Analysis III (BAUG) **Assignment 10**

Prof. Dr. Alessandro Sisto
Organizer: Davide Spriano

Due 5th December 2018

Question 1:

Use the Laplace transform of derivatives to find the Laplace transform of the functions below.

(i) $f(t) = \cos^2(\pi t)$

(ii) $f(t) = t\sin(\frac{\pi}{2}t)$

Question 2:

Express each of the following functions as a single formula using the Heaviside function.

(i) $f(t) = \begin{cases} t & \text{for } t \leq 1 \\ t^2 & \text{for } 1 \leq t \end{cases}$

(ii) $f(t) = \begin{cases} 0 & \text{for } t \leq 2\pi \\ \sin(t) & \text{for } 2\pi \leq t \end{cases}$

(iii) $f(t) = \begin{cases} 2 & \text{for } t < 1 \\ 2+t & \text{for } 1 \leq t < 4 \\ 2+e^t & \text{for } 4 \leq t \end{cases}$

(iv) $f(t) = \begin{cases} 3 & \text{for } t < 2 \\ 3\cos(2t-3) & \text{for } t \geq 2 \end{cases}$

(v) $f(t) = \begin{cases} t/2 & \text{for } t \leq 6 \\ 3 & \text{for } t > 6 \end{cases}$

Question 3:

Compute the following functions.

(i) What is the Laplace transform of $f(t) = \begin{cases} t^2 & \text{for } t < 5 \\ t^2 + 3\sin(2t-10) & \text{for } t \geq 5 \end{cases}$?

(ii) What is the inverse Laplace transform of $F(s) = \frac{e^{-10s}}{s+2}$?

(iii) What is the inverse Laplace transform of $F(s) = \frac{e^{-10s}}{s(s+1)(s+2)}$?

(iv) What is the Laplace transform of $f(t) = \begin{cases} t/2 & \text{for } t \leq 6 \\ 3 & \text{for } t > 6 \end{cases}$?

(v) What is the Laplace inverse of $F(s) = \frac{e^{-6s}}{s^2(s^2+1)}$?

(vi) What is the Laplace inverse of $F(s) = \frac{e^{-2s}}{(s+1)^2+1}$?

- ☐ $f(t) = u(t-2)\sin(t-1)$
- ☐ $f(t) = u(t-2)e^{-t}\sin(t-2)$
- ☐ $f(t) = u(t-2)e^{-t+2}\sin(t-2)$
- ☐ $f(t) = u(t-2)e^{-t+1}\sin(t-1)$
- ☐ $f(t) = u(t-2)\sin(t-2)$

Question 4:

Use the Laplace transform to solve the following initial value problems.

(i) $x''(t) + 3x'(t) + 2x(t) = \begin{cases} 1 & \text{for } t < 10 \\ 0 & \text{for } t \geq 10 \end{cases}, \qquad x(0) = 0, \qquad x'(0) = 0$

(ii) $x''(t) + 4x(t) = 4t, \qquad x(0) = 1, \qquad x'(0) = 1$
[Exam question, 2012]

(iii) $x''(t) + x(t) = \begin{cases} t/2 & \text{for } t \leq 6 \\ 3 & \text{for } t > 6 \end{cases}, \qquad x(0) = 0, \qquad x'(0) = 1$

Question 5:

For each of the following pairs of functions f and g compute the corresponding convolution $f * g$.

(i) $f(t) = e^{-\omega t}$ and $g(t) = \sin(\omega t)$ for all $t \geq 0$ where ω is a positive real number.

(ii) $f(t) = t^a$, $g(t) = t^b$ for all $t \geq 0$ where a and b are positive integers.

(iii) $f(t) = g(t) = \sin(\omega t)$ for all $t \geq 0$ where ω is some positive real number.

(iv) $f(t) = e^{at}$, $g(t) = e^{bt}$ for all $t \geq 0$ where a and b are distinct real numbers.

(v) $f(t) = \frac{1}{\sqrt{t}}$, $g(t) = t^2$ for all $t \geq 0$.

Question 6:

For each of the following functions f, find the solution to

$$x''(t) + \omega^2 x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0$$

where ω is a positive real number.

(i) $f(t) = e^{-\omega t}$ for $t \geq 0$.

(ii) $f(t) = \sin(\omega t)$ for $t \geq 0$.

(iii) $f(t) = 1$ for $t \geq 0$.

Question 7:

[Exercise 1.2 Lecture 10]

(a) Check that the following identities are correct, where f, g and h are functions and c is a constant:

  (i) $f * g = g * f$.
  (ii) $f * (g + h) = f * g + f * h$.
  (iii) $(f * g) * h = f * (g * h)$.

(b) For each of the following, find an explicit function $f(t)$ such that:

  (i) $f * 1 \neq f$,
  (ii) $f * f \not\geq 0$.