Analysis III (BAUG) **Assignment 10**
Prof. Dr. Alessandro Sisto Due 5th December 2018
Organizer: Davide Spriano

Question 1:

Use the Laplace transform of derivatives to find the Laplace transform of the functions below.

(i) $f(t) = \cos^2(\pi t)$

Solution:
$f'(t) = -2\pi\cos(\pi t)\sin(\pi t) = -\pi\sin(2\pi t)$, so

$$\begin{aligned}\mathcal{L}(f')(s) = sF(s) - f(0) &= \mathcal{L}(-\pi\sin(2\pi t))(s)\\ &= -\pi\mathcal{L}(\sin(2\pi t))(s) = -\frac{2\pi^2}{s^2+4\pi^2}.\end{aligned}$$

Here $f(0) = 1$, therefore $F(s) = \frac{s^2+2\pi^2}{s^3+4\pi^2 s}$.

(ii) $f(t) = t\sin(\frac{\pi}{2}t)$

Solution:

$$\begin{aligned}f(t) &= t\sin(\frac{\pi}{2}t), \quad f(0) = 0\\ f'(t) &= \sin(\frac{\pi}{2}t) + \frac{\pi}{2}t\cos(\frac{\pi}{2}t), \quad f'(0) = 0\\ f''(t) &= \frac{\pi}{2}\cos(\frac{\pi}{2}t) + \frac{\pi}{2}\cos(\frac{\pi}{2}t) - \frac{\pi^2}{4}t\sin(\frac{\pi}{2}t) = \pi\cos(\frac{\pi}{2}t) - \frac{\pi^2}{4}f(t)\end{aligned}$$

Therefore

$$\mathcal{L}(f'')(s) = s^2F(s) - sf(0) - f'(0) = s^2F(s)$$

is equal to

$$s^2F(s) = \mathcal{L}(f'')(s) = \mathcal{L}(\pi\cos(\frac{\pi}{2}t) - \frac{\pi^2}{4}f(t))(s) = \frac{\pi s}{s^2+\frac{\pi^2}{4}} - \frac{\pi^2}{4}F(s).$$

Hence,

$$F(s) = \frac{\pi s}{\left(s^2+\frac{\pi^2}{4}\right)^2}$$

Question 2:

Express each of the following functions as a single formula using the Heaviside function.

(i) $f(t) = \begin{cases} t & \text{for } t \leq 1 \\ t^2 & \text{for } 1 \leq t \end{cases}$

(ii) $f(t) = \begin{cases} 0 & \text{for } t \leq 2\pi \\ \sin(t) & \text{for } 2\pi \leq t \end{cases}$

(iii) $f(t) = \begin{cases} 2 & \text{for } t < 1 \\ 2 + t & \text{for } 1 \leq t < 4 \\ 2 + e^t & \text{for } 4 \leq t \end{cases}$

(iv) $f(t) = \begin{cases} 3 & \text{for } t < 2 \\ 3\cos(2t - 3) & \text{for } t \geq 2 \end{cases}$

(v) $f(t) = \begin{cases} t/2 & \text{for } t \leq 6 \\ 3 & \text{for } t > 6 \end{cases}$

Solution:

(i) $f(t) = t + (t^2 - t)u(t-1)$

(ii) $f(t) = \sin(t)u(t - 2\pi)$

(iii) $f(t) = 2 + tu(t-1) + (e^t - t)u(t-4)$

(iv) $f(t) = 3 + \big(3\cos\big(2(t-3)\big) - 3\big)u(t-2)$

(v) $f(t) = \frac{t}{2} + u(t-6)(3 - \frac{t}{2})$

Question 3:

Compute the following functions.

(i) What is the Laplace transform of $f(t) = \begin{cases} t^2 & \text{for } t < 5 \\ t^2 + 3\sin(2t - 10) & \text{for } t \geq 5 \end{cases}$?

Solution:
We rewrite f using the Heaviside function: $f(t) = t^2 + u(t-5)3\sin(2t-10)$.

The Laplace transfrom of the first term is $\frac{2}{s^3}$.

For the second term, we know that the Laplace transform of $u(t-c)g(t-c) = e^{-cs}G(s)$. Here $c = 5$ and $g(t-5) = 3\sin(2t-10) = 3\sin(2(t-5))$. So $g(t) = 3\sin(2t)$ and $G(s) = \frac{3\cdot 2}{s^2+4}$.

Hence the Laplace transform is $\frac{2}{s^3} + e^{-5s}\frac{6}{s^2+4}$.

(ii) What is the inverse Laplace transform of $F(s) = \frac{e^{-10s}}{s+2}$?

Solution:
We know that the Laplace transform of $u(t-c)g(t-c)$ is $e^{-cs}G(s)$. Here $c = 10$ and $G(s) = \frac{1}{s+2}$, so $g(t) = e^{-2t}$ and $g(t-c) = g(t-10) = e^{-2(t-10)} = e^{-2t+20}$. So the inverse is $f(t) = u(t-10)e^{-2t+20}$.

(iii) What is the inverse Laplace transform of $F(s) = \frac{e^{-10s}}{s(s+1)(s+2)}$?

Solution:
Once again, the inverse will be of the form $u(t-c)g(t-c)$ where $c = 10$ and $G(s) = \frac{1}{s(s+1)(s+2)}$. But what is the Laplace inverse of this $G(s)$?

We apply the method of partial fractions: we want to write $G(s)$ as $\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$.

Bringing this to a common denominator, we get that the numerator is

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1) = (A+B+C)s^2 + (3A+2B+C)s + 2A,$$

hence, comparing the coefficients, we see that $2A = 1$, $A+B+C = 0$, $3A+2B+C = 0$, and therefore $A = 1/2$, $B = -1$, $C = 1/2$ and $D = 0$.

So $G(s) = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}$ with Laplace inverse $g(t) = \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}$.

Hence the inverse of $F(s)$ above is

$$f(t) = u(t-10)g(t-10) = u(t-10)\left(\frac{1}{2} - e^{-t+10} + \frac{e^{-2t+20}}{2}\right)$$

(iv) What is the Laplace transform of $f(t) = \begin{cases} t/2 & \text{for } t \leq 6 \\ 3 & \text{for } t > 6 \end{cases}$?

Solution:
$x(t) = \frac{t}{2} + u(t-6)(3 - \frac{t}{2})$.

The Laplace transfrom of the first term is $\frac{1/2}{s^2}$. For the second term, we know that the Laplace transform of $u(t-c)g(t-c) = e^{-cs}G(s)$. Here $c = 6$ and $g(t-6) = 3 - \frac{t}{2} = -\frac{1}{2}(t-6)$. So $g(t) = -\frac{t}{2}$ and $G(s) = \frac{-1/2}{s^2}$.

Hence the Laplace transform is $\frac{1/2}{s^2} - e^{-6s}\frac{1/2}{s^2} = \frac{1-e^{-6s}}{2s^2}$.

(v) What is the Laplace inverse of $F(s) = \frac{e^{-6s}}{s^2(s^2+1)}$?

Solution:
Once again, the inverse will be of the form $u(t-c)g(t-c)$ where $c = 6$ and $G(s) = \frac{1}{s^2(s^2+1)}$. But what is the Laplace inverse of this $G(s)$?

We apply the method of partial fractions: we want to write $G(s)$ as $\frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$.
Bringing this to a common denominator, we get that the numerator is

$$1 = As^3 + Bs^2 + As + B + Cs^3 + Ds^2 = (A+C)s^3 + (B+D)s^2 + As + B,$$

hence, comparing the coefficients, we see that $A = 0$, $B = 1$ and $C = 0$ and $D = -1$.

So $G(s) = \frac{1}{s^2} - \frac{1}{s^2+1}$ with Laplace inverse $g(t) = t - \sin(t)$.

Hence the inverse of $F(s)$ above is

$$f(t) = u(t-6)g(t-6) = u(t-6)(t-6-\sin(t-6))$$

(vi) What is the Laplace inverse of $F(s) = \frac{e^{-2s}}{(s+1)^2+1}$?

- ☐ $f(t) = u(t-2)\sin(t-1)$
- ☐ $f(t) = u(t-2)e^{-t}\sin(t-2)$
- ☐ $f(t) = u(t-2)e^{-t+2}\sin(t-2)$
- ☐ $f(t) = u(t-2)e^{-t+1}\sin(t-1)$
- ☐ $f(t) = u(t-2)\sin(t-2)$

Solution:
Once again, the inverse will be of the form $u(t-c)g(t-c)$ where $c = 2$ and $G(s) = \frac{1}{(s+1)^2+1}$. The Laplace inverse of $G(s)$ is $g(t) = e^{-t}\sin(t)$, so $g(t-1) = e^{-t+1}\sin(t-1)$ and

$$f(t) = u(t-1)e^{-t+1}\sin(t-1)$$

Question 4:

Use the Laplace transform to solve the following initial value problems.

(i) $x''(t) + 3x'(t) + 2x(t) = \begin{cases} 1 & \text{for } t < 10 \\ 0 & \text{for } t \geq 10 \end{cases}, \qquad x(0) = 0, \qquad x'(0) = 0$

Solution:
We see that $x''(t)+3x'(t)+2x(t) = 1-u(t-10)$. Applying the Laplace transform, the ODE turns into

$$\begin{aligned} \mathcal{L}\{x''\} + 3\mathcal{L}\{x'\} + 2\mathcal{L}\{x\} &= s^2X(s) - sx(0) - x'(0) + 3sX(s) - 3x(0) + 2X(s) \\ &= \frac{1-e^{-10s}}{s} = \frac{1}{s} - \frac{e^{-10s}}{s} = \mathcal{L}\{u(t-10)\} \end{aligned}$$

Then we have

$$(s^2+3s+2)X(s) - sx(0) - x'(0) - 3x(0) = (s+1)(s+2)X(s) = \frac{1-e^{-10s}}{s}$$

and hence

$$X(s) = \frac{1-e^{-10s}}{s(s+1)(s+2)}.$$

By the previous exercise, we know that the Laplace inverse of $\frac{1}{s(s+1)(s+2)}$ is

$$\frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}$$

and the Laplace inverse of $\frac{e^{-10s}}{s(s+1)(s+2)}$ is

$$u(t-10)\left(\frac{1}{2} - e^{-t+10} + \frac{e^{-2t+20}}{2}\right),$$

so the solution of this initial value problem is

$$x(t) = \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} - u(t-10)\left(\frac{1}{2} - e^{-t+10} + \frac{e^{-2t+20}}{2}\right)$$

(ii) $x''(t) + 4x(t) = 4t, \qquad x(0) = 1, \qquad x'(0) = 1$

[Exam question, 2012]

Solution:

Applying the Laplace transform, the ODE turns into

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = s^2X(s) - sx(0) - x'(0) + 4X(s) = \frac{4}{s^2} = \mathcal{L}\{4t\}$$

Then we have

$$(s^2 + 4)X(s) - sx(0) - x'(0) = (s^2 + 4)X(s) - (s + 1) = \frac{4}{s^2}$$

and hence

$$X(s) = \frac{s^3 + s^2 + 4}{s^2(s^2 + 4)}.$$

We apply the method of partial fractions: we want to write $F(s)$ as $\frac{As+B}{s^2} + \frac{Cs+D}{s^2+4}$. Bringing this to a common denominator, we get that the numerator is

$$s^3 + s^2 + 4 = (As + B)(s^2 + 4) + (Cs + D)s^2 = (A + C)s^3 + (B + D)s^2 + 4As + 4B,$$

hence, comparing the coefficients, we see $1 = A + C$, $1 = B + D$, $0 = 4A$ and $4 = 4D$. The solution is $A = 0$, $B = 1$, $C = 1$ and $D = 0$.

So $F(s) = \frac{1}{s^2} + \frac{s}{s^2+4}$. The Laplace inverse of the first term is t and the inverse of the second term is $\cos(2t)$ so $f(t) = t + \cos(2t)$.

(iii) $x''(t) + x(t) = \begin{cases} t/2 & \text{for } t \le 6 \\ 3 & \text{for } t > 6 \end{cases}, \qquad x(0) = 0, \qquad x'(0) = 1$

Solution:

We see that $x''(t) + x(t) = \frac{t}{2} + u(t - 6)(3 - \frac{t}{2})$. Applying the Laplace transform (and using the previous exercise), the ODE turns into

$$\mathcal{L}\{x''\} + \mathcal{L}\{x\} = s^2X(s) - sx(0) - x'(0) + X(s) = \frac{1 - e^{-6s}}{2s^2} = \mathcal{L}\{\frac{t}{2} + u(t-6)(3 - \frac{t}{2})\}$$

Then we have

$$(s^2 + 1)X(s) - sx(0) - x'(0) = (s^2 + 1)X(s) - 1 = \frac{1 - e^{-6s}}{2s^2}$$

and hence

$$X(s) = \frac{1}{2} \cdot \frac{1 + 2s^2 - e^{-6s}}{s^2(s^2+1)}.$$

In the previous exercise we saw that the Laplace inverse of $\frac{e^{-6s}}{s^2(s^2+1)}$ is $u(t-6)(t-6-\sin(t-6))$.

A similar computation shows that the Laplace inverse of $\frac{1+2s^2}{s^2(s^2+1)}$ is $t+\sin(t)$. Hence the solution of this initial value problem is

$$x(t) = \frac{1}{2}\Big(t + \sin(t) - u(t-6)(t-6-\sin(t-6))\Big)$$

Question 5:

For each of the following pairs of functions f and g compute the corresponding convolution $f * g$.

(i) $f(t) = e^{-\omega t}$ and $g(t) = \sin(\omega t)$ for all $t \geq 0$ where ω is a positive real number.

Solution:
Recall from class that:

$$f * g = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\}\right\}$$

Moreover, we know that for $s \geq 0$

$$\mathcal{L}\left\{f\right\}(s) = \frac{1}{s+\omega}$$

and that

$$\mathcal{L}\left\{g\right\}(s) = \frac{\omega}{s^2+\omega^2}.$$

Thus, it follows that

$$\left(\mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\}\right)(s) = \frac{1}{s+\omega} \cdot \frac{\omega}{s^2+\omega^2} = \frac{1}{2\omega}\left(\frac{1}{s+\omega} - \frac{s}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right).$$

Applying the inverse Laplace transform to both sides of the equation above, we conclude by linearity of the Laplace transform that:

$$(f * g)(t) = \frac{1}{2\omega}\left(e^{-\omega t} - \cos(\omega t) + \sin(\omega t)\right)$$

(ii) $f(t) = t^a$, $g(t) = t^b$ for all $t \geq 0$ where a and b are positive integers.

Solution:
Recall from class that:

$$f * g = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\}\right\}$$

Moreover, we know that for $s > 0$

$$\mathcal{L}\left\{t^n\right\}(s) = \frac{n!}{s^{n+1}}$$

for any positive integer n. Thus, it follows that

$$\left(\mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\}\right)(s) = \frac{a!}{s^{a+1}} \cdot \frac{b!}{s^{b+1}} = \frac{a!b!}{(a+b+1)!} \cdot \frac{(a+b+1)!}{s^{a+b+2}}.$$

Since $\mathcal{L}^{-1}\left\{\frac{(a+b+1)!}{s^{a+b+2}}\right\}(t) = t^{a+b+1}$ we conclude by linearity of the Laplace transform that:

$$(f * g)(t) = \frac{a!b!}{(a+b+1)!} \cdot t^{a+b+1}$$

(iii) $f(t) = g(t) = \sin(\omega t)$ for all $t \geq 0$ where ω is some positive real number.

Solution:
By definition of convolution we have for every $t \geq 0$:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t \sin(\omega\tau)\sin(\omega(t-\tau))\,d\tau.$$

It follows from the trigonometric identity

$$\sin(\theta)\sin(\varphi) = \frac{1}{2}\left(\cos(\theta-\varphi) - \cos(\theta+\varphi)\right)$$

that

$$\sin(\omega\tau)\sin(\omega(t-\tau)) = \frac{1}{2}\left(\cos(\omega(2\tau-t)) - \cos(\omega t)\right).$$

Thus, we conclude that

$$\begin{aligned}
(f * g)(t) =& \frac{1}{2}\int_0^t \cos(\omega(2\tau-t))\,d\tau - \frac{1}{2}\int_0^t \cos(\omega t)\,d\tau \\
=& \frac{1}{2}\left[\frac{1}{2\omega}\sin(\omega(2\tau-t))\right]_0^t - \frac{1}{2}t\cos(\omega t) \\
=& \frac{1}{2}\left(\frac{1}{2\omega}\sin(\omega t) - \frac{1}{2\omega}\sin(-\omega t)\right) - \frac{1}{2}t\cos(\omega t) = \frac{1}{2\omega}\sin(\omega t) - \frac{1}{2}t\cos(\omega t)
\end{aligned}$$

(iv) $f(t) = e^{at}$, $g(t) = e^{bt}$ for all $t \geq 0$ where a and b are distinct real numbers.

Solution:
By definition of convolution we have for every $t \geq 0$:

$$\begin{aligned}(f * g)(t) =& \int_0^t f(\tau)g(t-\tau)\,d\tau \\ =& \int_0^t e^{a\tau}e^{b(t-\tau)}\,d\tau \\ =& e^{bt}\int_0^t e^{(a-b)\tau}\,d\tau \\ =& e^{bt}\left[\frac{1}{a-b}e^{(a-b)\tau}\right]_0^t = \frac{e^{at}-e^{bt}}{a-b}\end{aligned}$$

(v) $f(t) = \frac{1}{\sqrt{t}}$, $g(t) = t^2$ for all $t \geq 0$.

Solution:
By definition of convolution we have for every $t \geq 0$:

$$\begin{aligned}(f * g)(t) =& \int_0^t f(\tau)g(t-\tau)\,d\tau \\ =& \int_0^t \frac{1}{\sqrt{\tau}}(t-\tau)^2\,d\tau \\ =& \int_0^t \frac{1}{\sqrt{\tau}}\left(t^2 - 2t\tau + \tau^2\right)\,d\tau \\ =& \left[2t^2\tau^{1/2} - \frac{4}{3}t\tau^{3/2} + \frac{2}{5}\tau^{5/2}\right]_0^t \\ =& 2t^{5/2} - \frac{4}{3}t^{5/2} + \frac{2}{5}t^{5/2} = \frac{16}{15}t^{5/2}\end{aligned}$$

Question 6:

For each of the following functions f, find the solution to

$$x''(t) + \omega^2 x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0$$

where ω is a positive real number.

(i) $f(t) = e^{-\omega t}$ for $t \geq 0$.

Solution:
In lectures, we have seen that the solution for this problem is given by

$$x(t) = \frac{1}{\omega} \int_0^t f(\tau) \sin(\omega(t - \tau))\, d\tau.$$

Note that if $g(t) = \sin(\omega t)$ for $t \geq 0$ then the integral above is by definition equal to $(f * g)(t)$. This convolution is computed in part a) of the previous exercise where we see that

$$(f * g)(t) = \frac{1}{2\omega} \left(e^{-\omega t} - \cos(\omega t) + \sin(\omega t)\right).$$

Thus, we can conclude that

$$x(t) = \frac{1}{2\omega^2} \left(e^{-\omega t} - \cos(\omega t) + \sin(\omega t)\right)$$

(ii) $f(t) = \sin(\omega t)$ for $t \geq 0$.

Solution:
In lectures, we have seen that the solution for this problem is given by

$$x(t) = \frac{1}{\omega} \int_0^t f(\tau) \sin(\omega(t - \tau))\, d\tau.$$

Note that if $g(t) = \sin(\omega t)$ for $t \geq 0$ then the integral above is by definition equal to $(f * g)(t)$. This convolution is computed in part c) of the previous exercise where we see that

$$(f * g)(t) = \frac{1}{2\omega} \sin(\omega t) - \frac{1}{2} t \cos(\omega t).$$

Thus, we can conclude that

$$x(t) = \frac{1}{2\omega^2} \sin(\omega t) - \frac{1}{2\omega} t \cos(\omega t)$$

(iii) $f(t) = 1$ for $t \geq 0$.

Solution:
In lectures, we have seen that the solution for this problem is given by

$$x(t) = \frac{1}{\omega}\int_0^t \sin(\omega\tau) f(t-\tau)\, d\tau.$$

Since $f(t) = 1$ for $t \geq 0$, it follows that

$$x(t) = \frac{1}{\omega}\int_0^t \sin(\omega\tau)\, d\tau = \frac{1}{\omega}\left[-\frac{1}{\omega}\cos(\omega\tau)\right]_0^t = \frac{1}{\omega^2} - \frac{1}{\omega^2}\cos(\omega t)$$

Question 7:

[Exercise 1.2 Lecture 10]

(a) Check that the following identities are correct, where f, g and h are functions and c is a constant:

(i) $f * g = g * f$.

Solution:
Considering the change of variables $\eta(\tau) = t - \tau$, we get:

$$f * g = \int_0^t f(\tau) g(t-\tau) d\tau = \int_t^0 f(t-\eta) g(\eta)(-1) d\eta = g * f.$$

(ii) $f * (g + h) = f * g + f * h$.

Solution:

$$f * (g+h) = \int_0^t f(\tau)(g(t-\tau) + h(t-\tau)) d\tau =$$
$$= \int_0^t f(\tau) g(t-\tau) d\tau + \int_0^t f(\tau) h(t-\tau) d\tau = f * g + f * h.$$

(iii) $(f * g) * h = f * (g * h)$.

Solution:
We will use the change of variables $u(\alpha) = \alpha - \beta$.

$$\begin{aligned}(f * g) * h(t) &= \int_0^t (f * g)(\alpha) h(t-\alpha) d\alpha = \int_0^t \int_0^\alpha f(\beta) g(\alpha - \beta) h(t-\alpha) d\beta d\alpha = \\ &= \int_0^t \int_\beta^t f(\beta) g(\alpha-\beta) h(t-\alpha) d\alpha d\beta = \\ &= \int_0^t f(\beta) \int_\beta^t g(\alpha-\beta) h(t-\alpha) d\alpha d\beta = \\ &= \int_0^t f(\beta) \int_0^{t-\beta} g(u) h(t-u-\beta) du d\beta = \\ &= \int_0^t f(\beta)(g * h)(t-\beta) = f * (g * h).\end{aligned}$$

(b) For each of the following, find an explicit function $f(t)$ such that:

(i) $f * 1 \neq f$,

Solution:
Let $f = 1$. Then

$$1 * 1(t) = \int_0^t 1 d\alpha = t.$$

(ii) $f * f \not\geq 0$.

Solution:
Let $f(t) = \sin(t)$. Then:

$$\begin{aligned}f * f(2\pi) &= \int_0^{2\pi} \sin(\alpha)\sin(2\pi - \alpha) d\alpha = \int_0^{2\pi} \sin(\alpha)\sin(-\alpha) d\alpha = \\ &= \int_0^{2\pi} -\sin^2(\alpha) = -\frac{1}{2}\int_0^{2\pi} 1 - \cos(2\alpha) d\alpha = -\pi\end{aligned}$$