Multiple choice questions

(1) Let f be a C^1 vector field on $\mathbf{R}^2$ such that the line integrals

$$\int_\gamma f \cdot d\vec{s}$$

are zero for all circles centered at 0 in the plane. Is the vector field conservative?

YES □ NO □

(2) Let f be a continuous function on $\mathbf{R}^2$ such that $f \geqslant 0$ and

$$0 \leqslant \int_0^1 f(x,y)dx \leqslant 1$$

for all $y \in [0,1]$. Is it true that

$$\int_X f(x,y)dxdy \leqslant \text{Area}(X)$$

for all compact subsets X of $[0,1]^2$?

YES □ NO □

(3) Does the improper integral $\displaystyle\int_{[0,1]\times[0,+\infty[} e^{x^2-y}dxdy$ exist?

YES □ NO □

(4) Let D be the disc centered at $0 \in \mathbf{R}^2$ with radius $3/2$. For a continuous function f in $\mathbf{R}^2$, we have

$$\int_D f(x,y)dxdy = \int_0^{3/2}\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta)drd\theta.$$

YES □ NO □

Quick computations

(1) For which values of $a \in \mathbf{R}$, is the vector field $f(x,y) = ((xy+a)e^{xy}, x^2e^{xy})$ on $\mathbf{R}^2$ conservative?

(2) Let f be the vector field on $\mathbf{R}^2$ defined by

$$f(x,y) = \Big(\frac{2x}{(x^2+y^4+1)^2}, \frac{4y^3}{(x^2+y^4+1)^2}\Big).$$

Compute the line integral

$$\int_\gamma f \cdot d\vec{s}$$

where $\gamma(t) = (\sin(\pi\cos(2\pi t)), (1-t)e^t + t)$ for $0 \leqslant t \leqslant 1$.

(3) Compute the integral

$$\int_D (|x| - y^2)dxdy$$

where D is the disc centered at $(0,1)$ with radius 2.

(4) Let $f(x,y) = (e^{xy}, x)$. Compute the line integral

$$\int_\gamma f \cdot d\vec{s}$$

along the square with vertices $(0,0)$, $(0,2)$, $(2,2)$ and $(2,0)$ oriented counter-clockwise.

Solutions to multiple choice questions

(1) Let f be a C^1 vector field on $\mathbf{R}^2$ such that the line integrals

$$\int_\gamma f \cdot d\vec{s}$$

are zero for all circles centered at 0 in the plane. Is the vector field conservative?

NO – the integral must be zero along all closed parameterized curves. For instance, if we take a vector field that is perpendicular to the tangent vector and is not conservative, then $f \cdot d\vec{s}$ will be zero along each circle, so all integrals will be zero. An example is $f(x, y) = (x^2, xy)$.

(2) Let f be a continuous function on $\mathbf{R}^2$ such that $f \geqslant 0$ and

$$0 \leqslant \int_0^1 f(x, y)dx \leqslant 1$$

for all $y \in [0, 1]$. Is it true that

$$\left|\int_X f(x, y)dxdy\right| \leqslant \text{Area}(X)$$

for all compact subsets X of $[0, 1]^2$?

YES – We know that

$$\int_X f(x, y)dxdy \leqslant \int_{[0,1]^2} f(x, y)dxdy$$

since $f \geqslant 0$, and by Fubini's Theorem, this means that

$$\int_X f(x, y)dxdy \leqslant \int_0^1 \Big(\int_0^1 f(x, y)dx\Big)dy \leqslant 1$$

by properties of the one-dimensional Riemann integral.

(3) Does the improper integral

$$\int_{[0,1]\times[0,+\infty[} e^{x^2-y}dxdy$$

exist?

YES – By definition, we must check if the limit of

$$\int_{[0,1]\times[0,R]} e^{x^2-y}dxdy$$

as $R \to +\infty$ exists. By Fubini's Theorem, this is

$$\int_0^1 e^{x^2}\Big(\int_0^R e^{-y}dy\Big)dx = (1 - e^{-R})\int_0^1 e^{x^2}dx$$

which converges to $\int_0^1 e^{x^2}dx$.

(4) Let D be the disc centered at $0 \in \mathbf{R}^2$ with radius $3/2$. For a continuous function f in $\mathbf{R}^2$, we have

$$\int_D f(x,y)dxdy = \int_0^{3/2} \int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta)drd\theta.$$

NO – the correct change of variable formula in polar coordinates is

$$\int_D f(x,y)dxdy = \int_{-\pi}^{\pi} \int_0^{3/2} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Solutions to quick computations

(1) For which values of $a \in \mathbf{R}$, is the vector field $f(x,y) = ((xy+a)e^{xy}, x^2e^{xy})$ on $\mathbf{R}^2$ conservative?

Since $\mathbf{R}^2$ is star-shaped and f is C^1, the vector field is conservative if and only if

$$\partial_y((xy+a)e^{xy}) = \partial_x(x^2e^{xy}),$$

which translates to the condition

$$axe^{xy} + xe^{xy} + x^2ye^{xy} = 2xe^{xy} + x^2ye^{xy},$$

which holds if and only if $a = 1$.

(2) Let f be the vector field on $\mathbf{R}^2$ defined by

$$f(x,y) = \Big(\frac{2x}{(x^2+y^4+1)^2}, \frac{4y^3}{(x^2+y^4+1)^2}\Big).$$

Compute the line integral

$$\int_\gamma f \cdot d\vec{s}$$

where $\gamma(t) = (\sin(\pi\cos(2\pi t)), (1-t)e^t + t)$ for $0 \leqslant t \leqslant 1$.

Observe that γ is a closed curve, since $\gamma(0) = (0,1) = \gamma(1)$, and that f is C^1 and conservative because $f = \nabla g$, where $g(x,y) = -1/(x^2+y^4+1)$. So the line integral is zero.

(3) Compute the integral

$$\int_D (|x| - y^2)dxdy$$

where D is the disc centered at $(0,1)$ with radius 2.

Using polar coordinates centered at $(0,1)$, so that $(x,y) = (r\cos\theta, 1+r\sin\theta)$ with $0 \leqslant r \leqslant 2$ and $0 \leqslant \theta \leqslant 2\pi$, this integral becomes

$$\int_0^2\int_0^{2\pi}(|r\cos\theta| - (1+r\sin\theta)^2)rdrd\theta.$$

The first term is

$$\int_0^2 r^2\Big(\int_0^{2\pi}|\cos\theta|d\theta\Big)dr = \frac{8}{3}\times 2\int_{-\pi/2}^{\pi/2}\cos\theta d\theta = \frac{32}{3},$$

and the second is

$$\int_0^2\int_0^{2\pi}(r + 2r^2\sin\theta + r^3\sin^2\theta)drd\theta = 4\pi + 0 + 4\int_0^{2\pi}\sin^2\theta d\theta = 8\pi,$$

so the integral is $32/3 - 8\pi$.

(4) Let $f(x,y) = (e^{xy}, x)$. Compute the line integral

$$\int_\gamma f \cdot d\vec{s}$$

along the square with vertices $(0,0)$, $(0,2)$, $(2,2)$ and $(2,0)$ oriented counter-clockwise.

We use Green's Theorem: since the path γ is the boundary, positively oriented, of the square $[0,2]^2$, the line integral is equal to

$$\begin{aligned}\int_{[0,2]^2} (\partial_x x - \partial_y(e^{xy}))dxdy &= \int_0^2 \int_0^2 (1 - xe^{xy})dxdy \\ &= 4 - \int_0^2 x\Big(\int_0^2 e^{xy}dy\Big)dx = 4 - \int_0^2 x \times \frac{1}{x}(e^{2x} - 1)dx \\ &= 2 - \int_0^2 e^{2x}dx = 2 - \frac{1}{2}(e^4 - 1) = \frac{3 - e^4}{2}.\end{aligned}$$