Recall $n \geq 2$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$f(x) = (f_1(x), \ldots, f_m(x))$$

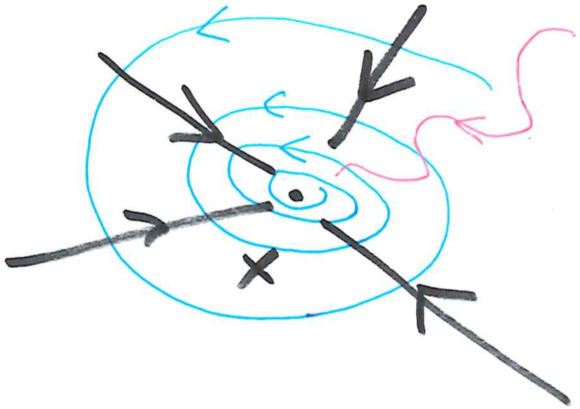
Continuity: f continuous

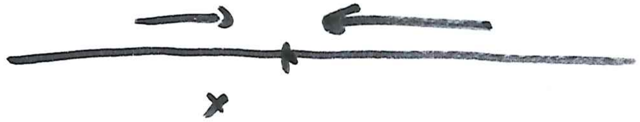
$$\Updownarrow$$

if $x \in \mathbb{R}^{n}$, $(x_k) \longrightarrow x$ in $\mathbb{R}^n$

$\Rightarrow (f(x_k)) \longrightarrow f(x)$ in $\mathbb{R}^m$

Difference with $n = 1$:



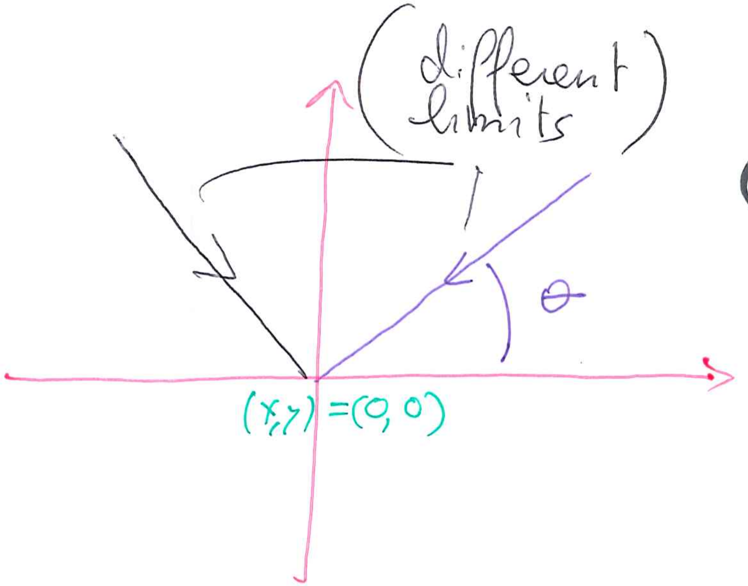


Different ways to converge to x might lead to different limits, or no limits.

<u>Ex.</u> (3.2.6) $n = 2, \quad m = 1$

$$f(x,y) = \frac{xy}{x^2+y^2}, \quad f(0,0) = 0$$

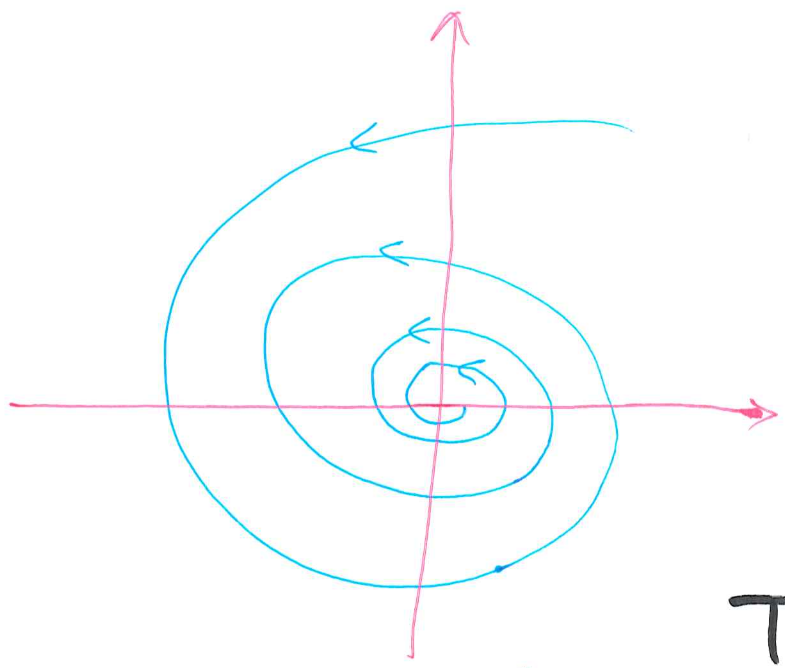
$(x,y) = (0,0)$ [on $\mathbb{R}^2 - \{(0,0)\}$, f is continuous]



$$(0,0) \longleftarrow (x_k, y_k) = \left(\frac{\cos\theta}{k}, \frac{\sin\theta}{k} \right)$$

$$f(x_k, y_k) = \frac{(\cos\theta \sin\theta)/\cancel{k^2}}{\frac{\cos^2\theta + \sin^2\theta}{\cancel{k^2}}}$$

$$= \cos\theta \sin\theta$$



$$(x_k, y_k) = \left(\frac{\cos k}{k}, \frac{\sin k}{k} \right) \underset{}{\downarrow} (0,0)$$

$$f(x_k, y_k) = \cos k \sin k = \frac{1}{2} \sin(2k)$$

The set of values of $f(x_k, y_k)$ can be arbitrarily close to any $t \in [-\frac{1}{2}, \frac{1}{2}]$. <u>No limit</u> as $k \to +\infty$!

f is <u>not</u> continuous at $(0,0)$.

<u>Recall:</u> $f : [a,b] \longrightarrow \mathbb{R}$ continuous

$\Longrightarrow$ the maximum and minimum of f are achieved.

<u>Analogue in $\mathbb{R}^n$</u>

<u>Th. 3.2.11.</u> $X \subset \mathbb{R}^n$ <u>compact</u>

$f : X \longrightarrow \mathbb{R}$ <u>continuous</u>

$\Rightarrow$ there is $x_+ \in X$, $x_- \in X$

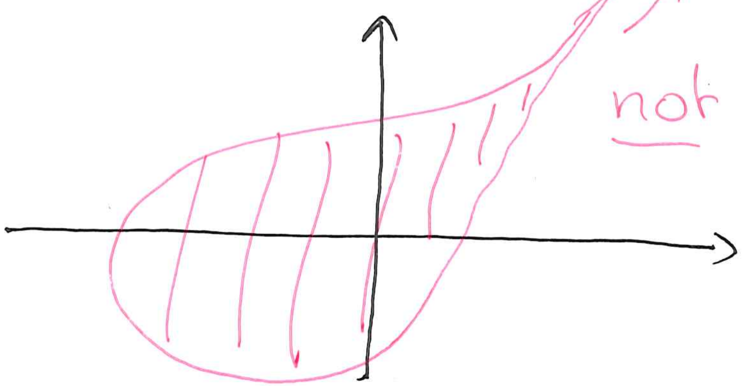
s.t. $f(x_+) = \sup_{x \in X} f(x)$, $f(x_-) = \inf_{x \in X} f(x)$

Compact: bounded + closed

$\|x\|$ is bounded for $x \in X$

Ex. $]0,1[^n$ bounded

$\mathbb{R}^n$ not bounded



Def. $X \subset \mathbb{R}^n$ is closed if:

if $x_k \in X$, $k \geq 1$,

and $x_k \longrightarrow x$,

then $x \in X$.

Ex. $\mathbb{R}^n$ closed

$]0,1[^n$ not closed

$x_k = \left(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k}\right) \qquad k \geq 2$

$x_k \in \,]0,1[^n \quad ; \quad x_k \longrightarrow (0, \dots, 0)$

but $(0, \dots, 0) \notin \,]0,1[^n$

(so $]0,1[^n$ <u>not</u> closed)

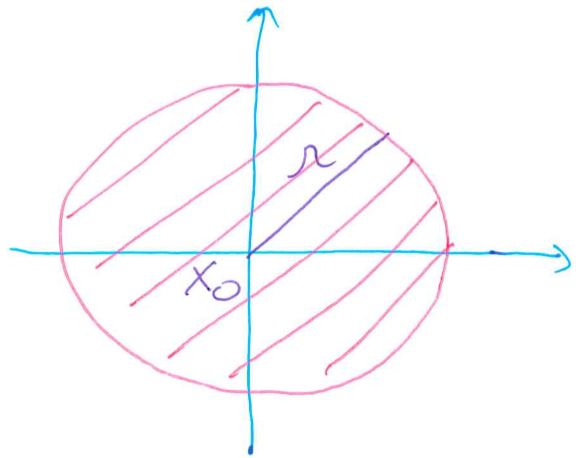
<u>Ex.</u> (3.2.10) $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ continuous

$$\mathbb{R} \supset [a,b]$$

$\{x \in \mathbb{R}^n \mid f(x) \in [a,b]\}$ is closed in $\mathbb{R}^n$

For instance: $x_0 \in \mathbb{R}^n$, $r \geq 0$

$$D = \{ x \in \mathbb{R}^n \mid \|x - x_0\| \leq r \}$$



$$= \{ x \in \mathbb{R} \mid f(x) \in [0, r] \}$$

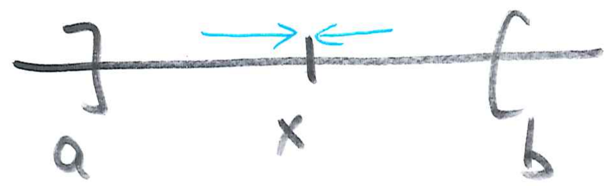
where $f(x) = \|x - x_0\|$

f continuous $\Longrightarrow$ D is <u>closed</u>.

Here D is also bounded, so <u>compact</u>.

Next: differential calculus on $\mathbb{R}^n$

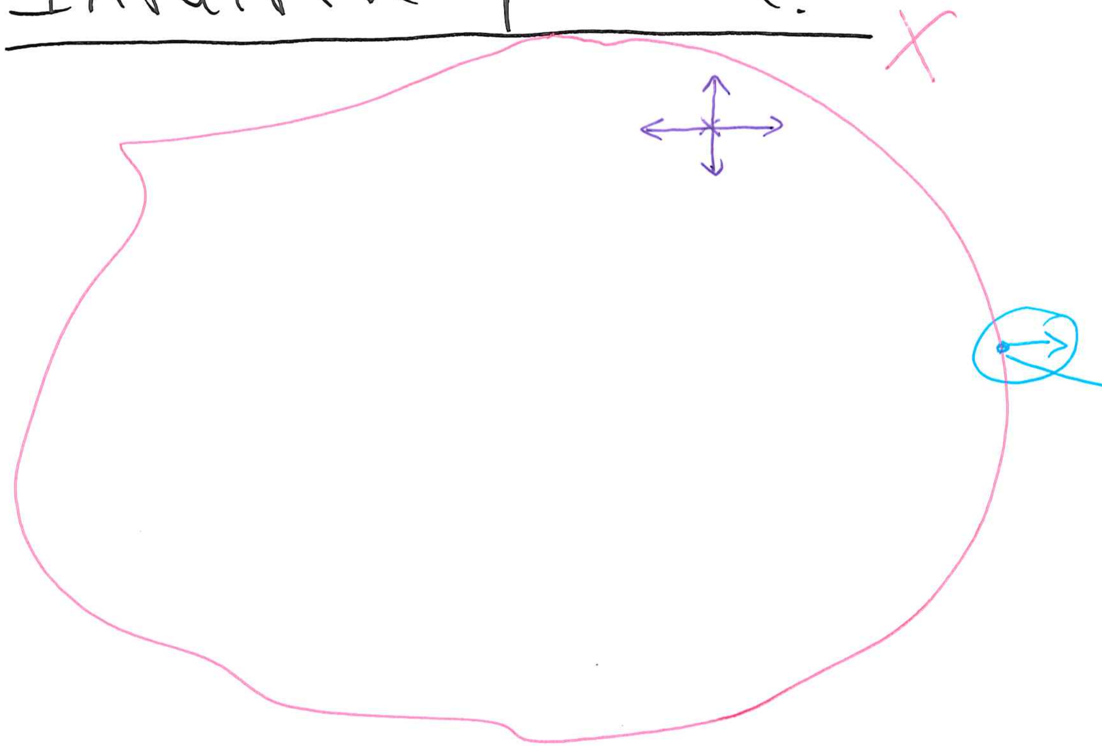
3_ Partial derivatives



Recall: derivative defined for $f : I \to \mathbb{R}$ where I is open.

Def. (3.3.1) $X \subset \mathbb{R}^n$ is **open** if for any $x \in X$, there is some $\delta > 0$ so that any $y \in \mathbb{R}^n$ with $|y_i - x_i| < \delta$ (for all i) is in X.

<u>Intuitive picture:</u>



cannot be in X if X is open

"If we move a little bit away from x, we stay inside X"

<u>Warning!</u> A set might not be open <u>and</u> not closed!
$([0,1[)$

Examples

(1) $\mathbb{R}^n$ is open (and closed) (<u>not</u> compact)

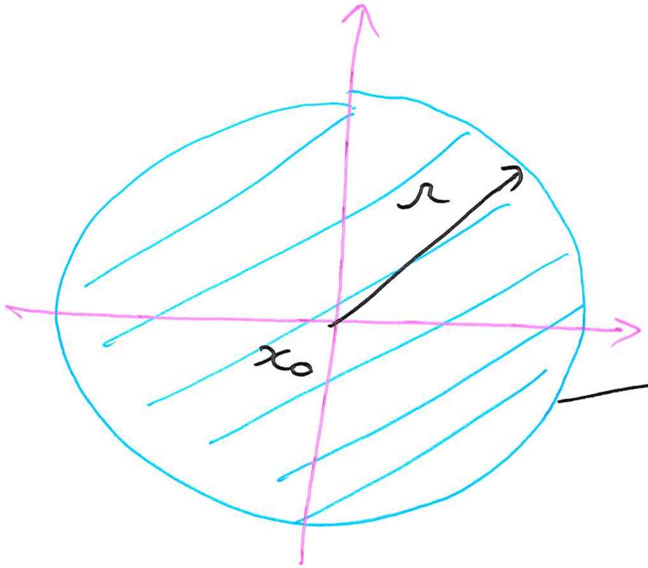
(2) (3.3.2) $X \subset \mathbb{R}^n$ is <u>open</u>

$$\Updownarrow$$

$\{x \in \mathbb{R}^n \mid x \notin X\}$ is <u>closed</u>

(3) (3.3.3) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous <u>and</u> $I \subset \mathbb{R}$ open (interval) then $\{x \in \mathbb{R}^n \mid f(x) \in I\}$ is <u>open</u>.

For example $x_0 \in \mathbb{R}^n$, $r > 0$



$$E = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$$

$$= \{x \in \mathbb{R}^n \mid f(x) \in]-r, r[\}$$

boundary not included

where $f(x) = \|x - x_0\|$ is continuous

<u>Or:</u> $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \underset{(x_1 \in]0, +\infty[)}{x_1 > 0}\}$ is open

Def. (3.3.5)

$X \subset \mathbb{R}^n$ open

$f : X \longrightarrow \mathbb{R}^m$

$1 \leq i \leq n$

f has a partial derivative with respect to x_i (on X)

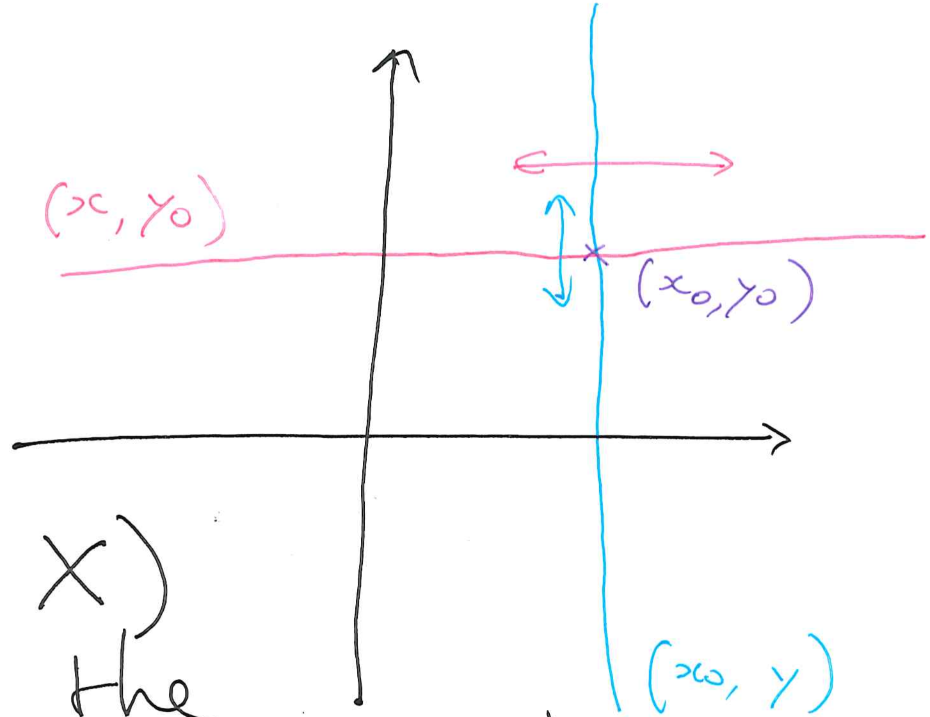
if: for all $x_0 \in X$, the function $g(t) = f(x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots)$

defined on $I = \{t \in \mathbb{R} \mid (x_{0,1}, \dots, t, x_{0,i+1}, \dots)$ is in $X\}$

is differentiable at $t = x_{0,i}$.

($I \subset \mathbb{R}$ contains an open interval containing $x_{0,i}$)

single variable



Notation:

\partial

$$g'(x_{0,i}) = \begin{cases} \dfrac{\partial f}{\partial x_i}(x_0) \\ \partial_{x_i} f(x_0) \\ \partial_i f(x_0) \end{cases}$$

fixed point in X

variable

Example $n = 2$, $m = 1$

$$f(x,y) = 12 \exp(\cos(x^2 y))$$

f has partial derivatives on $\mathbb{R}^2$ with respect to x and y

$$\frac{\partial f}{\partial x} = 12 \exp(\cos(x^2 y))(2xy)(-\sin(x^2 y))$$

[y considered fixed]

$$\frac{\partial f}{\partial y} = 12 \exp(\cos(x^2 y))\, x^2 (-\sin(x^2 y))$$

Recall: $f : \mathbb{R} \longrightarrow \mathbb{R}^m$

$$f(t) = (f_1(t), \dots, f_m(t))$$

is differentiable $\iff$ each f_i is and

$$f' = (f_1', \dots, f_m')$$

so

$$f : X \longrightarrow \mathbb{R}^m$$

$$f(x) = (f_1(x), \ldots, f_m(x))$$

has partial derivative w.r.t. x_1

$$\Updownarrow$$

each f_i does, and

$$\frac{\partial f}{\partial x_1} = \left(\frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_m}{\partial x_1} \right)$$

Def. (3.3.9)

$X \subset \mathbb{R}^n$ open

$f : X \longrightarrow \mathbb{R}^m$, $f = (f_1, \dots, f_m)$

with all partial derivatives at x_0

The matrix with m rows, n columns

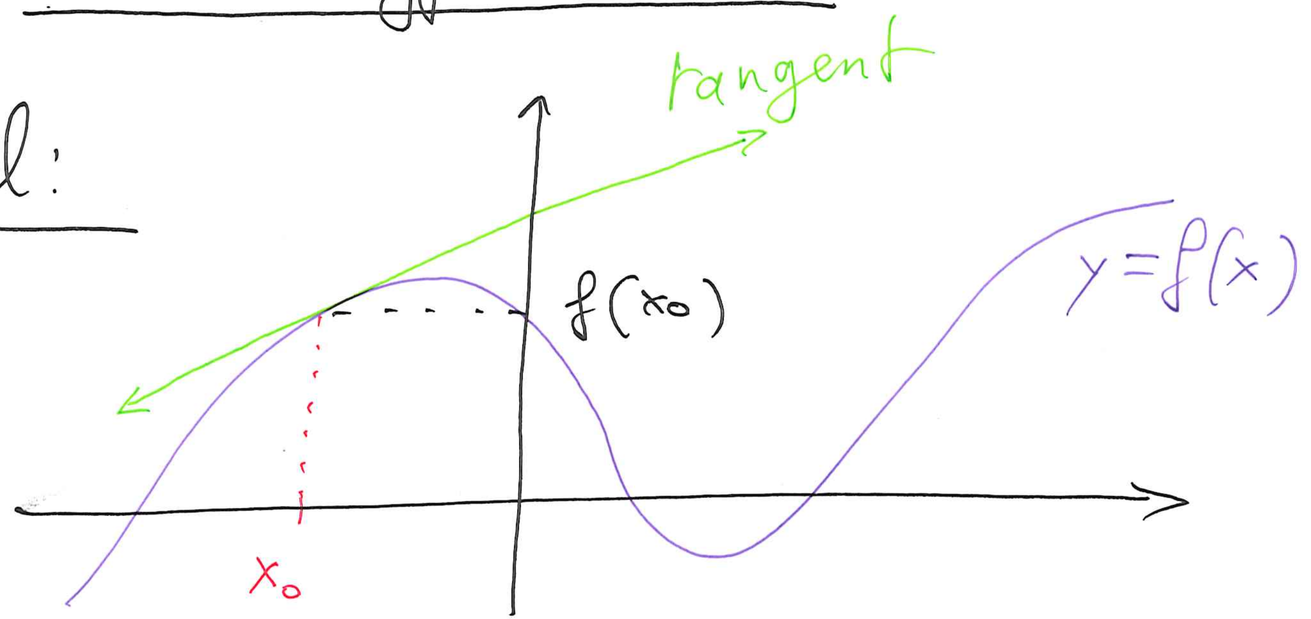
$$J_f(x_0) = \left(\frac{\partial f_i}{\partial x_j}(x_0) \right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

is called the Jacobian matrix of f at x_0

4 – The differential

$n = 1$

Recall:



f differentiable at x_0 means that

$$g(x) = \underset{\substack{\shortparallel \\ f(x_0)}}{y_0} + f'(x_0)(x - x_0)$$

is a very good approximation to $f(x)$ for x close to x_0.

Precisely: the error

$$E(x) = f(x) - \left(f(x_0) + f'(x_0)(x - x_0) \right)$$

is "much smaller" than $|x - x_0|$ when $x \longrightarrow x_0$. $\left[\lim_{x \to x_0} \frac{E(x)}{|x - x_0|} = 0. \right]$

Ex. (3.4.1 (2))

$n = 2, \quad m = 1$

$$g(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \quad \text{if } (x,y) \neq (0,0)$$

$$g(0,0) = 0$$

Facts:

1. g is continuous on $\mathbb{R}^2$
2. g has partial derivatives on $\mathbb{R}^2$
3. $\frac{\partial g}{\partial x}(0,0) = \frac{\partial g}{\partial y}(0,0) = 0$

$$\frac{xy}{\sqrt{x^2+y^2}}$$

1 g is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$

As $(x,y) \longrightarrow (0,0)$

$$|g(x,y)| = \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \frac{\frac{1}{2}(x^2+y^2)}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2}$$

$$= \frac{1}{2}\|(x,y)\|$$

so $\lim_{(x,y)\to(0,0)} g(x,y) = 0$

$$\frac{xy}{\sqrt{x^2+y^2}}$$

2 + 3 g has partials outside $(0,0)$

At $(0,0)$:

$$g(x,0) = \frac{0}{\sqrt{x^2}} = 0$$

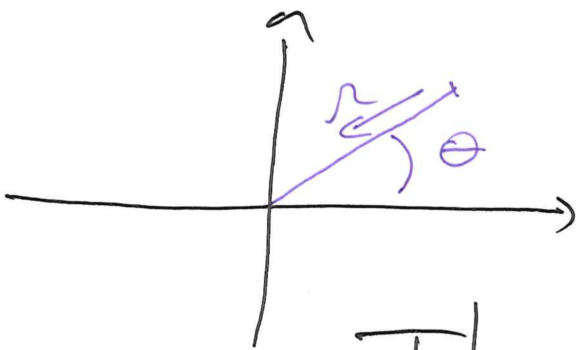
$\Rightarrow \frac{\partial g}{\partial x}(0)$ exists and is $= 0$

$$g(0,y) = \frac{0}{\sqrt{y^2}} = 0$$

$$\Rightarrow \frac{\partial g}{\partial y}(0) = 0.$$

Point: ~~the~~ any approximation to $g(x,y)$ constructed using $\frac{\partial g}{\partial x}(0)$, $\frac{\partial g}{\partial y}(0)$ will be zero, whereas

$$g(r\cos\theta, r\sin\theta) \qquad (r \to 0)$$



$$= \frac{r^2 \cos\theta \sin\theta}{r}$$

$$= r \cos\theta \sin\theta$$

The 0 function is <u>not</u> a very good approximation!

<u>Def.</u> (3.4.2)

$X \subset \mathbb{R}^n$, $f : X \longrightarrow \mathbb{R}^m$

$u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ <u>linear</u>

$x_0 \in X$

f is differentiable at x_0 with differential $df(x_0) = u$ if

$$0 = \lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{1}{\|x - x_0\|} \underbrace{\left(f(x) - f(x_0) - u(\underbrace{x - x_0}_{\in \mathbb{R}^n}) \right)}_{\in \mathbb{R}^m}$$