Analysis III (BAUG) **Assignment 11**

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The last exercise of this series is concerned with the proof of Fourier's Theorem. It is harder than the other exercises, but is not impossible. You need to be familiar with trigonometric identities and properties of sin and cos such as orthogonality.

Question 1:

Show, using the definition of convolution, that for any distinct real numbers a and b we have:

$$(f * g)(t) = \frac{a}{b^2 - a^2} (\cos(at) - \cos(bt))$$

if $f(t) = \sin(at)$ and $g(t) = \cos(bt)$ for $t \geq 0$.

Question 2:

Find the solution $x : [0, \infty) \to \mathbb{R}$ to the following integral equations by using the Laplace transform:

(i) $x(t) = \cos(t) + \int_0^t x(\tau)\, d\tau$

[Exam question, 2015]

(ii) $6x(t) = 2t^3 + \int_0^t x(\tau)(t - \tau)^3\, d\tau$

[Exam question, 2015]

Question 3:

Solve the following ODE on $[0, 6\pi]$ with boundary conditions

$$\begin{aligned} \text{ODE}: \quad & y''''(x) + 4y(x) = H(x - 2\pi) - H(x - 4\pi) \\ \text{BC}: \quad & y''(0) = 0, \;\; y'''(0) = 0, \\ & y''(6\pi) = 0, \;\; y'''(6\pi) = 0. \end{aligned}$$

where $H(x)$ is the Heaviside step function. You can use as a fact the following inverses of Laplace transforms:

$$\alpha(x) = \mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4}\right\}(x) = \cos(x)\cosh(x)$$
$$\beta(x) = \mathcal{L}^{-1}\left\{\frac{s^2}{s^4+4}\right\}(x) = \frac{1}{2}\left(\sin(x)\cosh(x) + \cos(x)\sinh(x)\right)$$
$$\gamma(x) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^4+4)}\right\}(x) = \frac{1-\alpha(x)}{4}$$

and the following values:

$$\alpha''(6\pi) = 0, \quad \alpha'''(6\pi) = -2\sinh(6\pi)$$
$$\beta''(6\pi) = \sinh(6\pi), \quad \beta'''(6\pi) = 0$$
$$\gamma''(4\pi) = 0, \quad \gamma'''(4\pi) = \frac{1}{2}\sinh(4\pi)$$
$$\gamma''(2\pi) = 0, \quad \gamma'''(2\pi) = \frac{1}{2}\sinh(2\pi)$$

Question 4:

What is the inverse Laplace transform of $\frac{s}{s^2-6s+10}$?

- ☐ $e^{3t}\cos(t)$
- ☐ $e^{-3t}\cos(t)$
- ☐ $e^{-3t}(\cos(t) + 3\sin(t))$
- ☐ $e^{3t}(\cos(t) + 3\sin(t))$

Suppose that $y(t)$ satisfies

$$y'' - 2y' - y = 1 \qquad y(0) = -1 \quad y'(0) = 1$$

What is the Laplace transform of $y(t)$?

- ☐ $\frac{1}{s^2-2s-1}$
- ☐ $\frac{1}{s(s^2-2s-1)}$
- ☐ $\frac{-s+3}{s^2-2s-1} + \frac{1}{s(s^2-2s-1)}$
- ☐ $\frac{s+1}{s^2-2s-1} + \frac{1}{s(s^2-2s-1)}$

Question 5:

Consider the ODE:

ODE: $f''(t) + \omega_0^2 f(t) = \cos(\eta t)$;

IC: $f(0) = f'(0) = 0$.

Suppose that $\eta \neq \omega_0$.

(i) Check that the following is the solution of the ODE:

$$f(t) = \frac{1}{\eta^2 - \omega_0^2}(\cos(\omega_0 t) - \cos(\eta t)).$$

(ii) How does the solution behave when $|\eta|$ is very close to $|\omega_0|$?

Question 6:

Suppose we have a beam of length 4 (parametrized by $0 \leq x \leq 4$) embedded in the wall at both ends, and suppose $EI = 1$.

For each of the scenarios below, find the deflection curve by first writing up the corresponding ODE with the corresponding boundary conditions (see hints for answer), and then solving the obtained boundary value problem using the Laplace transform.

(i) We apply force $F = 2$ downwards at the point $x = 3$.

(ii) The force we apply is described by the function

$$f(x) = \begin{cases} 0 & \text{for } x < 2 \\ -2 & \text{for } 2 \leq x. \end{cases}$$

Question 7:

The goal of this exercise is to prove Fourier's Theorem. More precisely, let $f\colon [-\pi, \pi]$ be a continuous function, and let FS_f be the Fourier series of f. We will show that for each $t \in [-\pi, \pi]$, we have $\mathrm{FS}_f(t) = f(t)$.

(a) Consider the family of functions $h_k(t) = c_k \left(\frac{1+\cos(t)}{2}\right)^k$, where c_k is such that

$$\int_{-\pi}^{\pi} h_k(t)dt = 1.$$

Show that for each k the function h_k is the finite sum of terms of the form $\sin(nt)$ and $\cos(nt)$ plus a constant (possibly with some coefficients).

(b) Note that the function h_1 has value 1 at $t = 0$, and 0 at $t = \pi, -\pi$. The functions h_k looks more and more like a very high spike which has it maximum at $t = 0$ and is zero almost everywhere else. In particular, you can assume that $\lim_{k\to\infty} h_k(t) = \delta(t)$. Using this fact, show that $f(0) - \mathrm{FS}_f(0) = 0$.

(c) Show that for each $s \in [-\pi, \pi]$ we have $f(s) - \mathrm{FS}_f(s) = 0$. *Hint: How can you write $\delta(t - s)$?*