Analysis III (BAUG) **Assignment 2**

Prof. Dr. Alessandro Sisto — Due 4th October 2018

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Question 1

Consider the following PDEs:

$$u_x(x,t) + e^{-2t}u_{xt}(x,t) = F(x,t) \tag{1}$$
$$u_x(x,t) + e^{-2t}u_{xt}(x,t) = 0. \tag{2}$$

and suppose $v(x,t)$ satisfies (1) and $w(x,t)$ satisfies (2).

(i) Is $v(x,t) + w(x,t)$ a solution of (1)?

(ii) Let α and β be any constants. Is $\alpha v(x,t) + \beta w(x,t)$ a solution of (1)?

(iii) Let β be any constant. Is $v(x,t) + \beta w(x,t)$ a solution of (1)?

(iv) Let α and β be any constants. Is $\alpha v(x,t) + \beta w(x,t)$ a solution of the following:

$$u_x(x,t) + e^{-2t}u_{xt}(x,t) = \alpha F(x,t)?$$

Question 2

Let $v(x,t)$ be a solution of $u_t = k \cdot u_{xx}$ for some k. Show that the following statements hold:

(i) For any constants a, t', x', the function $w(x,t) = v(ax - x', a^2t - t')$ satisfies $u_t = k \cdot u_{xx}$.

(ii) The function

$$w(x,t) = \frac{1}{\sqrt{t}} \cdot e^{-x^2/4kt} \cdot v\left(\frac{x}{t}, \frac{-1}{t}\right)$$

satisfies $u_t = k \cdot u_{xx}$.

Question 3

[Remark 1.1 Lecture 2]

Let $\alpha \in \mathbf{R}, L > 0$ and assume that $u(x,t)$ is a solution for:

$$\text{IBVP (a)} : \begin{cases} u_t(x,t) = \alpha^2 u_{xx}(x,t) & \text{in } \Omega = (0,L) \times (0,\infty), \\ u(0,t) = u(L,t) = 0 & \text{for all } t > 0, \\ u(x,0) = \phi(x) & \text{for all } x \in (0,L). \end{cases}$$

Show that the function $v(x,t)$ defined as $v(x,t) = u(x,t) + c$ is a solution of

$$\text{IBVP (b)} : \begin{cases} v_t(x,t) = \alpha^2 v_{xx}(x,t) & \text{in } \Omega = (0,L) \times (0,\infty), \\ v(0,t) = v(L,t) = c & \text{for all } t > 0, \\ v(x,0) = \phi(x) + c & \text{for all } x \in (0,L). \end{cases}$$

Question 4

Use the general formula from lectures to solve the following PDE with the initial condition $\phi(x)$ described below.

$$
\begin{aligned}
PDE: &\quad u_t(x,t) = 4u_{xx}(x,t) &\quad 0 < x < 10, t > 0 \\
BC: &\quad u(0,t) = u(10,t) = 0 &\quad t \geq 0 \\
IC: &\quad u(x,0) = \phi(x) &\quad 0 \leq x \leq 10
\end{aligned}
$$

In particular, for each of the following $\phi(x)$ determine for which values of n the coefficient a_n is nonzero, and compute it.

(i) $\phi(x) = \sin(5\pi x) - 3\sin(\pi x)$

(ii) $\phi(x) = 2\sin(\frac{7\pi x}{2})$

(iii) $\phi(x) = \sin(3\pi x) + 2\cos(\frac{(6x+5)\pi}{10})$

(iv) $\phi(x) = 6\cos^2(\pi x - \frac{\pi}{4}) - 3$

Question 5

Decide if the following statements are true.

1. If $v(x,t)$ is a solution of $u_{xx}(x,t) + u_{tt}(x,t) = 0$ then so is $v(-x,t)$. [Exam question, 2008-9]
2. The PDE $u_{tt} - (1+t^2)u_{xx} = 0$ is linear and hyperbolic. [Exam question, 2008-9]

Question 6

Note! This question is harder then the ones above, but it is more interesting. It is not compulsory to solve it, but we encourage you to try to do at least some parts of it.

Consider a laterally insulated thin rod as the standard example in the lecture. Suppose that the initial temperature of the rod ranges between -10 and 10 degrees. The physical intuition suggests that, while time passes, the temperature will stay in the range between -10 and 10 degrees. Indeed, we would be very surprised if, at some point, the temperature of a segment of the rod will be -50 degrees.

In this exercise we will show that the solutions of the IBVP presented in the lecture agree with our physical intuition. We will show the, so called, *maximum principle* that states that for a PDE u defined on a set $\Omega = [0,L] \times [0,1]$, the maximal and minimal value of u during time coincide with the ones of u at the initial state (that is, with the ones of $\phi(x)$). This will be done by parts.

Part I Let u be a function that satisfy the following IBVP, that we will call IBVP (c):

PDE $u_t - u_{xx} = c > 0$ for $(x,t) \in \Omega = [0,L] \times [0,1]$ (note the c!);

BC $u(0,t) = u(L,t) = 0$ for all $t > 0$;

IC $u(x,0) = \phi(x)$.

Let (x_0, t_0) be a minimum for the value of u. Show that (x_0, t_0) cannot be a point of $(0, L) \times (0, 1)$.

Part II Show that if (x_0, t_0) is a minimum, it cannot happen that $t_0 = 1$. In particular deduce that if u satisfies the above PDE, then the minimal value of u coincides with $\min_{x\in[0,L]}\{\phi(x)\}$.

Part III We want now to substitute the PDE setting $c = 0$. Let u be a function that satisfy the following IBVP, that we will call IBVP (0):

PDE $u_t - u_{xx} = 0$ (not c!) for $(x,t) \in \Omega = [0, L] \times [0, 1]$;

BC $u(0,t) = u(L,t) = 0$ for all $t > 0$;

IC $u(x,0) = \phi(x)$.

Show that the minimal value of u coincide with $\min_{x\in[0,L]}\{\phi(x)\}$. *(Hint: modify the function u so that the IBVP (c) is satisfied, for some arbitrarily small c. Then apply Part II).*

Part IV Show that for a function u satisfying IBVP (0), the maximum and minimal value of u coincide with the maximal and minimal value of ϕ.