Analysis III (BAUG) **Assignment 12**

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The first 4 questions of this exercise sheet review the topics connected to beams (static and dynamics). You should try to solve at least one point for each of them.

Question 5 is Exercise 4.1 of Lecture 10. Giving a full correct solution may be not easy, but it is a good exercise to think at least a little bit about it.

Question 1:

For each of the following pairs of functions f, g, find the solution to the following IBVP:

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = -u_{xxxx}(x,t) && \text{for } 0 < x < 1 \text{ and } t > 0 \\ \text{BC}: \quad & u(0,t) = 0, \ \ u_{xx}(0,t) = 0, \ \ u(1,t) = 0, \ \ u_{xx}(1,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: \quad & u(x,0) = f(x), \ \ u_t(x,0) = g(x) && \text{for } 0 \leq x \leq 1 \end{aligned}$$

(i) $f(x) = g(x) = \sin(\pi x)$ for $x \in [0,1]$.

Solution:
Recall from the lectures that the solution to this general IBVP is given by:

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x) \left(a_n \sin((n\pi)^2 t) + b_n \cos((n\pi)^2 t) \right)$$

where the constants a_n and b_n satisfy:

$$\begin{cases} u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \\ u_t(x,0) = g(x) = \sum_{n=1}^{\infty} (n\pi)^2 a_n \sin(n\pi x) \end{cases}$$

In this case, since $f(x) = g(x) = \sin(\pi x)$, it is clear that $b_1 = 1$, $b_n = 0$ for $n \geq 2$, $a_1 = \frac{1}{\pi^2}$ and $a_n = 0$ for $n \geq 2$. Thus, we obtain that the solution in this case is:

$$u(x,t) = \sin(\pi x) \left(\frac{1}{\pi^2} \sin(\pi^2 t) + \cos(\pi^2 t) \right)$$

(ii) $f(x) = 1 - x^2$ and $g(x) = 0$ for $x \in [0,1]$.

Solution:
Recall from the lectures that the solution to this general IBVP is given by:

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(a_n \sin((n\pi)^2 t) + b_n \cos((n\pi)^2 t)\right)$$

where the constants a_n and b_n satisfy:

$$\begin{cases} u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \\ u_t(x,0) = g(x) = \sum_{n=1}^{\infty} (n\pi)^2 a_n \sin(n\pi x) \end{cases}$$

In this case, since $g(x) = 0$, it is clear that $a_n = 0$ for $n \geq 1$. Note that the b_n's are the coefficients of the sine Fourier series of f (in the interval $[0,1]$). Thus, we have:

$$\begin{aligned} b_n =& 2\int_0^1 (1-x^2)\sin(n\pi x)\,dx \\ =& 2\left[-(1-x^2)\frac{1}{n\pi}\cos(n\pi x)\right]_0^1 - 2\int_0^1 2x\frac{1}{n\pi}\cos(n\pi x)\,dx \\ =& \frac{2}{n\pi} - \frac{4}{n\pi}\left[x\frac{1}{n\pi}\sin(n\pi x)\right]_0^1 + \frac{4}{n\pi}\int_0^1 \frac{1}{n\pi}\sin(n\pi x)\,dx \\ =& \frac{2}{n\pi} + \frac{4}{n^2\pi^2}\left[-\frac{1}{n\pi}\cos(n\pi x)\right]_0^1 = \frac{2}{n\pi} + \frac{4\,(1-(-1)^n)}{n^3\pi^3} \end{aligned}$$

We conclude that the solution in this case is:

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{n\pi} + \frac{4\,(1-(-1)^n)}{n^3\pi^3}\right)\sin(n\pi x)\cos((n\pi)^2 t)$$

Question 2:

Suppose we have a beam of length 10 (parametrized by $0 \leq x \leq 10$) embedded in the wall at one end, free at the other, and suppose $EI = 1$.

For each of the scenarios below, find the deflection curve by first writing up the corresponding ODE with the corresponding boundary conditions (see hints for answer), and then solving the obtained boundary value problem using the Laplace transform.

(i) We apply force $F = 1$ downwards at the point $x = 5$.

Solution:
The ODE we want to solve is $y''''(x) = -\delta(x-5)$ with boundary conditions $y(0) = y'(0) = y''(10) = y'''(10) = 0$.

Applying the Laplace transform to this ODE, we get

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0) = s^2(s^2+1)Y(s)-sy''(0)-y'''(0) = -e^{-5s}.$$

Let $A = y''(0)$ and $B = y'''(0)$, then $s^4Y(s) - As - B = -e^{-5s}$, hence

$$Y(s) = \frac{As + B - e^{-5s}}{s^4} = \frac{A}{s^3} + \frac{B}{s^4} - \frac{e^{-5s}}{s^4}$$

Applying the inverse Laplace transform, we obtain

$$y(x) = \frac{A}{2}x^2 + \frac{B}{6}x^3 - \frac{1}{6}(x-5)^3u(x-5).$$

This also needs to satisfy the boundary conditions at $x = 10$, and using that $u(x-5) = 1$ around $x = 10$ we get $y''(x) = A + Bx - (x-5)$ and $y'''(x) = B - 1$, so

$$0 = y''(10) = A + 10B - 5$$
$$0 = y'''(10) = B - 1.$$

Solving this we get $B = 1$ and $A = -5$, hence the solution of the ODE is

$$y(x) = -\frac{5}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{6}(x-5)^3u(x-5).$$

(ii) The force we apply is described by the function

$$f(x) = \begin{cases} 0 & \text{for } x < 4 \\ 8 - 2x & \text{for } 4 \leq x. \end{cases}$$

Solution:
The ODE we want to solve is $y''''(x) = -2u(x-4)(x-4)$ with boundary conditions $y(0) = y'(0) = y''(10) = y'''(10) = 0$.

Applying the Laplace transform to this ODE, we get

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4Y(s) - sy''(0) - y'''(0) = -\frac{2e^{-4s}}{s^2}.$$

Let $A = y''(0)$ and $B = y'''(0)$, then $s^4Y(s) - As - B = -\frac{2e^{-4s}}{s^2}$, hence

$$Y(s) = \frac{As^3 + Bs^2 - 2e^{-4s}}{s^5} = \frac{A}{s^3} + \frac{B}{s^4} - \frac{2e^{-4s}}{s^6}.$$

Taking the Laplace inverse, we get

$$y(x) = \frac{A}{2}x^2 + \frac{B}{6}x^3 - \frac{1}{60}(x-4)^5u(x-4).$$

This function must also satisfy the remaining two boundary conditions. Using that $u(x-4) = 1$ around the point $x = 10$, we have $y''(x) = A + Bx - \frac{1}{3}(x-4)^3$ and $y'''(x) = B - (x-4)^2$. Therefore,

$$0 = y''(10) = A + 10B - 72$$

and

$$0 = y'''(10) = B - 36.$$

Solving this, we get $B = 36$ and $A = -288$, so the deflection curve is

$$y(x) = -144x^2 + 6x^3 - \frac{1}{60}(x-4)^4u(x-4).$$

Question 3:

Suppose we have a beam of length 3 (parametrized by $0 \leq x \leq 3$) simply supported at both ends, and suppose $EI = 4$

For each of the scenarios below, find the deflection curve by first writing up the corresponding ODE with the corresponding boundary conditions (see hints for answer), and then solving the obtained boundary value problem using the Laplace transform.

(i) We simultaneously apply force $F = 1$ downwards at both of the points $x = 1$ and $x = 2$.

Solution:
The ODE we want to solve is $4y''''(x) = -\delta(x-1) - \delta(x-2)$ with boundary conditions $y(0) = y''(0) = y(3) = y''(3) = 0$.

Applying the Laplace transform to this ODE, we get

$$4s^4Y(s) - 4s^3y(0) - 4s^2y'(0) - 4sy''(0) - 4y'''(0) = 4s^4Y(s) - 4s^2y'(0) - 4y'''(0) = -e^{-s} - e^{-2s}.$$

Let $A = y'(0)$ and $B = y'''(0)$, then $4s^4Y(s) - 4As^2 - 4B = -e^{-s} - e^{-2s}$, hence

$$Y(s) = \frac{4As^2 + 4B - e^{-s} - e^{-2s}}{4s^4} = \frac{A}{s^2} + \frac{B}{s^4} - \frac{e^{-s}}{4s^4} - \frac{e^{-2s}}{4s^4}.$$

Taking the Laplace inverse, we get

$$y(x) = Ax + \frac{B}{6}x^3 - \frac{1}{24}(x-1)^3u(x-1) - \frac{1}{24}(x-2)^3u(x-2).$$

This function must also satisfy the remaining two boundary conditions :

$$0 = y(3) = 3A + \frac{9B}{2} - \frac{1}{3} - \frac{1}{24}$$

and, since $u(x-1) = u(x-2) = 1$ around the point $x = 3$, $y''(x) = Bx - \frac{1}{4}(x-1) - \frac{1}{4}(x-2)$ around $x = 3$, thus

$$0 = y''(3) = 3B - \frac{1}{2} - \frac{1}{4}.$$

Solving this, we get $A = -1/4$ and $B = 1/4$, so the deflection curve is

$$y(x) = -\frac{1}{4}x + \frac{1}{24}x^3 - \frac{1}{24}(x-1)^3u(x-1) - \frac{1}{24}(x-2)^3u(x-2).$$

(ii) The force we apply is described by the function

$$f(x) = \begin{cases} 1 & \text{for } x < 2 \\ 0 & \text{for } 2 \leq x. \end{cases}$$

Solution:
The ODE we want to solve is $4y''''(x) = 1 - u(x-2)$ with boundary conditions $y(0) = y''(0) = y(3) = y''(3) = 0$.

Applying the Laplace transform to this ODE, we get

$$4s^4Y(s) - 4s^3y(0) - 4s^2y'(0) - 4sy''(0) - 4y'''(0) = 4s^4Y(s) - 4s^2y'(0) - 4y'''(0) = \frac{1-e^{-2s}}{s}.$$

Let $A = y'(0)$ and $B = y'''(0)$, then $4s^4Y(s) - 4As^2 - 4B = \frac{1-e^{-2s}}{s}$, hence

$$Y(s) = \frac{4As^3 + 4Bs + 1 - 1e^{-2s}}{4s^5} = \frac{A}{s^2} + \frac{B}{s^4} + \frac{1}{4s^5} - \frac{e^{-2s}}{4s^5}.$$

Taking the Laplace inverse, we get

$$y(x) = Ax + \frac{B}{6}x^3 + \frac{1}{96}x^4 - \frac{1}{96}(x-2)^4 u(x-2).$$

This function must also satisfy the remaining two boundary conditions :

$$0 = y(3) = 3A + \frac{9B}{2} + \frac{27}{32} - \frac{1}{96} = 3A + \frac{9B}{2} + \frac{5}{6}$$

and, since $u(x-2) = 1$ around the point $x = 3$, $y''(x) = Bx + \frac{1}{8}x^2 - \frac{1}{8}(x-2)^2$ around $x = 3$, thus

$$0 = y''(3) = 3B + \frac{9}{8} - \frac{1}{8} = 3B + 1.$$

Solving this, we get $A = 2/9$ and $B = -1/3$, so the deflection curve is

$$y(x) = \frac{2}{9}x - \frac{1}{18}x^3 + \frac{1}{96}x^4 - \frac{1}{96}(x-2)^4 u(x-2).$$

Question 4:

Suppose we have a water tower (beam column) of length 10 (parametrized by $0 \leq x \leq 10$) and load $W = 10$ and $EI = 10$. Suppose that the lateral forces are described by the following function:

$$f(x) = \begin{cases} 0 & \text{for } x < 5 \\ 20 & \text{for } 5 \leq x. \end{cases}$$

What is the corresponding ODE for the deflection curve. Solve the ODE to find the deflection curve.

You can use the following formulas in your solutions:

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = 1 - \cos(x)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = x - \sin(x)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^3(s^2+1)}\right) = -1 + \frac{x^2}{2} + \cos(x)$$

Solution:

Here $f(x) = 20u(x-5)$, so the ODE we want to solve is $y''''(x) + y''(x) = 2u(x-5)$ with boundary conditions $y(0) = y'(0) = y''(10) = y'''(10) = 0$.

Applying the Laplace transform to this ODE, we get

$$\begin{aligned} &s^4 Y(s) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0) + s^2 Y(s) - sy(0) - y'(0) \\ &= s^2(s^2+1)Y(s) - sy''(0) - y'''(0) = \frac{2e^{-5s}}{s}. \end{aligned}$$

Let $A = y''(0)$ and $B = y'''(0)$, then $s^2(s^2+1)Y(s) - As - B = \frac{2e^{-5s}}{s}$, hence

$$Y(s) = \frac{As^2 + Bs + 2e^{-5s}}{s^3(s^2+1)} = \frac{A}{s(s^2+1)} + \frac{B}{s^2(s^2+1)} + \frac{2e^{-5s}}{s^3(s^2+1)}$$

Using the formulas above, we see that the inverse Laplace transform is

$$y(x) = A - A\cos(x) + Bx - B\sin(x) - 2u(x-5) + (x-5)^2 u(x-5) + 2\cos(x-5)u(x-5).$$

Now this also needs to satisfy the boundary conditions at $x = 10$, and using that $u(x-5) = 1$ around $x = 10$ we get $y''(x) = A\cos(x) + B\sin(x) + 2 - 2\cos(x-5)$ and $y'''(x) = -A\sin(x) + B\cos(x) + 2\sin(x-5)$, so

$$0 = y''(10) = A\cos(10) + B\sin(10) + 2 - 2\cos(5)$$

$$0 = y'''(10) = -A\sin(10) + B\cos(10) + 2\sin(5).$$

We can solve this e.g. by multiplying the first equation by $\sin(10)$, the second by $\cos(10)$ and adding them together. Then we get:

$$0 = B(\sin^2(10) + \cos^2(10)) + 2\sin(10) - 2(\sin(10)\cos(5) - \sin(5)\cos(10)) = B + 2\sin(10) - 2\sin(5)$$

and hence $B = 2\sin(5) - 2\sin(10)$. Similarly, multiplying the first one by $\cos(10)$, the second one by $\sin(10)$ and looking at the difference, we get

$$0 = A(\cos^2(10) + \sin^2(10)) + 2\cos(10) - 2(\cos(10)\cos(5) + \sin(10)\sin(5)) = A + 2\cos(10) - 2\cos(5),$$

hence $A = 2\cos(5) - 2\cos(10)$. Plugging this back in the above formula for $y(x)$, we obtain the deflection curve.

$$\begin{aligned} y(x) =& 2\cos(5) - 2\cos(10) - (2\cos(5) - 2\cos(10))\cos(x) + (2\sin(5) - 2\sin(10))x \\ &- (2\sin(5) - 2\sin(10))\sin(x) - 2u(x-5) + (x-5)^2 u(x-5) + 2\cos(x-5)u(x-5). \end{aligned}$$

Question 5:

The goal of this exercise is to show that for any $a > 0$, the Dirac delta function $\delta(t-a)$ behave as the "derivative" of the Heaviside step function $u(t-a)$. Recall that the delta function is defined to be such that for each integrable function f, the following holds:

$$\int_0^\infty f(t)\delta(t-a)dt = f(a).$$

Show that the "derivative" of the Heaviside step function behaves as δ.

Solution:
There are several ways to show this. Here we present 2 of them. If your solution uses different ideas and you want to know if it is correct, you can write directly to davide.spriano@math.ethz.ch .

With the definition of δ Let $u'(t-a)$ be the derivative of $u(t-a)$ (note that the derivative of $u(t-a)$ is not well defined, but pretend it was). Let $f\colon [0,\infty) \to \mathbb{R}$ be any integrable function. Note that this implies $\lim_{t\to\infty} f(t) = 0$. Then using integration by parts we have:

$$\int_0^\infty f(t)u'(t-a)dt = f(t)u(t-a)|_0^\infty - \int_0^\infty f'(t)u(t-a)dt =$$
$$= 0 - 0 - \int_a^\infty f'(t)dt = f(t)|_a^\infty = f(a).$$

Thus the "derivative" of the Heaviside step function has to behave exactly like the delta function. In this sense, we say that $\delta(t-a)$ is the derivative of $u(t-a)$.

With Laplace We show that the delta function behaves like the derivative of the Heaviside function with respect to the Laplace transform. Again, let $a > 0$.

Recall that

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}.$$

Moreover, we have that

$$\mathcal{L}\{u'(t-a)\} = s\left(\frac{e^{-as}}{s}\right) - u(0-a) = e^{-as} = \mathcal{L}\{\delta(t-a)\}.$$

In particular, δ behaves as the derivative of u with respect to the Laplace transform.