Multiple choice questions

(1) If x_0 is a critical point of a C^1 function $f\colon \mathbf{R}^2 \to \mathbf{R}$, then f is maximal or minimal at x_0.

YES □ NO □

(2) Let $f(x,y) = (f_1(x,y), f_2(x,y), f_3(x,y))$ and $g(u,v,w)$ be differentiable functions $\mathbf{R}^2 \to \mathbf{R}^3$ and $\mathbf{R}^3 \to \mathbf{R}$ respectively. We have

$$\frac{\partial(g \circ f)}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial f_1}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial f_2}{\partial y}.$$

YES □ NO □

(3) If $f\colon \mathbf{R}^2 \to \mathbf{R}$ is of class C^2, then its second Taylor polynomial at $(0,0)$ is

$$f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)y^2 + \frac{1}{2}\frac{\partial^2 f}{\partial xy}(0,0)xy.$$

YES □ NO □

(4) If $f\colon \mathbf{R}^3 \to \mathbf{R}$ is of class C^2, $\nabla f(0,0,0) = 0$, and the Hessian matrix of f at $(0,0,0)$ is

$$\begin{pmatrix} 5 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{pmatrix}$$

then f has

A local minimum □ A local maximum □
A saddle point □

at $(0,0,0)$.

Quick computations

(1) Compute the gradient of $f(x, y, z) = xy^2 \exp(\cos(xz) - y)$.

(2) Compute the Hessian of $f(x, y) = 2 \exp(x^2 - y^2)$ at $(x, y) = (0, 0)$.

(3) Compute the critical points of $f(x, y, z) = x^3 - (y - 1)ze^x$.

Solutions to multiple choice questions

(1) If x_0 is a critical point of a C^1 function $f\colon \mathbf{R}^2 \to \mathbf{R}$, then f is maximal or minimal at x_0.

NO – the function could have a saddle point, or only a local extremum.

(2) Let $f(x,y) = (f_1(x,y), f_2(x,y), f_3(x,y))$ and $g(u,v,w)$ be differentiable functions $\mathbf{R}^2 \to \mathbf{R}^3$ and $\mathbf{R}^3 \to \mathbf{R}$ respectively. We have

$$\frac{\partial(g\circ f)}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial f_1}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial f_2}{\partial y}.$$

NO – the correct formula, from the Chain Rule, is

$$\frac{\partial(g\circ f)}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial f_1}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial f_2}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial f_3}{\partial x}.$$

(3) If $f\colon \mathbf{R}^2 \to \mathbf{R}$ is of class C^2, then its second Taylor polynomial at $(0,0)$ is

$$f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)y^2 + \frac{1}{2}\frac{\partial^2 f}{\partial xy}(0,0)xy.$$

NO – the factor 1/2 is wrong for the xy term; the correct polynomial is

$$f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)y^2 + \frac{\partial^2 f}{\partial xy}(0,0)xy.$$

(4) If $f\colon \mathbf{R}^3 \to \mathbf{R}$ is of class C^2, $\nabla f(0,0,0) = 0$, and the Hessian matrix of f at $(0,0,0)$ is

$$\begin{pmatrix} 5 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{pmatrix}$$

then f has

A local minimum at $(0,0,0)$ – Indeed, the three principal minor matrices are

$$5, \quad \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 5 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{pmatrix}$$

with determinants 5, 6 and 7; since all are > 0, this implies that the Hessian is positive definite, which means that f has a local minimum at the critical point.

Solutions to quick computations

(1) Compute the gradient of $f(x,y,z) = xy^2 \exp(\cos(xz) - y)$.

We compute that

$$\nabla f(x,y,z) = e^{\cos(xz)-y} \begin{pmatrix} y^2 - xy^2 z \sin(xz) \\ 2xy - xy^2 \\ -x^2y^2 \sin(xz) \end{pmatrix}.$$

(2) Compute the Hessian of $f(x,y) = 2\exp(x^2 - y^2)$ at $(x,y) = (0,0)$.

We compute that

$$\nabla f(x,y) = \begin{pmatrix} 4x \exp(x^2 - y^2) \\ -4y \exp(x^2 - y^2) \end{pmatrix}$$

and then

$$\mathrm{Hess}_f(x,y) = e^{x^2-y^2} \begin{pmatrix} 4 + 8x & -8xy \\ -8xy & -4 + 8y \end{pmatrix}$$

So

$$\mathrm{Hess}_f(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}.$$

(3) Compute the critical points of $f(x,y,z) = x^3 - (y-1)ze^x$.

The gradient is

$$\nabla f(x,y,z) = \begin{pmatrix} 3x^2 - (y-1)ze^x \\ -ze^x \\ -(y-1)e^x \end{pmatrix}$$

so that (x,y,z) is a critical point if and only if

$$\begin{cases} 3x^2 - (y-1)ze^x = 0 \\ -ze^x = 0 \\ -(y-1)e^x = 0. \end{cases}$$

From the last two equations we get $y = 1$ and $z = 0$, and then the first gives $x = 0$. So $(0,1,0)$ is the only critical point.