Analysis III (BAUG) **Assignment 2**

Prof. Dr. Alessandro Sisto Due 4th October 2018

Organizer: Davide Spriano

Question 1

Consider the following PDEs:

$$u_x(x,t) + e^{-2t}u_{xt}(x,t) = F(x,t) \tag{1}$$

$$u_x(x,t) + e^{-2t}u_{xt}(x,t) = 0. \tag{2}$$

and suppose $v(x,t)$ satisfies (1) and $w(x,t)$ satisfies (2).

(i) Is $v(x,t) + w(x,t)$ a solution of (1)?

Solution:

Yes ✓

No

$(v+w)_x + e^{-2t}(v+w)_{xt} = v_x + e^{-2t}v_{xt} + w_x + e^{-2t}w_{xt} = F + 0$

(ii) Let α and β be any constants. Is $\alpha v(x,t) + \beta w(x,t)$ a solution of (1)?

Solution:

Yes

No ✓

$(\alpha v + \beta w)_x + e^{-2t}(\alpha v + \beta w)_{xt} = \alpha(v_x + e^{-2t}v_{xt}) + \beta(w_x + e^{-2t}w_{xt}) = \alpha \cdot F + \beta \cdot 0$

But αF is not necessarily equal to F.

(iii) Let β be any constant. Is $v(x,t) + \beta w(x,t)$ a solution of (1)?

Solution:

Yes ✓

No

$(v + \beta w)_x + e^{-2t}(v + \beta w)_{xt} = v_x + e^{-2t}v_{xt} + \beta(w_x + e^{-2t}w_{xt}) = F + \beta \cdot 0$

(iv) Let α and β be any constants. Is $\alpha v(x,t)+\beta w(x,t)$ a solution of the following:

$$u_x(x,t)+e^{-2t}u_{xt}(x,t)=\alpha F(x,t)?$$

Solution:

Yes ✓

No

$$(\alpha v+\beta w)_x+e^{-2t}(\alpha v+\beta w)_{xt}=\alpha(v_x+e^{-2t}v_{xt})+\beta(w_x+e^{-2t}w_{xt})=\alpha\cdot F+\beta\cdot 0$$

Question 2

Let $v(x,t)$ be a solution of $u_t=k\cdot u_{xx}$ for some k. Show that the following statements hold:

(i) For any constants a,t',x', the function $w(x,t)=v(ax-x',a^2t-t')$ satisfies $u_t=k\cdot u_{xx}$.

Solution:
Using the chain rule, we see that

$$\frac{\partial^2}{\partial x^2}\big(v(ax-x',a^2t-t')\big)=\frac{\partial}{\partial x}\big(av_x(ax-x',a^2t-t')\big)=a^2v_{xx}(ax-x',a^2t-t')$$

and

$$\frac{\partial}{\partial t}\big(v(ax-x',a^2t-t')\big)=a^2v_t(ax-x',a^2t-t'),$$

so

$$w_t(x,t)=a^2v_t(ax-x',a^2t-t')=a^2kv_{xx}(ax-x',a^2t-t')=kw_{xx}(x,t).$$

(ii) The function

$$w(x,t)=\frac{1}{\sqrt{t}}\cdot e^{-x^2/4kt}\cdot v\left(\frac{x}{t},\frac{-1}{t}\right)$$

satisfies $u_t=k\cdot u_{xx}$.

Solution:
Using the chain rule, we see that

$$\begin{aligned}
w_{xx}(x,t)&=\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{t}}\cdot e^{-x^2/4kt}\cdot v\left(\frac{x}{t},\frac{-1}{t}\right)\right)\\
&=\frac{\partial}{\partial x}\frac{1}{\sqrt{t}}\cdot e^{-x^2/4kt}\cdot\left(\frac{-2x}{4kt}v\left(\frac{x}{t},\frac{-1}{t}\right)+\frac{1}{t}v_x\left(\frac{x}{t},\frac{-1}{t}\right)\right)\\
&=\frac{1}{\sqrt{t}}\cdot e^{-x^2/4kt}\cdot\left(\left(\frac{x^2}{4k^2t^2}-\frac{1}{2kt}\right)v\left(\frac{x}{t},\frac{-1}{t}\right)-\frac{x}{kt^2}v_x\left(\frac{x}{t},\frac{-1}{t}\right)+\frac{1}{t^2}v_{xx}\left(\frac{x}{t},\frac{-1}{t}\right)\right)
\end{aligned}$$

and

$$w_t(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{t}} \cdot e^{-x^2/4kt} \cdot v\left(\frac{x}{t}, \frac{-1}{t}\right)\right)$$
$$= e^{-x^2/4kt} \cdot \left(\left(\frac{x^2}{4kt^{5/2}} - \frac{1}{2t^{3/2}}\right) v\left(\frac{x}{t}, \frac{-1}{t}\right) + \frac{1}{\sqrt{t}}\left(\frac{1}{t^2} v_t\left(\frac{x}{t}, \frac{-1}{t}\right) - \frac{x}{t^2} v_x\left(\frac{x}{t}, \frac{-1}{t}\right)\right)\right)$$

Now applying the condition on v, we get that w also satisfies the PDE.

Question 3

[Remark 1.1 Lecture 2]

Let $\alpha \in \mathbf{R}, L > 0$ and assume that $u(x,t)$ is a solution for:

$$\text{IBVP (a)} : \begin{cases} u_t(x,t) = \alpha^2 u_{xx}(x,t) & \text{in } \Omega = (0,L) \times (0,\infty), \\ u(0,t) = u(L,t) = 0 & \text{for all } t > 0, \\ u(x,0) = \phi(x) & \text{for all } x \in (0,L). \end{cases}$$

Show that the function $v(x,t)$ defined as $v(x,t) = u(x,t) + c$ is a solution of

$$\text{IBVP (b)} : \begin{cases} v_t(x,t) = \alpha^2 v_{xx}(x,t) & \text{in } \Omega = (0,L) \times (0,\infty), \\ v(0,t) = v(L,t) = c & \text{for all } t > 0, \\ v(x,0) = \phi(x) + c & \text{for all } x \in (0,L). \end{cases}$$

Solution:

We start by verifying the PDE:

$$v_t(x,t) = \frac{\partial}{\partial t}(u(x,t) + c) = u_t(x,y)$$
$$v_{xx}(x,t) = \frac{\partial^2}{\partial x^2}(u(x,t) + c) = u_{xx}(x,t).$$

Thus, we have $v_t(x,t) = \alpha^2 v_{xx}(x,t)$. Moreover, we have:

$$v(0,t) = u(0,t) + c = c,$$

and similarly for $v(L,t)$. Finally:

$$v(x,0) = u(x,0) + c = \phi(x) = c.$$

Thus $v(x,t)$ is a solution of IBVP(b).

Question 4

Use the general formula from lectures to solve the following PDE with the initial condition $\phi(x)$ described below.

$$
\begin{aligned}
PDE: &\quad u_t(x,t) = 4u_{xx}(x,t) &\quad 0 < x < 10, t > 0\\
BC: &\quad u(0,t) = u(10,t) = 0 &\quad t \geq 0\\
IC: &\quad u(x,0) = \phi(x) &\quad 0 \leq x \leq 10
\end{aligned}
$$

In particular, for each of the following $\phi(x)$ determine for which values of n the coefficient a_n is nonzero, and compute it.

(i) $\phi(x) = \sin(5\pi x) - 3\sin(\pi x)$

Solution:
From lectures, we know that the for constants $L > 0$, $a_1, \ldots, a_N$, for the IBVP

$$
\begin{aligned}
PDE: &\quad u_t(x,t) = \alpha^2 u_{xx}(x,t) &\quad (x,t) \in (0,L) \times (0,\infty)\\
BC: &\quad u(0,t) = u(L,t) = 0 &\quad \text{for all } t > 0\\
IC: &\quad u(x,0) = \sum_{n=1}^{N} a_n \sin(\frac{n\pi}{L}x) &\quad \text{for all } x \in [0,L]
\end{aligned}
$$

the solution is

$$u(x,t) = \sum_{n=1}^{N} a_n e^{-(\frac{n\pi}{L})^2\alpha^2 t} \sin(\frac{n\pi}{L}x).$$

Note that in our case $\alpha^2 = 4$ and $L = 10$. The initial condition is $\sum_{n=1}^{N} a_n \sin(\frac{n\pi}{10}x) = \sin(5\pi x) - 3\sin(\pi x)$.

There are two values n_1 and n_2 such that the corresponding coefficients are nonzero. Here for the first term, $a_{n_1} = 1$ and $\frac{n_1\pi}{L} = 5\pi$, and for the second term $a_{n_2} = -3$ and $\frac{n_2\pi}{L} = \pi$, while all other coefficients a_n are 0. Hence the corresponding solution is

$$e^{-(5\pi)^2 4t} \sin(5\pi x) - 3e^{-\pi^2 4t} \sin(\pi x) = e^{-100\pi^2 t} \sin(5\pi x) - 3e^{-4\pi^2 t} \sin(\pi x).$$

The nonzero coefficients are n_1 and n_2, where $\frac{n_1\pi}{10} = 5\pi$, so $n_1 = 50$, and $\frac{n_2\pi}{10} = \pi$, so $n_2 = 10$.

(ii) $\phi(x) = 2\sin(\frac{7\pi x}{2})$

Solution:
Using the formula, the solution is

$$2e^{-(7\pi/2)^2 4t} \sin(\frac{7\pi x}{2}) = 2e^{-49\pi^2 t} \sin(\frac{7\pi x}{2}).$$

The corresponding nonzero coefficient n satisfies $\frac{n\pi}{10} = \frac{7\pi}{2}$, so $n = 35$.

(iii) $\phi(x) = \sin(3\pi x) + 2\cos(\frac{(6x+5)\pi}{10})$

Solution:
Using $\cos(t + \frac{\pi}{2}) = -\sin(t)$, we see that

$$\phi(x) = \sin(3\pi x) - 2\sin(\frac{6x\pi}{10}).$$

Then, using the formula, the solution is

$$e^{-36\pi^2 t}\sin(3\pi x) - 2e^{-(6\pi/10)^2 4t}\sin(\frac{6\pi x}{10}).$$

The corresponding nonzero coefficient n_1 and n_2 satisfy $\frac{n_1\pi}{10} = 3\pi$ and $\frac{n_2\pi}{10} = \frac{6\pi}{10}$, so $n_1 = 30$ and $n_2 = 6$.

(iv) $\phi(x) = 6\cos^2(\pi x - \frac{\pi}{4}) - 3$

Solution:
Using $\cos(2t) = 2\cos^2(t) - 1$ and $\sin(t) = \cos(t - \pi/2)$, we see that

$$\phi(x) = 3\cos(2\pi x - \frac{\pi}{2}) = 3\sin(2\pi x).$$

The solution is then $3e^{-16\pi^2 t}\sin(2\pi x)$, and for the nonzero coefficient n, we have $\frac{n\pi}{10} = 2\pi$, giving $n = 20$.

Question 5

Decide if the following statements are true.

1. If $v(x,t)$ is a solution of $u_{xx}(x,t) + u_{tt}(x,t) = 0$ then so is $v(-x,t)$. [Exam question, 2008-9]

Solution:
Using the chain rule, we see that

$$\frac{\partial^2}{\partial x^2}\big(v(-x,t)\big) = \frac{\partial}{\partial x}\big(-v_x(-x,t)\big) = v_{xx}(-x,t)$$

and

$$\frac{\partial^2}{\partial t^2}\big(v(-x,t)\big) = \frac{\partial}{\partial t}\big(v_t(-x,t)\big) = v_{tt}(-x,t),$$

so

$$\frac{\partial^2}{\partial x^2}\big(v(-x,t)\big) + \frac{\partial^2}{\partial t^2}\big(v(-x,t)\big) = v_{xx}(-x,t) + v_{tt}(-x,t) = 0$$

by the assumption on $v(x,t)$. Hence the function $v(-x,t)$ also satisfies the PDE.

2. The PDE $u_{tt} - (1+t^2)u_{xx} = 0$ is linear and hyperbolic. [Exam question, 2008-9]

> **Solution:**
> Partial derivatives of the function u are not multiplied together, so the PDE is linear.
>
> $$B^2 - 4AC = 0 - 1 \cdot (-(1+t^2)) = 1 + t^2 > 0,$$
>
> so the equation is hyperbolic.

Question 6

Note! This question is harder then the ones above, but it is more interesting. It is not compulsory to solve it, but we encourage you to try to do at least some parts of it.

Consider a laterally insulated thin rod as the standard example in the lecture. Suppose that the initial temperature of the rod ranges between -10 and 10 degrees. The physical intuition suggests that, while time passes, the temperature will stay in the range between -10 and 10 degrees. Indeed, we would be very surprised if, at some point, the temperature of a segment of the rod will be -50 degrees.

In this exercise we will show that the solutions of the IBVP presented in the lecture agree with our physical intuition. We will show the, so called, *maximum principle* that states that for a PDE u defined on a set $\Omega = [0, L] \times [0, 1]$, the maximal and minimal value of u during time coincide with the ones of u at the initial state (that is, with the ones of $\phi(x)$). This will be done by parts.

Part I Let u be a function that satisfy the following IBVP, that we will call IBVP (c):

PDE $u_t - u_{xx} = c > 0$ for $(x, t) \in \Omega = [0, L] \times [0, 1]$ (note the c!);

BC $u(0, t) = u(L, t) = 0$ for all $t > 0$;

IC $u(x, 0) = \phi(x)$.

Let (x_0, t_0) be a minimum for the value of u. Show that (x_0, t_0) cannot be a point of $(0, L) \times (0, 1)$.

> **Solution:**
> Suppose that (x_0, t_0) lies in the interior of Ω. Then, since it is a minimum, we have that $u_t(x_0, t_0) = 0$ and $u_{xx}(x_0, t_0) \geq 0$. Indeed, the first derivative of a function around a minimum is zero, and the second is non negative.
>
> If we plug in the above value in the PDE, we get that $u_t(x_0, t_0) - u_{xx}(x_0, t_0) \leq 0$, which is a contradiction.

Part II Show that if (x_0, t_0) is a minimum, it cannot happen that $t_0 = 1$. In particular deduce that if u satisfies the above PDE, then the minimal value of u coincides with $\min_{x \in [0,L]}\{\phi(x)\}$.

Solution:
Suppose, by contradiction, that $(x_0, 1)$ is a minimum. This implies that x_0 is a minimum for the one-variable function $u(x, 1)$. Thus, $u_{xx}(x_0, 1) \geq 0$. Then we have that $u_t(x_0, 1) = u_{xx}(x_0, 1) + c > 0$. This implies that there is $t < 1$ such that $u(x_0, t) < u(x_0, 1)$, which is a contradiction.

Hence $(x_0, t_0) \in [0, L] \times \{0\} \cup \{0, L\} \times [0, 1]$. Since u is constant on $\{0, L\} \times [0, 1]$, we get the conclusion.

Part III We want now to substitute the PDE setting $c = 0$. Let u be a function that satisfy the following IBVP, that we will call IBVP (0):

PDE $u_t - u_{xx} = 0$ (not c!) for $(x, t) \in \Omega = [0, L] \times [0, 1]$;

BC $u(0, t) = u(L, t) = 0$ for all $t > 0$;

IC $u(x, 0) = \phi(x)$.

Show that the minimal value of u coincide with $\min_{x \in [0,L]}\{\phi(x)\}$. *(Hint: modify the function u so that the IBVP (c) is satisfied, for some arbitrarily small c. Then apply Part II).*

Solution:
Consider the family of functions $u_\epsilon(x, t)$ defined as

$$u_\epsilon(x, t) = u(x, t) - \epsilon x(L - x).$$

Since $\frac{\partial u_\epsilon}{\partial t} - \frac{\partial^2 u_\epsilon}{\partial x^2} = u_t - u_x x + 2\epsilon$, we get that each of the functions u_ϵ satisfy the following IBVP:

PDE $u_t - u_{xx} = 2\epsilon$ for $(x, t) \in \Omega = [0, L] \times [0, 1]$;

BC $u(0, t) = u(L, t) = 0$ for all $t > 0$;

IC $u(x, 0) = \phi(x) + \epsilon x(L - x)$.

In particular, we can apply part III to get that the minimal value assumed by u_ϵ coincide with $\min_{x \in [0,L]}\{\phi(x) + \epsilon x(L - x)\}$. Moreover, it is clear that for any point (x_1, t_1) is happens $u(x_1, t_1) \leq u_\epsilon(x_1, t_1)$. Considering the limit for $\epsilon \to 0$ gives the result.

Part IV Show that for a function u satisfying IBVP (0), the maximum and minimal value of u coincide with the maximal and minimal value of ϕ.

Solution:
The function $v = -u$ satisfy the IBVP obtained substituting ϕ with $-\phi$ in IBVP (0). Since the minimum of v is the maximum of u, and since the minimum of $-\phi$ is the maximum of ϕ, the solution follows from Part III applied to v.