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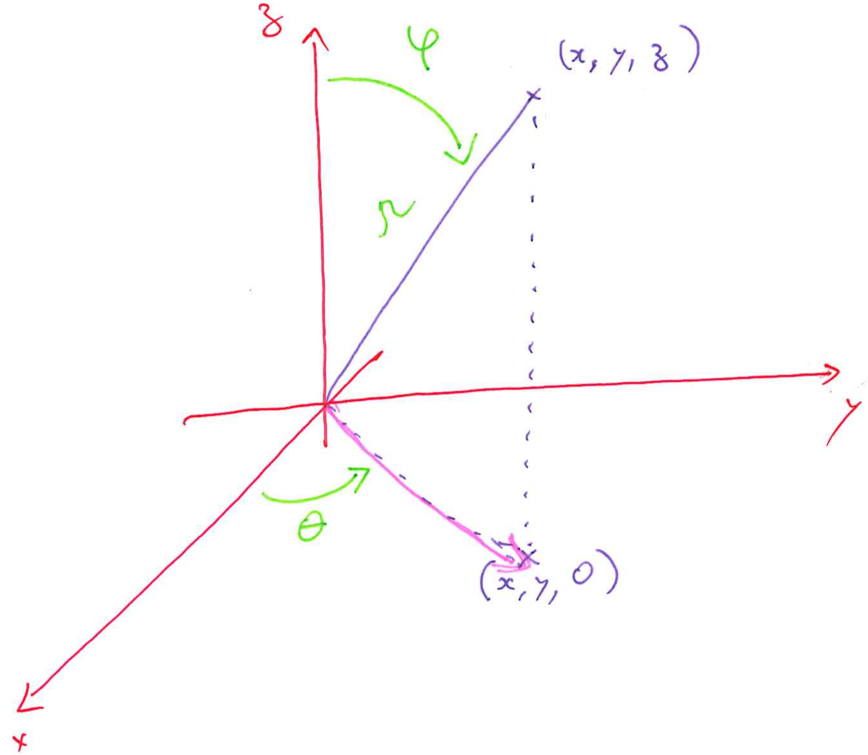
Ex. 3.10.3 (2)

(spherical coordinates)



($r \geq 0$)
r = distance to $(0,0,0)$

θ = angle $(\overrightarrow{Ox}, \overrightarrow{(x,y,0)})$
$(0 \leq \theta \leq 2\pi)$

φ = angle (z-axis, (x,y,z))
$(0 \leq \varphi \leq \pi)$

$$[0, +\infty[\times [0, 2\pi] \times [0, \pi] \xrightarrow{f} \mathbb{R}^3$$

$$(r, \theta, \varphi) \longmapsto \begin{pmatrix} r\cos\theta \sin\varphi \\ r\sin\theta \sin\varphi \\ r\cos\varphi \end{pmatrix}$$

surjective
<u>not</u> injective
$(0, \theta, \varphi) \mapsto (0,0,0)$

$f :]0, +\infty[\times]0, 2\pi[\times]0, \pi[\longrightarrow \mathbb{R}^3$ 131

is injective, but not surjective

(image missing $(0,0,0)$, >0 real axis, z-axis)

$$J_f(r, \theta, \varphi) = \left(\cdots \right)$$

and $\det J_f(r, \theta, \varphi) = -r^2 \sin(\varphi)$

$\neq 0$ whenever $r > 0$

and $\varphi \in]0, \pi[$. ~~and~~ $\cancel{\varphi \in]0, \pi[}$

Chapter IV

Integration in $\mathbb{R}^n$

$\hookrightarrow$ definition integrals of functions of ≥ 2 variables
[for instance to compute volumes of subsets of $\mathbb{R}^3$]

$\hookrightarrow$ relations between integrals in n and $n-1$ dimensions
"vector calculus"

4.1 – Line integrals

Def. 4.1.1

(1) $[a,b] \xrightarrow{f} \mathbb{R}^n$ continuous

$f(t) = (f_1(t), \dots, f_n(t))$, f_i continuous

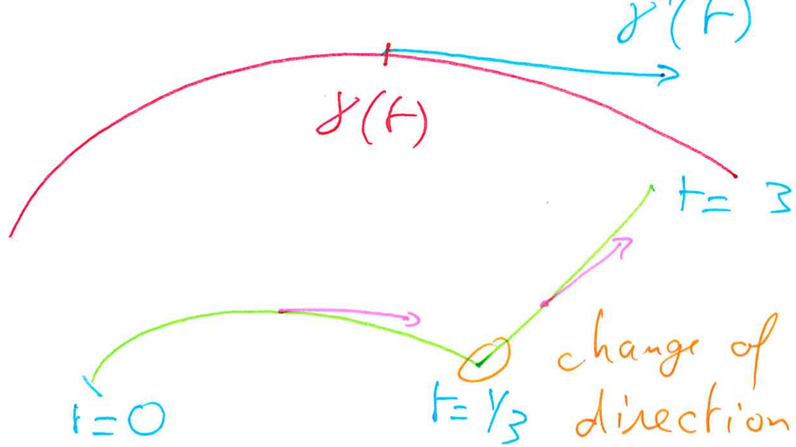
$$\int_a^b f(t)\,dt = \left(\int_a^b f_1(t)\,dt, \dots, \int_a^b f_n(t)\,dt\right)$$

This is an element in $\mathbb{R}^n$.

(2) A <u>parameterized curve</u> in $\mathbb{R}^n$ is

$$\gamma : [a,b] \longrightarrow \mathbb{R}^n \text{ s.t.}$$

(2.i) γ continuous



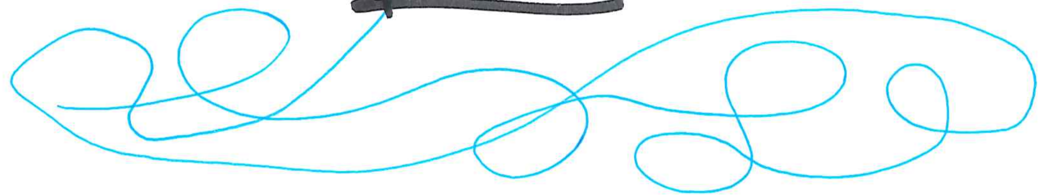
(2.ii) there are

$$t_0 = a < t_1 < \dots < t_k = b$$

s.t.

$$\gamma|\,]t_i, t_{i-1}[\text{ is } C^1$$

We say that γ is a <u>path</u> between $\gamma(a)$ and $\gamma(b)$.

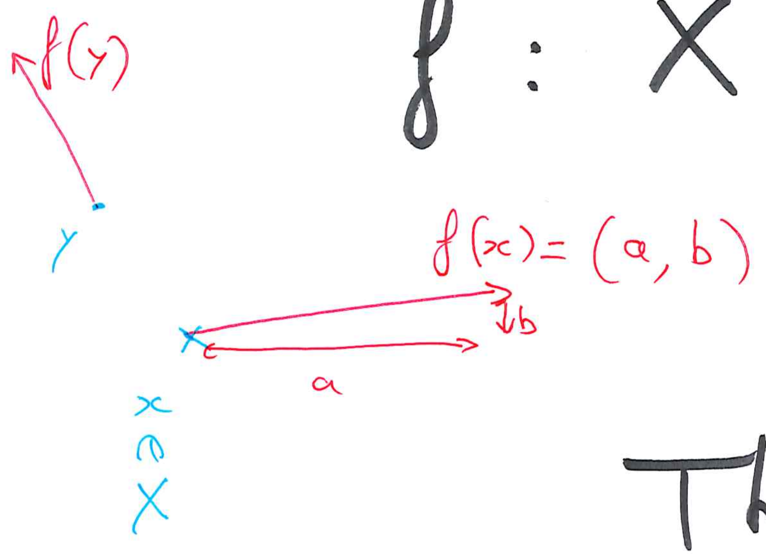


(3) $\gamma : [a, b] \longrightarrow \mathbb{R}^n$

$f : X \longrightarrow \mathbb{R}^n, \quad X \subset \mathbb{R}^n$

continuous

"vector field"

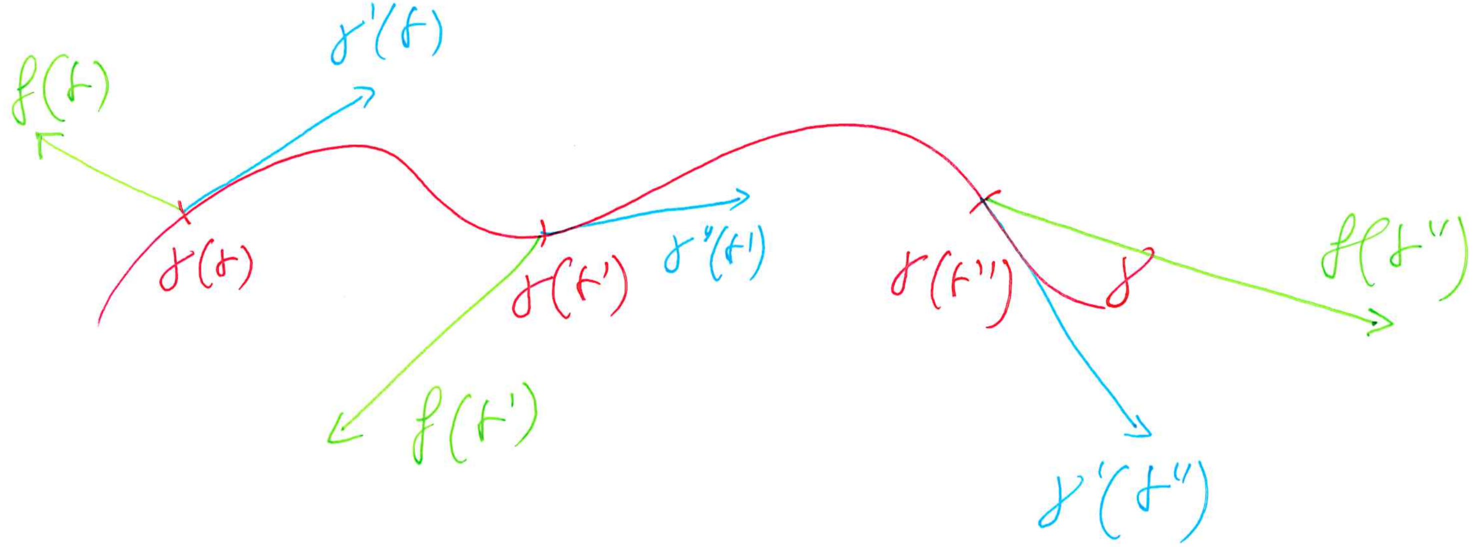


The <u>line integral of</u>

<u>f along γ</u> is

$$\int_\gamma f(\vec{s}) \cdot d\vec{s} = \underbrace{\int_a^b \underbrace{f(\gamma(t))}_{\in \mathbb{R}^n} \cdot \underbrace{\gamma'(t)}_{\in \mathbb{R}^n}}_{\text{scalar product in } \mathbb{R}} dt \in \mathbb{R}$$

$[\gamma([a,b]) \subset X]$



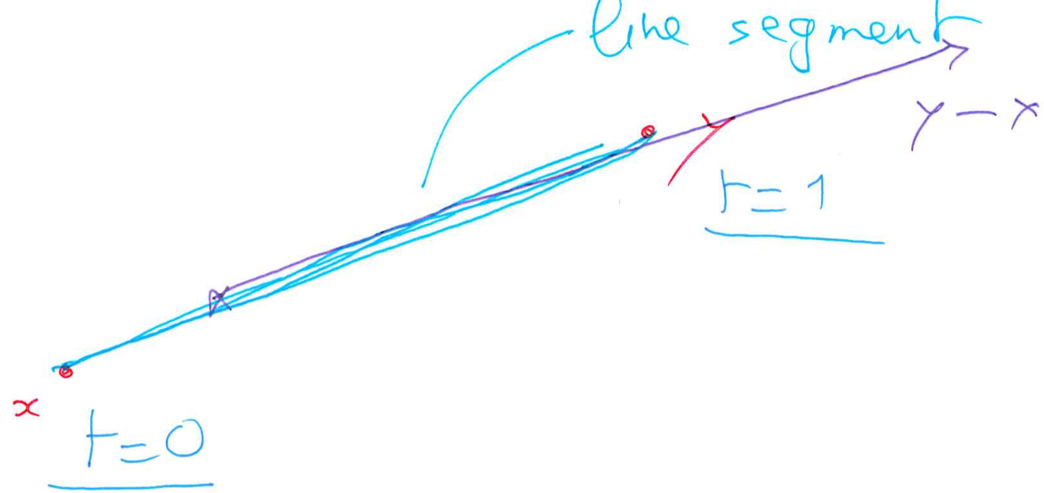
Examples -

(1) $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuous

$x, y \in \mathbb{R}^n$

$\gamma(t) = (1-t)x + ty$

$t \in [0, 1]$, $\gamma'(t) = y - x$



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For any f,

$$\int_\gamma f(s)\cdot d\vec{s} = \int_0^1 f((1-t)x+ty)\cdot(y-x)\,dt$$

$$= \sum_{i=1}^{n} (y_i - x_i)\int_0^1 f_i((1-t)x+ty)\,dt$$

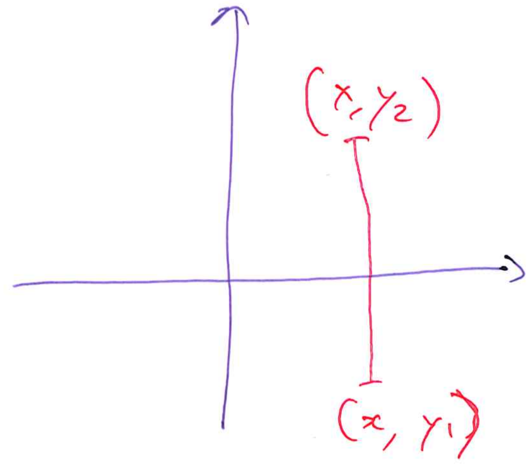
where

$$f(p) = (f_1(p), \dots, f_n(p))$$

$$x = (x_1, \dots, x_n)$$

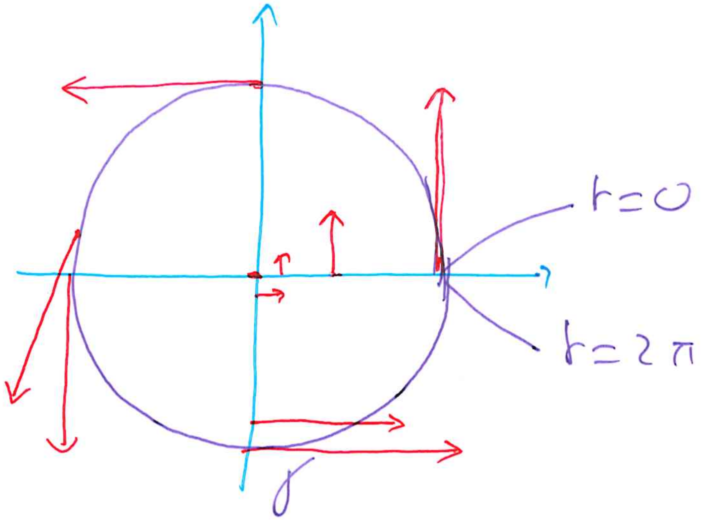
$$y = (y_1, \dots, y_n)$$

Ex.



(2)

$$\gamma(t) = (\cos(t), \sin(t))$$



$$a = 0 \leq t \leq 2\pi = b$$

$$\gamma(a) = \gamma(b)$$

("closed par. curve")

$$f(x, y) = (-y, x)$$

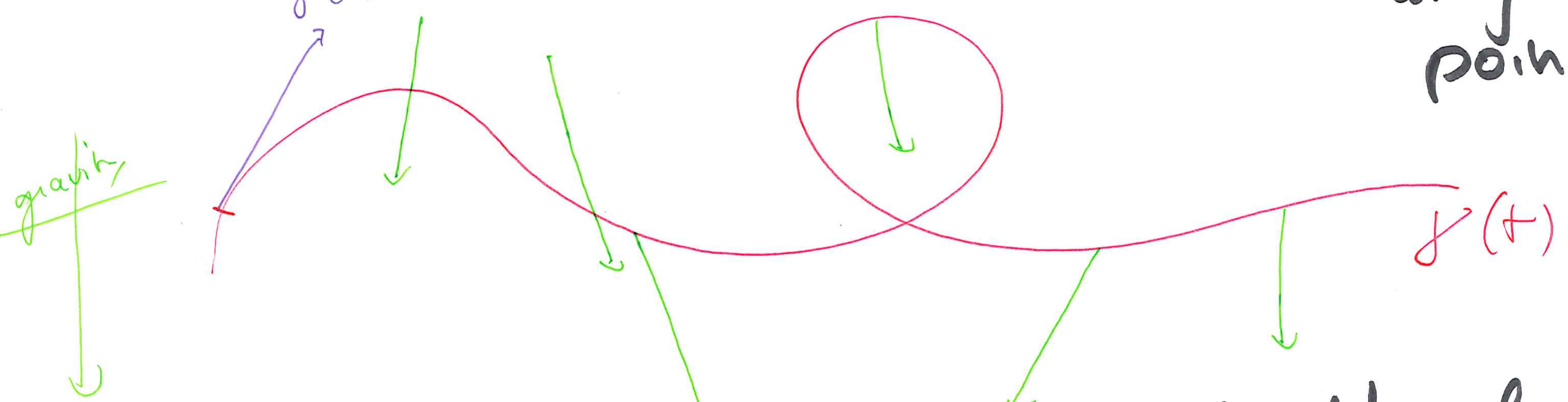
$$\int_\gamma f(s) \cdot d\vec{s} = \int_0^{2\pi} \underbrace{(-\cancel{\cos}\sin(t), \cos(t))}_{f(\gamma(t))} \cdot \underbrace{(-\sin(t), \cos(t))}_{\gamma'(t)} dt$$

$$= \int_0^{2\pi} \left(\sin^2(t) + \cos^2(t) \right) dt$$

$$= 2\pi$$

Physical interpretation

f = force at given point



$$\int_\gamma f(s) \cdot d\vec{s} = \text{"work" of the force } f \text{ along the trajectory}$$

(f is <u>all</u> forces) = kinetic energy at time b [Newton]

− kinetic energy at time a

Most important property of $\int_\gamma f(s) \cdot d\vec{s}$:

independance of parameterization!

<u>Def.</u> (4.1.4)

$$\gamma : [a, b] \longrightarrow \mathbb{R}^n$$

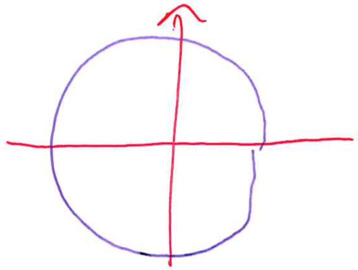
An <u>oriented</u> reparameterization of γ is $\sigma : [c, d] \longrightarrow \mathbb{R}^n$

s.t. $\gamma(t) = \sigma(\varphi(u))$, $c \leq u \leq d$

where $\varphi : [c, d] \longrightarrow [a, b]$ is C^1 strictly increasing, $\varphi(c) = a$ $\varphi(d) = b$

Ex. $\gamma_n(t) = \left(\cos(2\pi t^n), \sin(2\pi t^n)\right)$

$n \geq 1$ integer, $0 \leq t \leq 1$



repar. of the circle γ_1

Prop. 4.1.5 – γ, σ oriented reparameterization of γ

$f : X \longrightarrow \mathbb{R}^n$

$$\int_\gamma f(s) \cdot d\vec{s} = \int_\sigma f(s) \cdot d\vec{s}$$