Analysis III (BAUG) **Assignment 11**

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The last exercise of this series is concerned with the proof of Fourier's Theorem. It is harder than the other exercises, but is not impossible. You need to be familiar with trigonometric identities and properties of sin and cos such as orthogonality.

Question 1:

Show, using the definition of convolution, that for any distinct real numbers a and b we have:

$$(f * g)(t) = \frac{a}{b^2 - a^2} (\cos(at) - \cos(bt))$$

if $f(t) = \sin(at)$ and $g(t) = \cos(bt)$ for $t \geq 0$.

Solution:

By definition of convolution, we have that

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau = \int_0^t \sin(a\tau) \cos(b(t - \tau))\, d\tau$$

It follows from the trigonometric identity

$$\sin(\theta)\cos(\varphi) = \frac{1}{2}(\sin(\theta + \varphi) + \sin(\theta - \varphi))$$

that

$$\sin(a\tau)\cos(b(t-\tau)) = \frac{1}{2}(\sin((a-b)\tau + bt) + \sin((a+b)\tau - bt)).$$

Thus, we conclude that

$$\begin{aligned}
(f * g)(t) =& \frac{1}{2}\int_0^t \sin((a-b)\tau + bt)\, d\tau + \frac{1}{2}\int_0^t \sin((a+b)\tau - bt)\, d\tau \\
=& \frac{1}{2}\left[-\frac{1}{a-b}\cos((a-b)\tau + bt)\right]_0^t + \frac{1}{2}\left[-\frac{1}{a+b}\cos((a+b)\tau - bt)\right]_0^t \\
=& -\frac{1}{2(a-b)}(\cos(at) - \cos(bt)) - \frac{1}{2(a+b)}(\cos(at) - \cos(-bt)) \\
=& \frac{a}{b^2 - a^2}(\cos(at) - \cos(bt))
\end{aligned}$$

Question 2:

Find the solution $x : [0, \infty) \to \mathbb{R}$ to the following integral equations by using the Laplace transform:

(i) $x(t) = \cos(t) + \int_0^t x(\tau)\, d\tau$

[Exam question, 2015]

Solution:
Note first of all that if $y(t) = 1$ for $t \geq 0$ then

$$\int_0^t x(\tau)\, d\tau = \int_0^t x(\tau) y(t-\tau)\, d\tau = (x * y)(t)$$

Denote by $X(s)$ the Laplace transform of $x(t)$. By applying the Laplace transform to both sides of the integral equation we obtain:

$$X(s) = \mathcal{L}\{\cos(t)\}(s) + \mathcal{L}\left\{(x * y)(t)\right\}(s) = \mathcal{L}\{\cos(t)\}(s) + X(s)\mathcal{L}\{y(t)\}(s).$$

Since $\mathcal{L}\{\cos(t)\}(s) = \frac{s}{s^2+1}$ and $\mathcal{L}\{y(t)\}(s) = \frac{1}{s}$ it follows that

$$X(s) = \frac{s}{s^2+1} + \frac{X(s)}{s} \Leftrightarrow X(s) = \frac{s^2}{(s-1)(s^2+1)}.$$

The latter splits into partial fractions as follows:

$$\frac{s^2}{(s-1)(s^2+1)} = \frac{1}{2}\left(\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right)$$

and so

$$\begin{aligned} x(t) =& \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) \\ =& \frac{1}{2}e^t + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) \end{aligned}$$

(ii) $6x(t) = 2t^3 + \int_0^t x(\tau)(t-\tau)^3\, d\tau$

[Exam question, 2015]

Solution:
Note first of all that if $y(t) = t^3$ for $t \geq 0$ then

$$\int_0^t x(\tau)(t-\tau)^3 \, d\tau = \int_0^t x(\tau)y(t-\tau)\, d\tau = (x * y)(t)$$

Denote by $X(s)$ the Laplace transform of $x(t)$. By applying the Laplace transform to both sides of the integral equation we obtain:

$$6X(s) = 2\mathcal{L}\{t^3\}(s) + \mathcal{L}\left\{(x*y)(t)\right\}(s) = 2\mathcal{L}\{t^3\}(s) + X(s)\mathcal{L}\{t^3\}(s).$$

Since $\mathcal{L}\{t^3\}(s) = \frac{6}{s^4}$ it follows that

$$6X(s) = \frac{12}{s^4} + \frac{6X(s)}{s^4} \Leftrightarrow X(s) = \frac{2}{s^4 - 1} = \frac{2}{(s-1)(s+1)(s^2+1)}.$$

The latter splits into partial fractions as follows:

$$\frac{2}{(s-1)(s+1)(s^2+1)} = \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{s^2+1}$$

and so

$$\begin{aligned} x(t) =& \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) \\ =& \frac{1}{2}e^t - \frac{1}{2}e^{-t} - \sin(t) \end{aligned}$$

Question 3:

Solve the following ODE on $[0, 6\pi]$ with boundary conditions

$$\begin{aligned} \text{ODE}: \quad & y''''(x) + 4y(x) = H(x-2\pi) - H(x-4\pi) \\ \text{BC}: \quad & y''(0) = 0, \;\; y'''(0) = 0, \\ & y''(6\pi) = 0, \;\; y'''(6\pi) = 0. \end{aligned}$$

where $H(x)$ is the Heaviside step function. You can use as a fact the following inverses of Laplace transforms:

$$\begin{aligned}\alpha(x) =& \mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4}\right\}(x) = \cos(x)\cosh(x)\\ \beta(x) =& \mathcal{L}^{-1}\left\{\frac{s^2}{s^4+4}\right\}(x) = \frac{1}{2}\left(\sin(x)\cosh(x)+\cos(x)\sinh(x)\right)\\ \gamma(x) =& \mathcal{L}^{-1}\left\{\frac{1}{s(s^4+4)}\right\}(x) = \frac{1-\alpha(x)}{4}\end{aligned}$$

and the following values:

$$\begin{aligned}\alpha''(6\pi) =& 0, \quad \alpha'''(6\pi) = -2\sinh(6\pi)\\ \beta''(6\pi) =& \sinh(6\pi), \quad \beta'''(6\pi) = 0\\ \gamma''(4\pi) =& 0, \quad \gamma'''(4\pi) = \frac{1}{2}\sinh(4\pi)\\ \gamma''(2\pi) =& 0, \quad \gamma'''(2\pi) = \frac{1}{2}\sinh(2\pi)\end{aligned}$$

Solution:

We start by applying Laplace transform to both sides of the ODE. Let $Y(s)$ denote the Laplace transform of y. We know from the lectures that:

$$\mathcal{L}\{y''''(x)\}(s) = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0).$$

Since $y''(0) = y'''(0) = 0$, it follows that the Laplace transform of the left-hand side of the ODE is

$$\mathcal{L}\{y''''(x) + 4y(x)\}(s) = (s^4+4)Y(s) - s^3y(0) - s^2y'(0).$$

Moreover, we know that the Laplace transform of the right-hand side of the ODE is

$$\mathcal{L}\{H(x-2\pi) - H(x-4\pi)\}(s) = \frac{e^{-2\pi s} - e^{-4\pi s}}{s}.$$

Putting the two equations together we conclude that:

$$Y(s) = \frac{s^3}{s^4+4}y(0) + \frac{s^2}{s^4+4}y'(0) + \frac{1}{s(s^4+4)}(e^{-2\pi s} - e^{-4\pi s}).$$

Applying the inverse Laplace transform we then obtain:

$$\begin{aligned}y(x) =& \mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4}\right\}(x)y(0) + \mathcal{L}^{-1}\left\{\frac{s^2}{s^4+4}\right\}(x)y'(0) + \mathcal{L}^{-1}\left\{\frac{1}{s(s^4+4)}(e^{-2\pi s} - e^{-4\pi s})\right\}(x)\\ =& \alpha(x)y(0) + \beta(x)y'(0) + \gamma(x-2\pi)H(x-2\pi) - \gamma(x-4\pi)H(x-4\pi)\end{aligned}$$

where in the last equality we use the property of shifting to compute the last two terms.

Thus, to determine $y(x)$ it remains to compute the values of $y(0)$ and $y'(0)$. Note that for $x > b$ we have

$$y(x) = \alpha(x)y(0) + \beta(x)y'(0) + \gamma(x - 2\pi) - \gamma(x - 4\pi)$$

and so we have

$$0 = y''(6\pi) = \alpha''(6\pi)y(0) + \beta''(6\pi)y'(0) + \gamma''(4\pi) - \gamma''(2\pi) = \sinh(6\pi)y'(0) \Rightarrow y'(0) = 0$$

and

$$\begin{aligned} 0 = y'''(6\pi) =& \alpha'''(6\pi)y(0) + \beta'''(6\pi)y'(0) + \gamma'''(4\pi) - \gamma'''(2\pi) \\ =& -2\sinh(6\pi)y(0) + \frac{\sinh(4\pi) - \sinh(2\pi)}{2} \Rightarrow y(0) \\ =& \frac{\sinh(4\pi) - \sinh(2\pi)}{4\sinh(6\pi)} \end{aligned}$$

Question 4:

What is the inverse Laplace transform of $\frac{s}{s^2-6s+10}$?

- ☐ $e^{3t}\cos(t)$
- ☐ $e^{-3t}\cos(t)$
- ☐ $e^{-3t}(\cos(t) + 3\sin(t))$
- ☐ $e^{3t}(\cos(t) + 3\sin(t))$

Solution:

We have

$$\frac{s}{s^2 - 6s + 10} = \frac{s-3}{(s-3)^2+1} + \frac{3}{(s-3)^2+1}$$

The inverse Laplace transfrom of $\frac{s-3}{(s-3)^2+1}$ is $e^{3t}\cos t$.
The inverse Laplace transfrom of $\frac{3}{(s-3)^2+1}$ is $3e^{3t}\sin t$.
This shows that the inverse Laplace transfrom of $\frac{s}{s^2-6s+10}$ is $e^{-3t}(\cos(t) + 3\sin(t))$

Suppose that $y(t)$ satisfies

$$y'' - 2y' - y = 1 \qquad y(0) = -1 \quad y'(0) = 1$$

What is the Laplace transform of $y(t)$?

- ☐ $\frac{1}{s^2-2s-1}$
- ☐ $\frac{1}{s(s^2-2s-1)}$
- ☐ $\frac{-s+3}{s^2-2s-1} + \frac{1}{s(s^2-2s-1)}$
- ☐ $\frac{s+1}{s^2-2s-1} + \frac{1}{s(s^2-2s-1)}$

Solution:
Applying the Laplace transform, the ODE turns into

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} - \mathcal{L}\{y\} = s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] - Y(s) = s^{-1} = \mathcal{L}\{1\}$$

Plugging in $y(0)$ and $y'(0)$ and rearranging gives:

$$Y(S) = \frac{1+3s-s^2}{s(s^2-2s-1)} = \frac{-s+3}{s^2-2s-1} + \frac{1}{s(s^2-2s-1)}$$

Question 5:

Consider the ODE:

ODE: $f''(t) + \omega_0^2 f(t) = \cos(\eta t)$;
IC: $f(0) = f'(0) = 0$.

Suppose that $\eta \neq \omega_0$.

(i) Check that the following is the solution of the ODE:

$$f(t) = \frac{1}{\eta^2 - \omega_0^2}(\cos(\omega_0 t) - \cos(\eta t)).$$

Solution:
Computing we obtain:

$$f'(t) = \frac{1}{\eta^2 - \omega_0^2}\left(-\omega_0 \sin(\omega_0 t) + \eta \sin(\eta t)\right)$$

$$f''(t) = \frac{1}{\eta^2 - \omega_0^2}\left(-\omega_0^2 \cos(\omega_0 t) + \eta^2 \cos(\eta t)\right).$$

Thus we have

$$f''(t) + \omega_0^2 f(t) = \frac{1}{\eta^2 - \omega_0^2}(\eta^2 \cos(\eta t) - \omega_0^2 \cos(\eta t)) = \cos(\eta t).$$

(ii) How does the solution behave when $|\eta|$ is very close to $|\omega_0|$?

Solution:
There are 4 possible cases such that $|\eta|$ is very close to $|\omega_0|$. Indeed η can be either close to ω_0 or to $-\omega_0$, and $|\eta|$ can be either bigger or smaller than $|\omega_0|$). For simplicity, we assume that η is close to ω_0, and that $\eta > \omega_0$. The other cases are completely analogous.

We observe that if η is very close to ω_0, then the quantity $\frac{1}{\eta^2 - \omega_0^2}$ is very big. Let's call it M Moreover, let $\epsilon = \frac{\eta - \omega}{2}$.

Using the trigonometric identity $\cos(\theta) - \cos(\phi) = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$, we obtain that the solution has the form:

$$f(t) = -2M \sin(\epsilon t)\sin((\eta - \epsilon)t).$$

Thus the function $\sin((\eta - \epsilon)t)$ is multiplied by $-2M\sin(\epsilon t)$, that is, the function slowly grows to the maximum $-2M$, but then, instead of growing indefinitely, it grows back to zero. All of this happens over a period of $\frac{2\pi}{\epsilon}$.

Question 6:

Suppose we have a beam of length 4 (parametrized by $0 \le x \le 4$) embedded in the wall at both ends, and suppose $EI = 1$.

For each of the scenarios below, find the deflection curve by first writing up the corresponding ODE with the corresponding boundary conditions (see hints for answer), and then solving the obtained boundary value problem using the Laplace transform.

(i) We apply force $F = 2$ downwards at the point $x = 3$.

Solution:
The ODE we want to solve is $y''''(x) = -2\delta(x-3)$ with boundary conditions $y(0) = y'(0) = y(4) = y'(4) = 0$.

Applying the Laplace transform to this ODE, we get

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4Y(s) - sy''(0) - y'''(0) = -2e^{-3s}.$$

Let $A = y''(0)$ and $B = y'''(0)$, then $s^4Y(s) - As - B = -2e^{-3s}$, hence

$$Y(s) = \frac{As + B - e^{-3s}}{s^4} = \frac{A}{s^3} + \frac{B}{s^4} - \frac{2e^{-3s}}{s^4}.$$

Taking the Laplace inverse, we get

$$y(x) = \frac{A}{2}x^2 + \frac{B}{6}x^3 - \frac{1}{3}(x-3)^3u(x-3).$$

This function must also satisfy the remaining two boundary conditions :

$$0 = y(4) = 8A + \frac{32B}{3} - \frac{1}{3}$$

and, since $u(x-3) = 1$ around the point $x = 4$, $y'(x) = Ax + \frac{B}{2}x^2 - (x-3)^2$ around $x = 4$, thus

$$0 = y'(4) = 4A + 8B - 1.$$

Solving this, we get $A = -3/8$ and $B = 5/16$, so the deflection curve is

$$y(x) = -\frac{3}{16}x^2 + \frac{5}{96}x^3 - \frac{1}{3}(x-3)^3u(x-3).$$

(ii) The force we apply is described by the function

$$f(x) = \begin{cases} 0 & \text{for } x < 2 \\ -2 & \text{for } 2 \leq x. \end{cases}$$

Solution:
The ODE we want to solve is $y''''(x) = -2u(x-2)$ with boundary conditions $y(0) = y'(0) = y(4) = y'(4) = 0$.

Applying the Laplace transform to this ODE, we get

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4Y(s) - sy''(0) - y'''(0) = -\frac{2e^{-2s}}{s}.$$

Let $A = y''(0)$ and $B = y'''(0)$, then $s^4Y(s) - As - B = -\frac{2e^{-2s}}{s}$, hence

$$Y(s) = \frac{As^2 + Bs - 2e^{-2s}}{s^5} = \frac{A}{s^3} + \frac{B}{s^4} - \frac{2e^{-2s}}{s^5}.$$

Taking the Laplace inverse, we get

$$y(x) = \frac{A}{2}x^2 + \frac{B}{6}x^3 - \frac{1}{12}(x-2)^4u(x-2).$$

This function must also satisfy the remaining two boundary conditions :

$$0 = y(4) = 8A + \frac{32B}{3} - \frac{4}{3}$$

and, since $u(x-2) = 1$ around the point $x = 4$, $y'(x) = Ax + \frac{B}{2}x^2 - \frac{1}{3}(x-2)^3$ around $x = 4$, thus

$$0 = y'(4) = 4A + 8B - \frac{8}{3}.$$

Solving this, we get $A = -5/6$ and $B = 3/4$, so the deflection curve is

$$y(x) = -\frac{5}{12}x^2 + \frac{1}{8}x^3 - \frac{1}{12}(x-2)^4u(x-2).$$

Question 7:

The goal of this exercise is to prove Fourier's Theorem. More precisely, let $f\colon [-\pi, \pi]$ be a continuous function, and let FS_f be the Fourier series of f. We will show that for each $t \in [-\pi, \pi]$, we have $\mathrm{FS}_f(t) = f(t)$.

(a) Consider the family of functions $h_k(t) = c_k\left(\frac{1+\cos(t)}{2}\right)^k$, where c_k is such that

$$\int_{-\pi}^{\pi} h_k(t)dt = 1.$$

Show that for each k the function h_k is the finite sum of terms of the form $\sin(nt)$ and $\cos(nt)$ plus a constant (possibly with some coefficients).

Solution:
It is clear that the result holds for h_1. Suppose that h_k can be written as a sum of terms of the form $\cos(nt)$ and $\sin(nt)$. Then also $h_k\left(\frac{1+\cos(t)}{2}\right)$ can be written in such a way. Indeed, we obtain the result using the standard trigonometric identities that allows to pass from a product to a sum.

(b) Note that the function h_1 has value 1 at $t = 0$, and 0 at $t = \pi, -\pi$. The functions h_k looks more and more like a very high spike which has it maximum at $t = 0$ and is zero almost everywhere else. In particular, you can assume that $\lim_{k\to\infty} h_k(t) = \delta(t)$. Using this fact, show that $f(0) - \mathrm{FS}_f(0) = 0$.

Solution:
Recall that for each $n \in \mathbb{N}$, with $n > 0$, we have $\int_{-\pi}^{\pi} \cos(nt)dt = \int_{-\pi}^{\pi} \sin(nt)dt = 0$. To simplify notation, let $g(t) = f(t) - \mathrm{FS}_f(t)$. We get:

$$\int_{-\pi}^{\pi} g(t)dt = 2\pi a_0 - \int_{-\pi}^{\pi} a_0 dt = 0.$$

Similarly, using the orthogonality of the trigonometric polynomials (see Lecture 4), we obtain:

$$\int_{-\pi}^{\pi} g(t)\cos(nt)dt = \pi a_n - \pi a_n = 0$$
$$\int_{-\pi}^{\pi} g(t)\sin(nt)dt = \pi b_n - \pi b_n = 0.$$

Since the functions $h_h = k$ are of the form:

$$h_k = c_k(1 + d_1\cos(t) + e_1\sin(t) + \cdots + d_s\cos(st) + e_s\sin(st))$$

for some s (that is, they are sum of terms $\cos(nt)$ and $\sin(nt)$), we obtain

$$\int_{-\pi}^{\pi} g(t)h_k(t)dt =$$
$$= c_k\int_{-\pi}^{\pi} g(t)dt + c_k d_1\int_{-\pi}^{\pi} g(t)\cos(t)dt + \dots$$
$$\dots + c_k e_s\int_{-\pi}^{\pi} g(t)\sin(st)dt = 0$$

Taking the limit for $k \to \infty$, we obtain

$$0 = \int_{-\pi}^{\pi} g(t)\delta(t) = f(0) - \mathrm{FS}_f(0).$$

Thus they coincide on $t = 0$.

(c) Show that for each $s \in [-\pi, \pi]$ we have $f(s) - \mathrm{FS}_f(s) = 0$. *Hint: How can you write $\delta(t-s)$?*

Solution:
We want to "shift by s" the procedure of part (b). That is, we want to show that for each s and k, we have:

$$\int_{-\pi}^{\pi} g(t)h_k(t-s)dt = 0.$$

Since the limit of $h_K(t-s)$ converges to $\delta(t-s)$, if the above is true for all k the result follows.

Note that $h_k(t-s)$ can be written as a sum of terms of the form $\sin(n(t-s))$ or $\cos(n(t-s))$. Using standard trigonometric inequalities, we obtain:

$$\sin(n(t-s)) = \sin(nt - ns) = \sin(nt)\cos(ns) - \cos(nt)\sin(ns)$$
$$\cos(n(t-s)) = \cos(nt)\cos(ns) + \sin(nt)\sin(ns)$$

Note that $\cos(ns)$ and $\sin(ns)$ are constants! Thus we have

$$\int_{-\pi}^{\pi} g(t)\sin(n(t-s))dt = \cos(ns)\int_{-\pi}^{\pi} g(t)\sin(nt)dt - \sin(ns)\int_{-\pi}^{\pi} g(t)\cos(nt)dt = 0$$
$$\int_{-\pi}^{\pi} g(t)\cos(n(t-s))dt = \cos(ns)\int_{-\pi}^{\pi} g(t)\cos(nt)dt - \sin(ns)\int_{-\pi}^{\pi} g(t)\sin(nt)dt = 0$$

Taking the limit for $k \to \infty$, we obtain that for each s:

$$0 = \int_{-\pi}^{\pi} g(t)\delta(t-s)dt = g(s),$$

thus $\mathrm{FS}_f = f$.