<u>Examples.</u> (4.4.3)

(1) $\varphi(x) = x + x_0$ (translation) $x_0 \in \mathbb{R}^n$

$J_\varphi(x) = 1_n$ at all $x$

$$\Longrightarrow \int_{\overline{X}} f(x + x_0)\, dx = \int_{\overline{X} + x_0} f(x)\, dx$$

In particular:

$$\mathrm{Vol}(\overline{X}) = \mathrm{Vol}(\overline{X} + x_0)$$

("volume is invariant by translation")

(2) $\varphi(x) = Ax$, $A$ fixed invertible matrix

$J_\varphi(x) = A$ constant

$|\det J_\varphi(x)| = |\det(A)|$

so

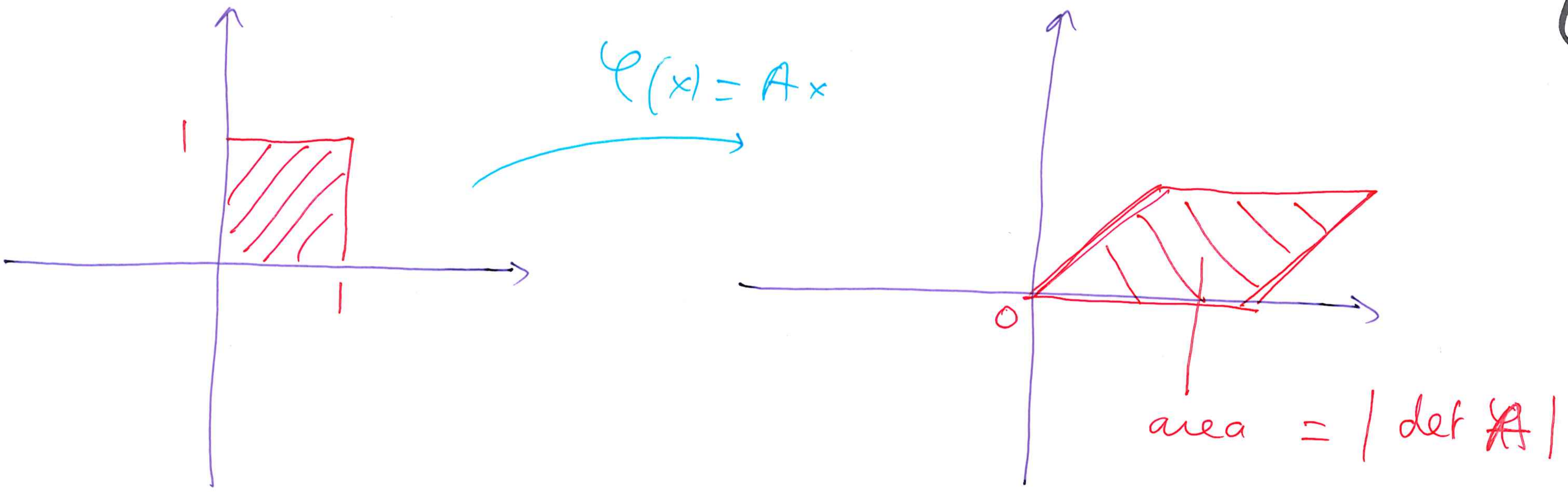
$$\int_{\overline{X}} f(Ax)\,dx = \frac{1}{|\det(A)|} \int_{A\overline{X}} f(y)\,dy$$

For $f = 1$:

$$\mathrm{Vol}(\overline{X}) = \frac{1}{|\det(A)|} \mathrm{Vol}(A\overline{X})$$

$$\mathrm{Vol}(A\overline{X}) = |\det(A)|\,\mathrm{Vol}(\overline{X})$$

$\overline{X} = [0,1]^n$: $\mathrm{Vol}(A[0,1]^n) = |\det(A)|$




(E.g.

$$A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, \quad A[0,1]^n = \begin{pmatrix} [0, a_1] \\ \vdots \\ [0, a_n] \end{pmatrix}$$

$(a_i \neq 0)$

$$\mathrm{Vol}(\text{—}) = |a_1| \cdots |a_n| = |\det A|$$

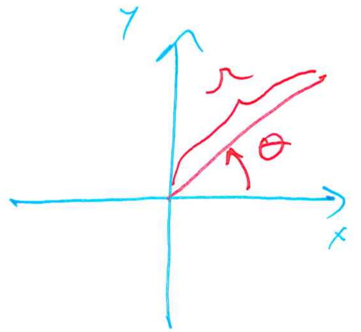
)

<u>N.B.</u> The sign of the determinant tells you if $\varphi(x) = Ax$ preserves orientation or not.

# Standard change of variables

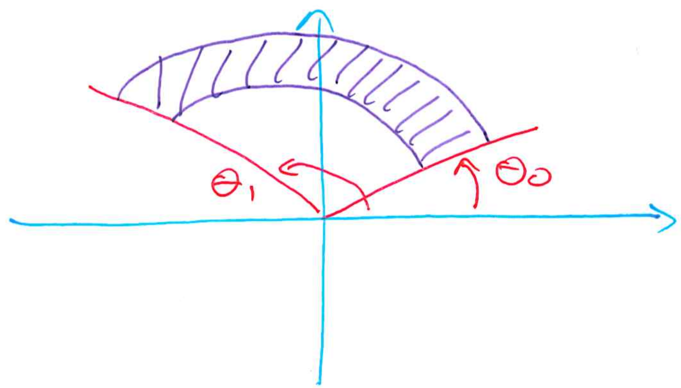
## (1) Polar coordinates in $\mathbb{R}^2$

$$\varphi(r,\theta) = (r\cos\theta, r\sin\theta)$$




$$\int_{\overline{X}} f(x,y)\,dx\,dy = \int_{\text{"}\overline{X}\text{ in polar coord"}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Useful if $\overline{X}$ "is a rectangle" in terms of $r, \theta$



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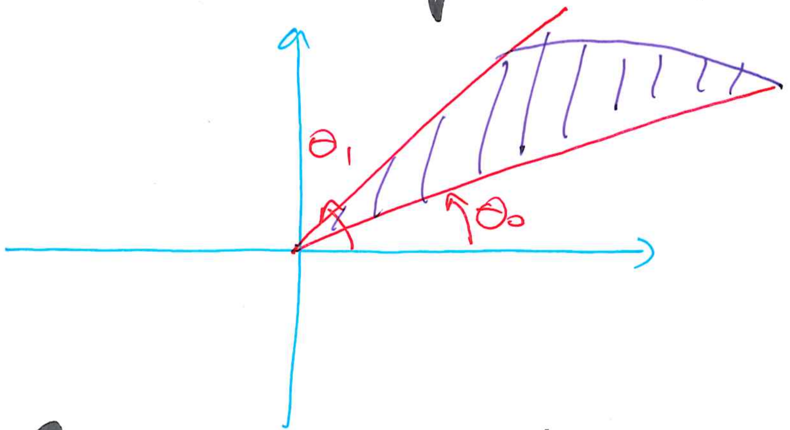
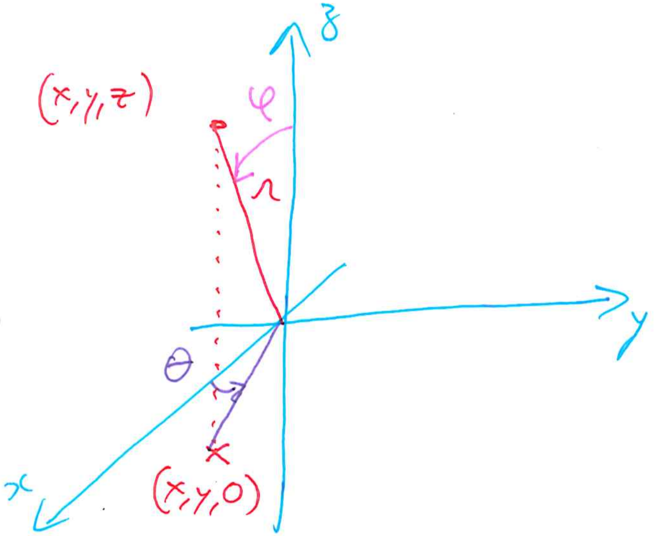
but also if only one of $r$ or $\theta$ is in an interval independant of the other variable




$\theta_0 \le \theta \le \theta_1$

$0 \le r \le g(\theta)$

## (2) Spherical coordinates in $\mathbb{R}^3$




$r > 0, \quad 0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

$$\psi(r,\theta,\varphi) = (r\cos\theta\sin\varphi,\ r\sin\theta\sin\varphi,\ r\cos\varphi)$$

We saw $|\det J_\psi| = +r^2 \sin(\varphi)$

$$\int_{\overline{X}} f(x,y,z)\,dx\,dy\,dz = \int_{\text{"}\overline{X} \text{ in spherical coordinates"}} f(r\cos\theta\sin\varphi,\ r\sin\theta\sin\varphi,\ r\cos\varphi)\, r^2\sin\varphi\,dr\,d\theta\,d\varphi$$

Useful whenever $\overline{X}$ has rectangular form in spherical coordinates:

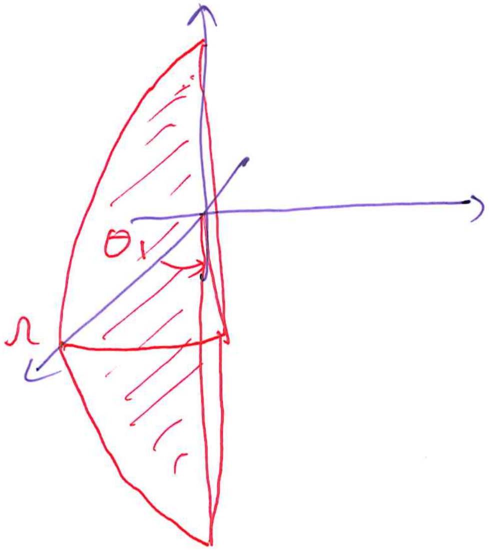
(1) balls centered at $O$

$$[0,r]\times[0,2\pi]\times[0,\pi]$$

(2) shells

$$[r_1,r_2]\times[0,2\pi]\times[0,\pi]$$

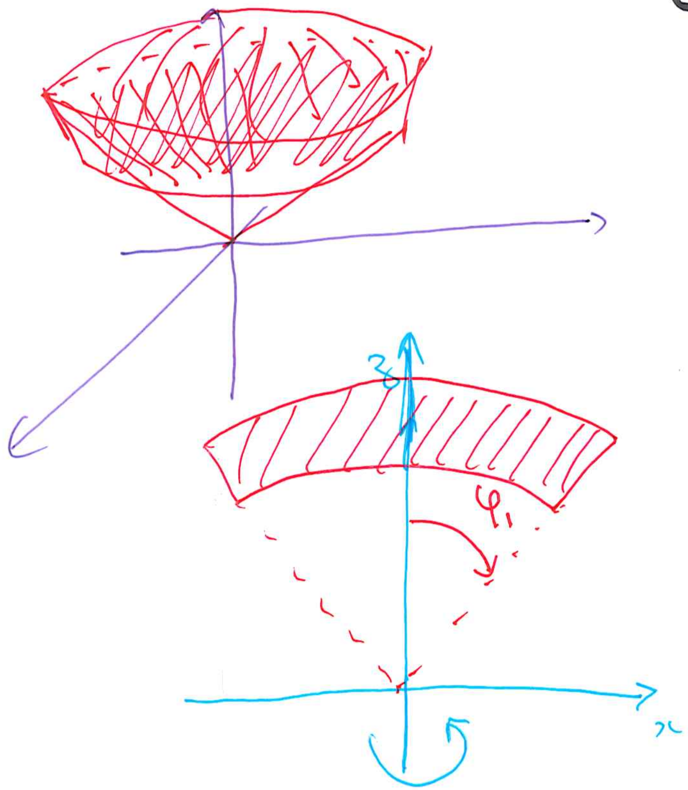
(3)




$[0,r] \times [0,\theta_1] \times [0,\pi]$

"orange piece"
(longitude in an interval)

(4)




$[0,r] \times [0,2\pi] \times [0,\varphi_1]$

[redacted]

"volcanic cone"

**Ex.** (4.4.4 (2))

$$\int_{\bar{X}} z^2 \, dx\,dy\,dz$$

$$\bar{X} = \{(x,y,z) \mid 1 \leq \|x\| \leq 2\}$$

(sph. coord)

$$\text{“}=\text{”} \; [1,2] \times [0,2\pi] \times [0,\pi]$$

$$\int_{\bar{X}} z^2 \, dx\,dy\,dz = \int_1^2 \int_0^{2\pi} \int_0^{\pi} r^2 \cos^2(\varphi) \, r^2 \sin(\varphi) \, dr\,d\theta\,d\varphi$$

$$= \left(\int_1^2 r^4 \, dr\right) 2\pi \left( \int_0^{\pi} \cos^2\varphi \sin\varphi \, d\varphi \right)$$

[ <u>Reminder:</u> how to compute

$$\int_a^b \cos^n(t)\,\sin^m(t)\,dt\ ?$$

Replace
$$\cos(t) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$
$$\sin(t) = \frac{1}{2i}\left(e^{it} - e^{-it}\right),$$

expand with binomial formula, end up with sum of $\int_a^b e^{ikt}\,dt$,

$$\int_a^b e^{ikt}\,dt = \begin{cases} b-a, & k=0 \\ \left[\frac{1}{ik}e^{ikt}\right]_a^b, & k \neq 0 \end{cases}$$ ]

Check: $\cos^2\varphi \sin\varphi = \frac{1}{4}\left[\sin(3\varphi) + \sin(\varphi)\right]$

$$\rightsquigarrow \int_{\overline{X}} z^2 \, dx\, dy\, dz = \frac{124\pi}{15}.$$

## 4.5 – Geometric applications of integrals

(a) Center of mass

(b) ~~Surfaces~~ Surface areas in $\mathbb{R}^3$

## a) Center of mass / barycenter

$\overline{X} \subset \mathbb{R}^n$ compact, with $Vol(\overline{X}) > 0$

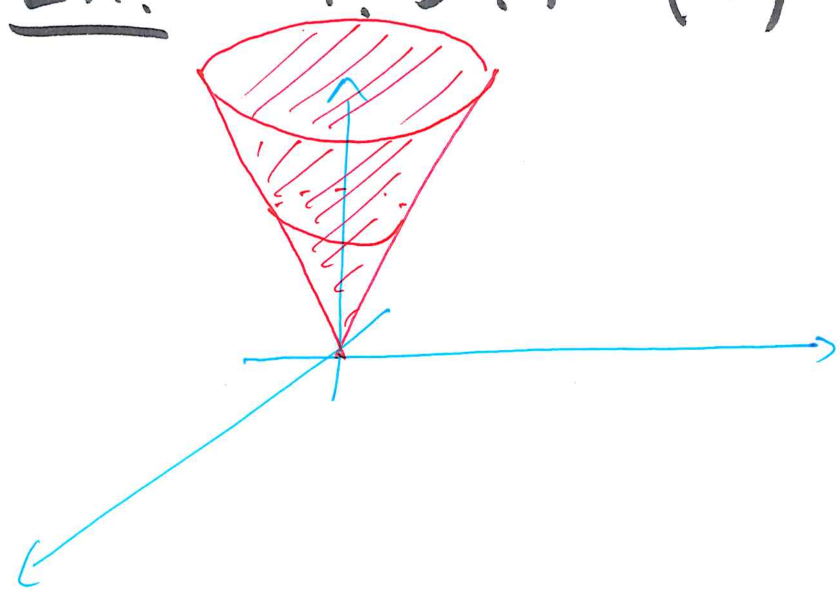
The center of mass of $\overline{X}$ is

$(\bar{x}_1, \ldots, \bar{x}_n)$ where

$$\bar{x}_i = \frac{1}{Vol(\overline{X})} \int_{\overline{X}} x_i \, dx_1 \ldots dx_n$$

("point where $\overline{X}$ is perfectly balanced")

Ex. 4.5.1 (1)



cone where the $0 \le z \le 1$

"slice" for fixed $z$ is a disc of radius $z$

$(\bar{x}, \bar{y}, \bar{z})$ center of mass

$\bar{x} = 0$ , $\bar{y} = 0$ (check)
(symmetry reasons)

$$\bar{z} = \frac{1}{V} \int_{\overline{X}} z \, dx\, dy\, dz$$ where $V = \mathrm{Vol}(\overline{X})$

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$$V = \int_{\overline{X}} dx\,dy\,dz = \int_0^1 \pi z^2\,dz = \frac{\pi}{3}$$

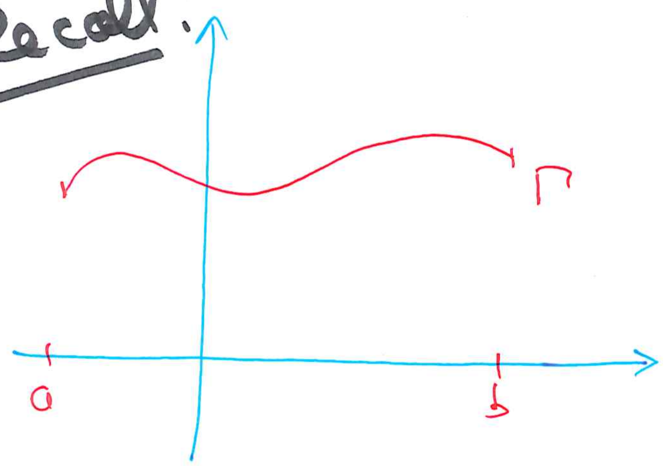
$$\bar{z} = \frac{3}{\pi}\int_{\overline{X}} z\,dx\,dy\,dz = \frac{3}{\pi}\int_0^1 z \times \left(\pi z^2\right) dz$$

$$= 3\int_0^1 z^3\,dz$$

$$= \frac{3}{4}$$

## b) Surface area

**Recall:**




$$f : [a,b] \longrightarrow \mathbb{R}$$

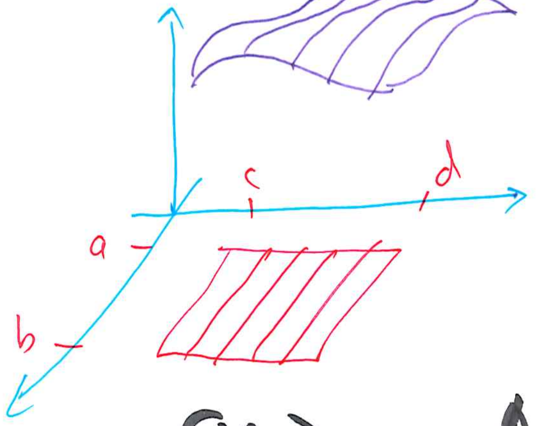
$$\Gamma = \text{graph of } f \subset \mathbb{R}^2$$

$$\text{length}(\Gamma) = \int_a^b \sqrt{1 + f'(t)^2}\, dt$$

For surfaces:

$$f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$$

$$\Gamma = \{ (x,y,z) \mid a \le x \le b,\ c \le y \le d,\ z = f(x,y) \} \subset \mathbb{R}^3$$

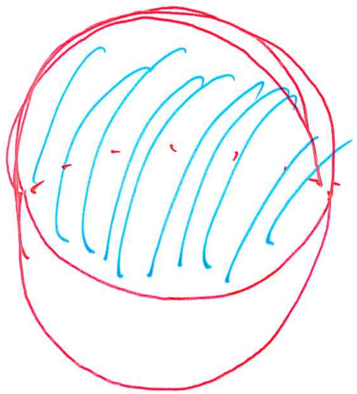



What is the area of $\Gamma$ in $\mathbb{R}^3$?

(*) $\text{Area}(\Gamma) = \int_a^b \int_c^d \sqrt{1+(\partial_x f)^2+(\partial_y f)^2}\, dx\,dy$

<u>Ex.</u> Area of sphere

$X = \{(x,y,z) \mid x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$



$\text{Area}(X) = 2\,\text{Area}(X \cap \{z \geq 0\})$

$= 2\,\text{Area}(\text{graph of } \sqrt{1-x^2-y^2}$ over disc $D$ of radius $1)$

The formula (*) extends to

$$\mathrm{Area}(X) = 2 \int_D \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dx\, dy$$

with $f(x,y) = \sqrt{1-x^2-y^2}$

$$\partial_x f = \frac{1}{\cancel{2}} \frac{-\cancel{2}x}{\sqrt{1-x^2-y^2}}, \quad \partial_y f = \frac{-y}{\sqrt{1-x^2-y^2}}$$

so

$$\mathrm{Area}(X) = 2 \int_D \sqrt{1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2}}\, dx\, dy$$

$$= 2 \int_D \sqrt{\frac{1}{1-x^2-y^2}}\, dx\, dy$$

$$= 2 \int_0^1 \int_0^{2\pi} \frac{1}{\sqrt{1-r^2}} \, r \, dr \, d\theta$$ 226

polar

$$= 4\pi \cdot \underbrace{\frac{1}{2} \int_0^1 \frac{1}{\sqrt{1-u} } \, du}_{\left[-\sqrt{1-u}\right]_0^1} \qquad u = r^2$$

$$= 4\pi$$

<u>Next:</u> important applications of integrals include relations between integrals in dimension $n$ and $n-1$:

"Stokes Theorem/Formula"

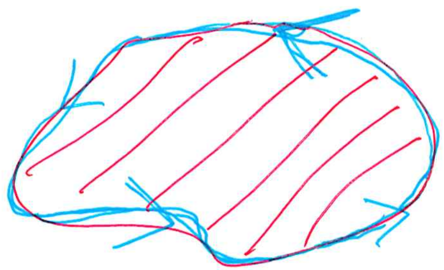
$$\int_{\partial X} f = \int_X Df$$

$f$: $C^1$-function on $\mathbb{R}^n$

$Df$: combination of 1st order partial derivatives

$X$: domain in $\mathbb{R}^n$ (dim. $n$)

$\partial X$: (oriented) boundary of $X$: $(n-1)$-dim.



**Recall:** Fundamental th. of calculus

orientation $\quad f(b) - f(a) = \int_a^b f'(t)\,dt$

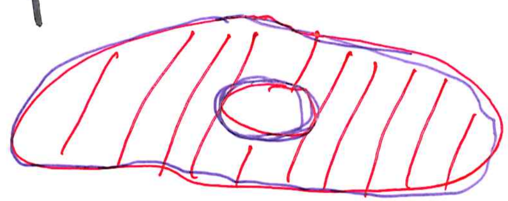

$X = [a, b]$

# 4.6 – Green formula

## Case $n = 2$

"**Theorem**" (4.6.3) "Green formula"

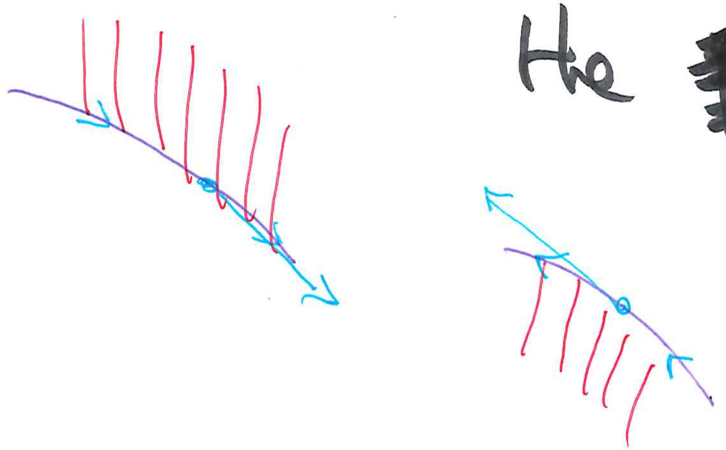
$X \subset \mathbb{R}^2$ compact



$\partial X$ = boundary of $X$

Assume: $\partial X$ is union of $k$ simple closed parameterized curves $\gamma_i$ (disjoint)

Assume: $X$ always lies to the left of $\gamma_i(t)$



Let $f = (f_1, f_2)$ be a vector field $C^1$ on $X$.

Then

$$\int_X \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy = \sum_{i=1}^{k} \int_{\gamma_i} f \cdot \vec{ds}$$