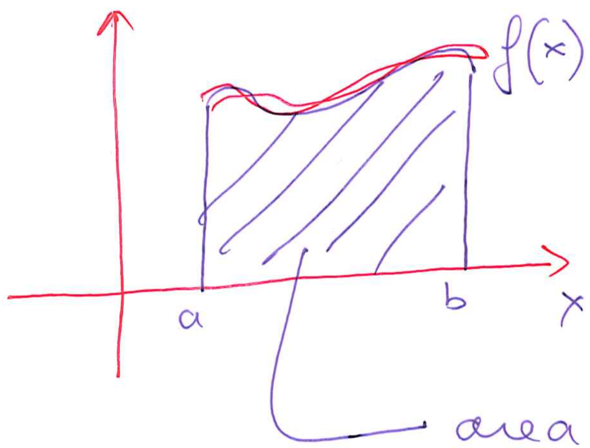
Ex. Suppose $f \geq 0$

$$\int_X f(x)\,dx = \mathrm{Vol}(Y)$$

where $Y = \{ (x,y) \in X \times \mathbb{R} \mid 0 \leq y \leq f(x) \} \subset \mathbb{R}^{n+1}$



$$\int_a^b f(x)\,dx$$

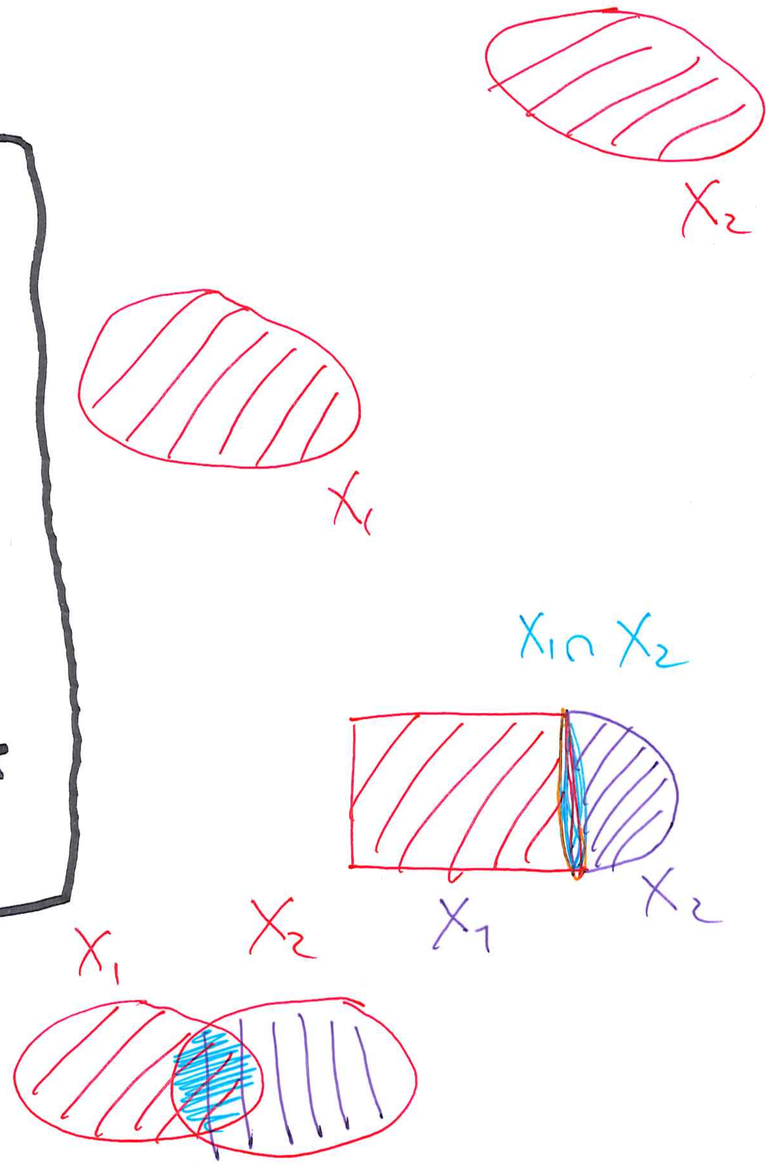
Proof:

$$\mathrm{Vol}(Y) = \int_Y dx\,dy = \int_X \underbrace{\left(\int_0^{f(x)} 1 \cdot dy \right)}_{f(x)} dx$$

Domain additivity (7)

$X = X_1 \cup X_2$

$$\boxed{\int_X f(x)\,dx + \int_{X_1 \cap X_2} f(x)\,dx = \int_{X_1} f(x)\,dx + \int_{X_2} f(x)\,dx}$$



Ex. nb. of elts in union of two finite sets

<u>Ex.</u> if $X_1 \cap X_2 = \emptyset$ then

$$\int_{X_1 \cup X_2} f(x)\,dx = \int_{X_1} f(x)\,dx + \int_{X_2} f(x)\,dx$$

<u>More generally</u>:

<u>Def.</u> (4.2.3)

(1) $1 \leq m \leq n$; a parameterized <u>m-set</u> is a continuous

$$f : [a_1, b_1] \times \dots \times [a_m, b_m] \longrightarrow \mathbb{R}^n$$

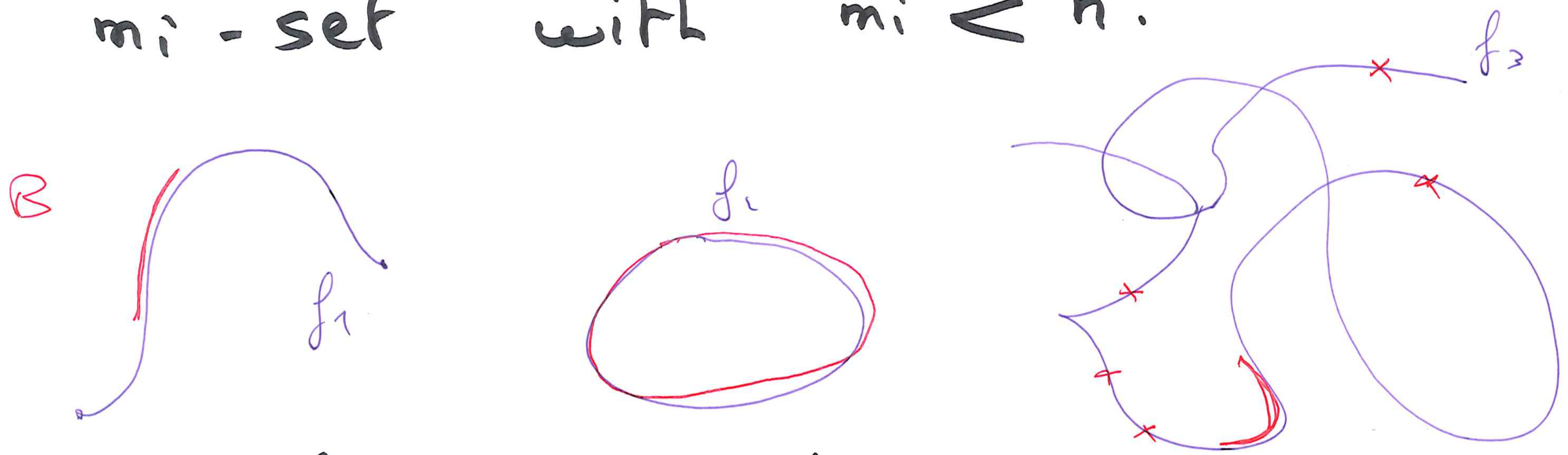
<u>C^1</u> on $]a_1, b_1[\times \dots \times]a_m, b_m[$

<u>Ex.</u> $m = 1 \iff$ parameterized curve

(2) $B \subset \mathbb{R}^n$ is <u>negligible</u> if

$$B \subset f_1(X_1) \cup \dots \cup f_k(X_k)$$

where $f_i : X_i \longrightarrow \mathbb{R}^n$ is an m_i-set with $m_i < n$.



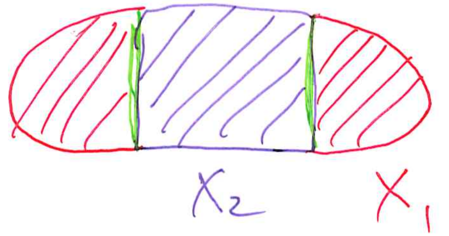
<u>Intuitively</u>: a subset of a "lower-dimensional" set

<u>Property:</u>

If $X \subset \mathbb{R}^n$ is negligible,
then for any $f : X \longrightarrow \mathbb{R}$ continuous,
we have
$$\int_X f(x)\,dx = 0.$$
(Esp. the volume is 0).

<u>Consequence:</u> if $X_1 \cap X_2$ is negligible
then
$$\int_{X_1 \cup X_2} f(x)\,dx = \int_{X_1} f(x)\,dx + \int_{X_2} f(x)\,dx$$

<u>Ex.</u>



(1)

$X_1 \cap X_2 =$ is negligible

(2) Any $\underbrace{B}_{\text{(bounded)}} \subset \mathbb{R}^n$ contained in a affine subspace $H \subset \mathbb{R}^n$ of dimension $< n$ is negligible

(3) The image of any parameterized curve in $\mathbb{R}^2$ is negligible.

We often use subdivisions of X to use specific adapted coordinates to simplify the integral.

4.3. Improper integrals

Recall (Analysis I)

$$\int_a^b f(x)\,dx \rightsquigarrow \int_a^{+\infty} f(x)\,dx$$

$$= \lim_{R\to+\infty} \int_a^R f(x)\,dx$$

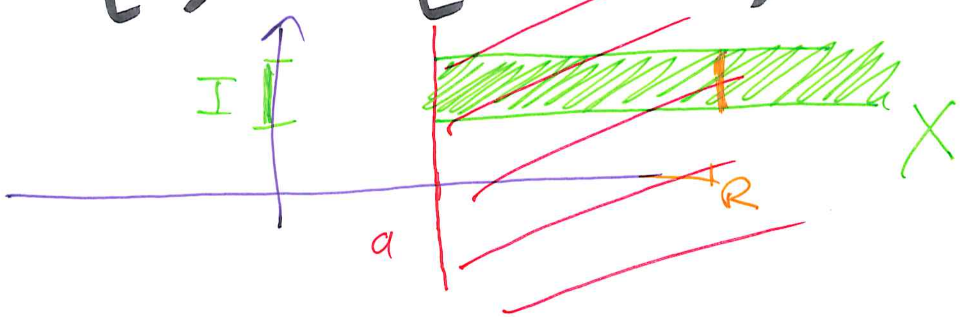
Ex.

$$\int f(x,y)\,dx\,dy \overset{?}{=} \lim_{R\to\infty}\int f$$

$$\overset{?}{=} \lim \int f$$

We use specific definitions, with $\underline{n=2}$.

① $X=[a,+\infty[\times I$, I bounded closed



We say that

$$\int_X f$$

converges if $\lim_{R \to \infty} \int_{[a,R] \times I} f$ exists,

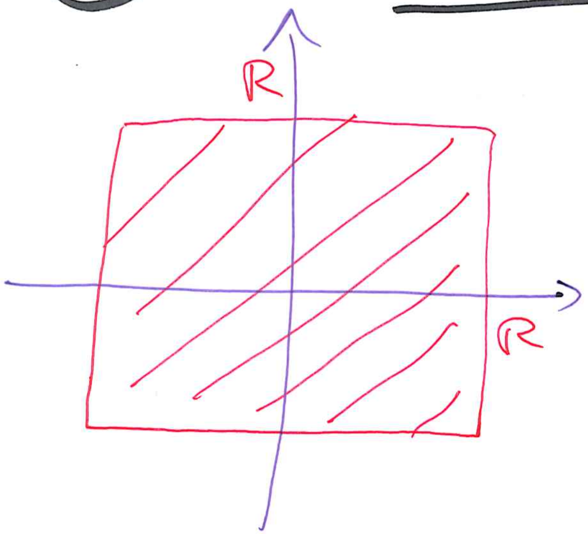
and we say that this limit is the integral.

If $f \geq 0$ then this integral is

$$\underbrace{\int_a^{+\infty}}_{\text{ordinary improper integral}} \left(\int_I f(x,y)\, dy \right) dx = \int_I \underbrace{\left(\int_a^{+\infty} f(x,y)\, dx \right)}_{\text{ordinary improper integral}} dy$$

(Not always true!)

(2) $\underline{X = \mathbb{R}^2}$: if $f \geq 0$ then

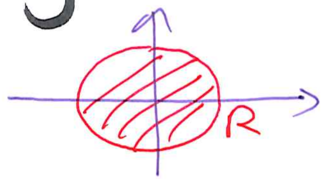


we say that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$
is integrable on $\mathbb{R}^2$
if

$$\lim_{R \to \infty} \int_{[-R,R]^2} f \quad \text{exists.}$$

One can show that then

$$\int_{\mathbb{R}^2} f = \lim_{R \to \infty} \int f$$



Ex. 4.3.2 (1)

$$\int_{[0,+\infty[\times[1,2]} x\, e^{-xy}\, dx\, dy = ?$$

$$= \lim_{R\to\infty} \int_0^R \underbrace{\left(\int_1^2 x\, e^{-xy}\, dy \right)}_{\cancel{x}\left[-\frac{1}{\cancel{x}} e^{xy} \right]_{y=1}^{y=2} = \left[-e^{-xy} \right]_{y=1}^{y=2} = -e^{-2x} + e^{-x}} dx$$

$$= \lim_{R\to\infty} \int_0^R \left(e^{-x} - e^{-2x} \right) dx = 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

4.4 - Change of variable

Recall:

$n=1$ (Analysis I)

$$\int f(g(x))\,g'(x)\,dx = \int f(y)\,dy$$

Goal: version with $n \geq 2$ variables

Change of variable

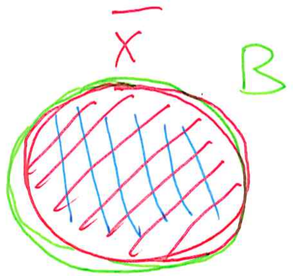
$\bar{X} \subset \mathbb{R}^n$, $\bar{Y} \subset \mathbb{R}^n$.

compact

$\varphi : \bar{X} \longrightarrow \bar{Y}$ continuous

$\bar{X} = X \cup B$, $\bar{Y} = Y \cup B'$

open negligible



X = open disc
B = circle

Assume $\varphi : X \longrightarrow Y$ is C^1, bijective

Th. (4.4.2)

Then
$$\int_{\overline{X}} f(\varphi(x))\,|\det J_\varphi(x)|\,dx = \int_{\overline{Y}} f(y)\,dy$$

Note: (1) φ bijective $\Rightarrow$ $\det J_\varphi(x) \neq 0 \;\forall$. on X

(2) if X is pathwise connected then $\det J_\varphi(x)$ has the same sign everywhere

(3) Mnemonic $y = \varphi(x)$
$\Rightarrow$ " $dy = |\det J_\varphi(x)|\,dx$ "

(4) $\det J_\varphi(x)$ on boundary is obtained by "using the same formula".

Ex. (4.4.3 (3)) $n = 2$

$$f(x,y) = e^{-(x^2+y^2)}$$

$\bar{X}$ = disc centered at O with radius $R > 0$

$$= \{(x,y) \mid x^2+y^2 \le R^2\}$$

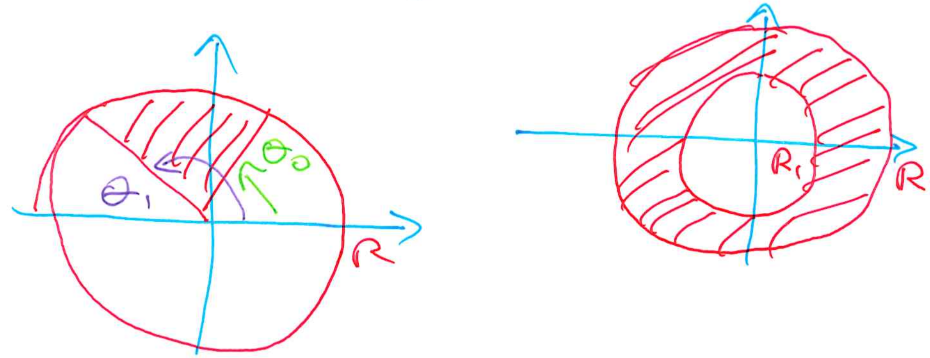
$$\bar{X} = X \cup B$$

X = open disc $= \{(x,y) \mid x^2+y^2 < R^2\}$

B = circle $= \{(x,y) \mid x^2+y^2 = R^2\}$ (negligible)

Integral over a disc: we think of polar coordinates!

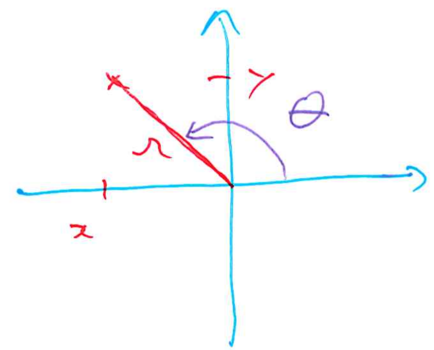
(same with sectors, annulus...



)

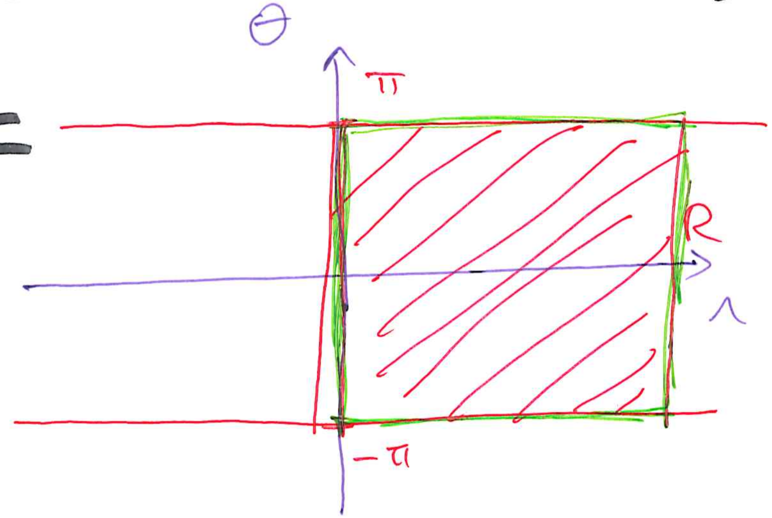
$$\varphi(r, \theta) = (r\cos\theta, r\sin\theta)$$

$r \in [0, R]$, $\theta \in [-\pi, \pi]$



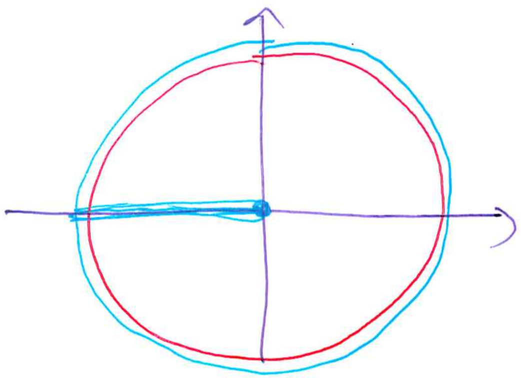
$\overline{Y} = [0, R] \times [-\pi, \pi]$

$Y =]0, R[\times]-\pi, \pi[$, φ C^1

$B' =$ 

four segments so negligible

$\varphi(Y) =$ open disc <u>minus</u>
the origin $(r = 0)$
the segment $]-1, 0[$ $(\theta = \pm\pi)$



The closed disc is
$\varphi(Y) \cup$ (negligible).

So we can apply the formula:

$$\int_{\overline{X}} f(y)\,dy = \int_{\overline{Y}} f(\varphi(x))\,|\det J_\varphi(x)|\,dx$$

Here

$$J_\varphi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

so $\det J_\varphi(r,\theta) = r > 0$ on Y.

<u>Note</u>: X and Y are interchanged wrt theory!

<u>So:</u>

$$\int_{\overline{X}} e^{-(x^2+y^2)}\,dx\,dy = \int_{\overline{Y}} e^{-r^2}\,r\,dr\,d\theta$$

(More generally

$$\int_{\substack{\text{disc of}\\ \text{radius } R}} f(x,y)\,dx\,dy = \int_0^R \int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta)\, r \; dr\,d\theta$$

for any f)

so

$$\int_{\bar{X}} e^{-(x^2+y^2)}\,dx\,dy = \left(\int_0^R r e^{-r^2}\,dr\right)\overbrace{\left(\int_{-\pi}^{\pi} d\theta\right)}^{2\pi}$$

$$= \left[-\frac{1}{2}e^{-r^2}\right]_0^R \times 2\pi$$

$$= \pi\left(1-e^{-R^2}\right)$$

Consequence

$$\lim_{R\to\infty} \int_{\substack{\text{disc}\\ \text{of radius}\\ R}} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \lim_{R\to\infty} \pi\left(1-e^{-R^2}\right) = \pi$$

So the integral

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy$$

exists and is $= \pi$.

Now compute

$$\int_{[-R,R]^2} \underbrace{e^{-(x^2+y^2)}}_{e^{-x^2}e^{-y^2}}\,dx\,dy$$

$$= \left(\int_{-R}^{R} e^{-x^2}dx\right)\cdot\left(\int_{-R}^{R} e^{-y^2}dy\right) = \left(\int_{-R}^{R} e^{-x^2}dx\right)^2$$

This converges to

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2$$

(one-variable improper integral).

Comparing, we get

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

("gaussian integral")