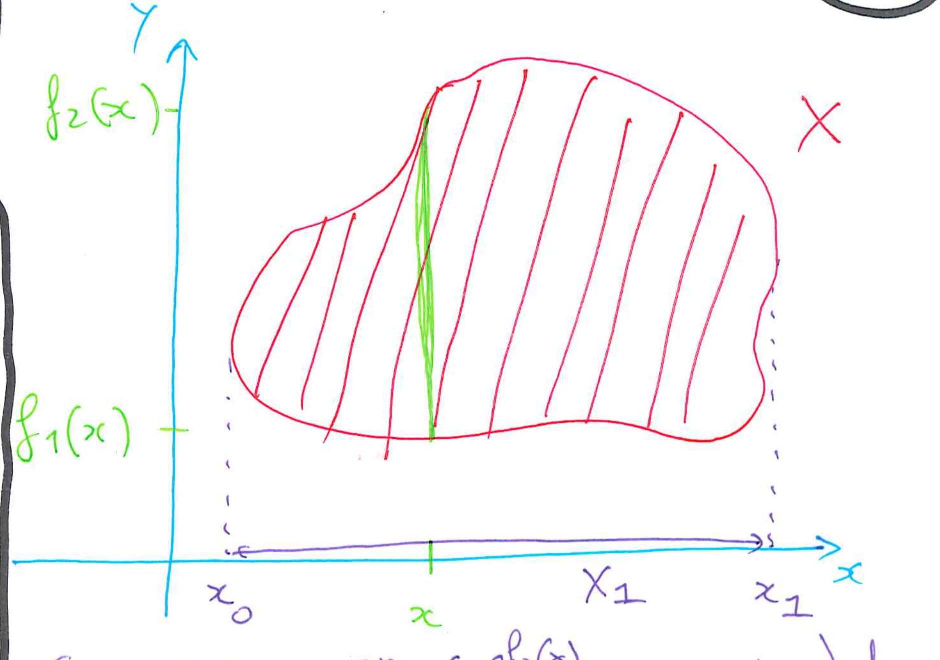
for $x_1 \in X_1$,

$Y_{x_1} = \{x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in X\}$ ("slices")

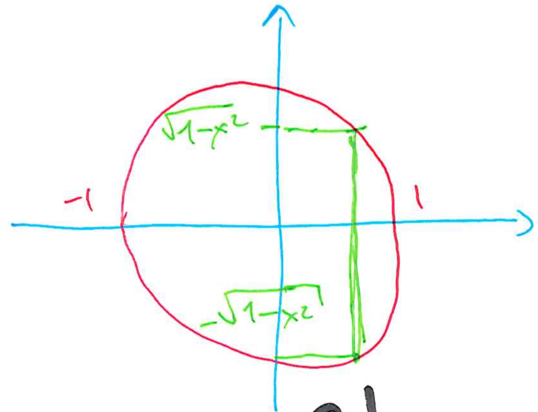
(compact)

<u>then</u>

$$\int_X f(\underbrace{x_1}_{\mathbb{R}^{n_1}}, \underbrace{x_2}_{\mathbb{R}^{n_2}})\, dx_1\, dx_2 = \int_{X_1} \overbrace{\left(\int_{Y_{x_1}} f(x_1, x_2)\, dx_2\right)}^{g(x_1)} dx_1$$



$$\int_X f(x,y)\, dx\, dy = \int_{x_0}^{x_1} \left(\int_{f_1(x)}^{f_2(x)} f(x,y)\, dy \right) dx$$



$$\text{area} = \int_{-1}^{1} 2\sqrt{1-x^2}\, dx = \pi$$

<u>if</u> $g : X_1 \longrightarrow \mathbb{R}$, $g(x_1) = \int_{Y_{x_1}} f(x_1, x_2)\, dx_2$

is <u>continuous</u>.

How to use this formula?

$$\int_{X_1} \left(\overbrace{\int_{Y_{x_1}} f(x_1, x_2)\, dx_2}^{g(x_1)} \right) dx_1 = \int_X f(x_1, x_2)\, dx_1 dx_2$$

- compute X_1
- compute Y_{x_1} for $x_1 \in X_1$
- compute $g(x_1)$
- check continuity ; if YES , done !

Summary:

6 conditions on $\int_X f(x)\,dx$

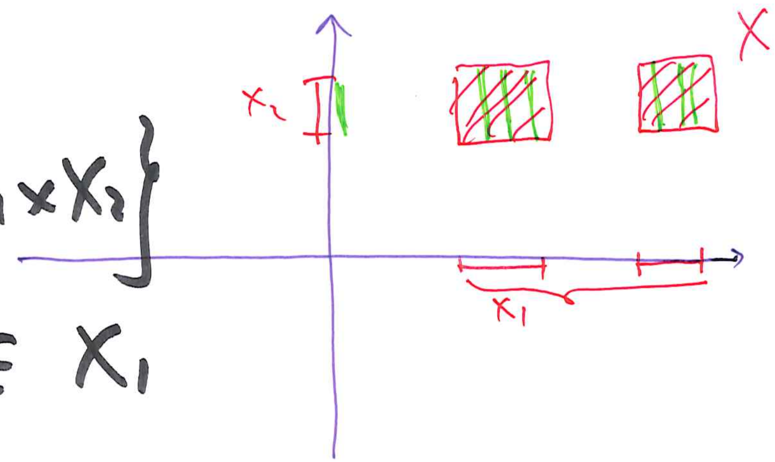
- one can show that these can only define one integral
- they <u>are</u> consistent (there exists an integral in this generality)
- in practice: use Fubini to reduce to integrals in lower dimensions
- in dimension 1, use Riemann integral

Examples:

(1) Important special case of Fubini's formula:

$$X = X_1 \times X_2 \,, \qquad X_1 \subset \mathbb{R}^{n_1}$$
$$X_2 \subset \mathbb{R}^{n_2}$$

the X_1 in Fubini



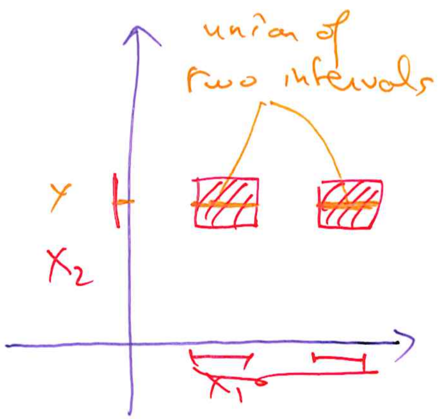
$$Y_{x_1} = \{ x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in X_1 \times X_2 \}$$
$$= X_2 \text{ for } \underline{\text{all}} \ x_1 \in X_1$$

(**Fact:** in that case,

$$g(x_1) = \int_{X_2} f(x_1, x_2)\, dx_2$$

is always continuous)

So:



$$\int_{X_1}\left(\int_{X_2} f(x_1,x_2)\,dx_2\right)dx_1$$

$$= \int_{X_1\times X_2} f(x_1,x_2)\,dx_1\,dx_2$$

$$= \int_{X_2}\left(\int_{X_1} f(x_1,x_2)\,dx_1\right)dx_2$$

for any f continuous on $X_1 \times X_2$.

(**Also** $\int_{X_1}\left(\int_{X_2}\left(\int_{X_3}\cdots\right)\right) = \int_{X_1\times X_2\times X_3}\cdots$)

Special case:

f has separated variables:

$$f(x_1, \ldots, x_n) = f_1(x_1) \cdot \ldots \cdot f_n(x_n)$$

f_i continuous on $X_i = [a_i, b_i]$

Then

$$\int_{[a_1,b_1]\times \cdots \times [a_n,b_n]} f \, dx = \int_{a_1}^{b_1} f_1 \times \int_{a_2}^{b_2} f_2 \times \cdots \times \int_{a_n}^{b_n} f_n$$

Ex. can compute integrals of polynomials over $[a_1, b_1] \times \cdots \times [a_n, b_n]$.

Ex.

$$\int_{[a_1,b_1]\times[a_2,b_2]} f_1(x_1)\,f_2(x_2)\;dx_1\,dx_2$$

$$\overset{\text{Fubini}}{=} \int_{a_1}^{b_1}\underbrace{\left(\int_{a_2}^{b_2} \underbrace{f_1(x_1)}_{\text{constant for fixed } x_1} f_2(x_2)\,dx_2\right)}_{f_1(x_1)\,\underbrace{\int_{a_2}^{b_2} f_2(x_2)\,dx_2}_{\text{constant}}} dx_1$$

$$= \left(\int_{a_2}^{b_2} f_2(x_2)\,dx_2\right)\cdot\int_{a_1}^{b_1} f_1(x_1)\,dx_1.$$

Example:

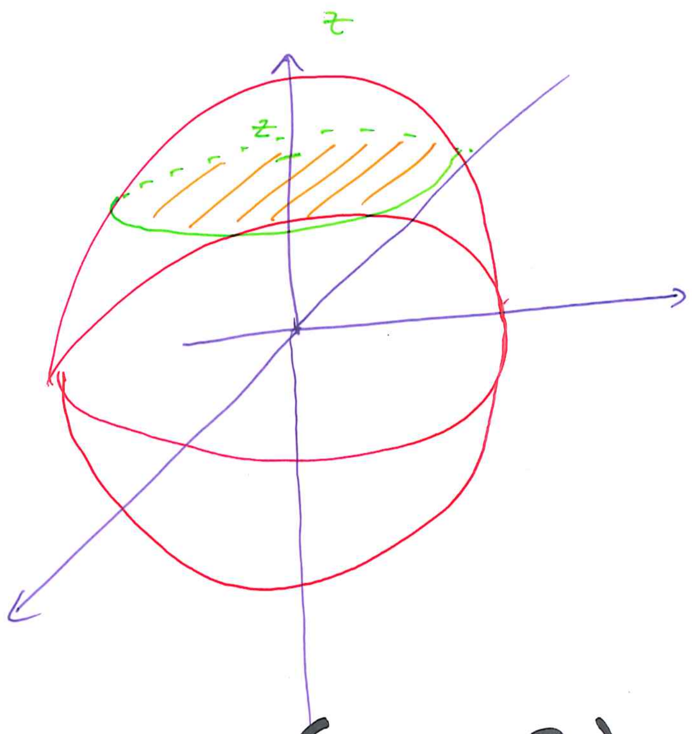
$$V = \mathrm{Vol}_3(\underbrace{\text{unit ball}}_{X = \{(x,y,z) \mid \sqrt{x^2+y^2+z^2} \leq 1\}})$$

$$V = \int_X dx$$

(Use z as variable)

$X_1 = [-1, 1]$

Y_z = disc of radius $\sqrt{1-z^2}$



so $V = \int_{-1}^{1} \pi(1-z^2)\,dz = \pi\left(2 - \frac{2}{3}\right) = \frac{4\pi}{3}$