Analysis III (BAUG) **Assignment 3**

Prof. Dr. Alessandro Sisto Due 11th October 2018

This exercise sheet has many Questions, it is ok to not solve all of them. The most important is Question 3, which has the "more standard exercises": it is important that you practice it.

Question 1

Let $a_0, \ldots, a_n$ be constants. Consider the function.

$$\phi(x) = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Show that $a_0 = \frac{1}{L}\int_0^L \phi(x)dx$.

Question 2

Consider the following IBVPs:

$$(1) \quad \begin{cases} \text{PDE}: & u_x(x,t) + e^{-t}u_{xt}(x,t) = 0 & \text{for } 0 < x < 3 \text{ and } t > 0 \\ \text{BC}: & u(0,t) = 0, \ \ u(3,t) = 0 & \text{for } t \geq 0 \end{cases}$$

$$(2) \quad \begin{cases} \text{PDE}: & u_x(x,t) + e^{-t}u_{xt}(x,t) = 0 & \text{for } 0 < x < 3 \text{ and } t > 0 \\ \text{BC}: & u(0,t) = t, \ \ u(3,t) = t^2 & \text{for } t \geq 0 \end{cases}$$

and suppose $v(x,t)$ satisfies (1) and $w(x,t)$ satisfies (2).

(i) Does $v(x,t) + w(x,t)$ satisfy (1)?

(ii) Does $v(x,t) + w(x,t)$ satisfy (2)?

(iii) Does $\alpha v(x,t) + \beta w(x,t)$ satisfy (2) for any constants α and β?

(iv) Does $\alpha v(x,t) + w(x,t)$ satisfy (2) for any constant α?

Question 3

For each of the corresponding physical situations, write the corresponding IBVP and solve it for the given initial condition.

A) Consider a laterally insulated rod of length 3, parametrised using the coordinate x with $x \in (0,3)$, and with diffusion coefficient $\alpha^2 = 4$. Suppose heat flows in this rod starting with an initial temperature distribution given by $\phi(x)$ for $x \in (0,3)$ whilst both ends have their temperature fixed at 0 degrees at all times.

- Write the corresponding IBVP.

- Solve the IBVP in the case where $\phi(x) = 10\sin(2\pi x) - 4\sin(4\pi x)$.

B) Consider a laterally insulated rod of length 4, parametrised using the coordinate x with $x \in (0, 4)$, and with diffusion coefficient $\alpha^2 = 4$. Suppose heat flows in this rod starting with an initial temperature distribution given by $\phi(x)$ for $x \in (0, 4)$ whilst both ends are insulated, meaning that they do not allow any heat to pass.

- Write the corresponding IBVP.
- Solve the IBVP in the case where $\phi(x) = 5 - 2\cos\left(\frac{\pi x}{4}\right) + 3\cos(\pi x)$.

C) Consider an insulated circular wire of length 10, parametrised using the coordinate x with $x \in (-5, 5)$, and with diffusion coefficient $\alpha^2 = 16$. Suppose heat flows in this wire starting from an initial temperature distribution given by $\phi(x)$ for $x \in (-5, 5)$.

- Write the corresponding IBVP.
- Solve the IBVP in the case where $\phi(x) = -3 - 4\sin(3\pi x) + 30\cos(2\pi x)$.

Question 4

The goal of this exercise is to derive the solution of the following IBVP:

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = u_{xx}(x,t) && \text{for } 0 < x < L \text{ and } t > 0 \\ \text{BC}: &\quad u_x(0,t) = 0, \quad u(L,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = \phi(x) && \text{for } 0 \leq x \leq L \end{aligned}$$

for a suitable initial distribution $\phi(x)$.

(i) Use the Ansatz $u(x,t) = f(x)g(t)$ and proceed as in the lecture (for a more precise reference, Lecture 2, part I) to obtain a non-zero solutions of the above PDE that depends on 3 parameters c_1, c_2, c_3.

(ii) Plug in the expressions for f and g obtained into the boundary conditions and obtain conclusions on the parameters c_1, c_2, c_3. Remember that we are only interested in non-zero solutions.

(iii) Conclude that for each integer n the function

$$u_n(x,t) = e^{-\left(\left(n+\frac{1}{2}\right)\frac{\pi}{L}\right)^2 t} \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi x}{L}\right)$$

satisfies the PDE and the boundary conditions of the IBVP above.

(iv) Using the superposition principle deduce that if $\phi(x) = \sum_{n=1}^{N} a_n \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi x}{L}\right)$, for some constants $a_1, \ldots, a_N$ and for some positive integer N, then

$$u(x,t) = \sum_{n=1}^{N} a_n e^{-\left(\left(n+\frac{1}{2}\right)\frac{\pi}{L}\right)^2 t} \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi x}{L}\right)$$

satisfies the IBVP above.

(v) Can you conjecture which heat problem this IBVP models? Use the interpretation of the BCs used in the examples you saw in lectures.

Question 5

1. Indicate which of the following PDEs is linear:

   (i) $10xu_{xt} + e^{x+t}u_x u = 0$

   (ii) $u_{xx} + t^{100}u_{tt} - x^3 u_x = e^{xt}u_t$

   (iii) $u_{xxy} - 20u_{xyy} + 100u_{xx} + e^{x^2}u = x + 3y$

   (iv) $u_{xx} + 30u_{xy} + 2u_x u_y = x^{10} + u^2$

2. Consider the following PDE:

   $$u_{xx} + 2u_{xy} + 2u_{yy} + u_y = 0$$

   This PDE is Linear/Nonlinear? Homogenous/not homogenous? Parabolic/elliptic/hyperbolic? [Exam question, 2012-2013]

3. Which of the following PDEs are homogeneous? (Check all that apply.)

   (i) $u_{xy} + e^{1+y^2}u_x + u = 2$

   (ii) $u_{xy} - u_{yz} + 100u_x - 2u_y = 30u$

   (iii) $u_{xxx} + e^{x^2}u_{xyx} - y^2 u_x = 0$

   (iv) $ye^x u_{xxy} + u_{xy} - 10u_y = e^{x+y^3}$

   (v) $u_{xy} + x^3 y u_{xx} + \log(1 + x^2) - 10yu_{yy} + u = 0$

Question 6

Note! This question is harder then the ones above, but it is more interesting. It is not compulsory to solve it, but we encourage you to try to do at least some parts of it.

Consider the following IVBP:

PDE $u_t - u_{xx} = 0$ for $(x, t) \in \Omega = [0, L] \times [0, \infty)$;

BC $u(0, t) = u(L, t) = 0$ for all $t > 0$;

IC $u(x, 0) = \phi(x)$.

The physical intuition tells us that, no matter which is the initial temperature $\phi(x)$ of the rod, after enough time the temperature will stabilize to 0 degrees. Indeed, for some explicit values of $\phi(x)$, we are able to compute an explicit solution and hence compute the limit $\lim_{t\to\infty} u(x, t)$ and observe that tends to zero, regardless of the point x. The goal of this exercise is to compute the, so called, solution at infinity for a general (!) function $\phi(x)$, and show that is, indeed, the constant function 0.

Part I Let $\psi, \psi' \colon [0, L] \to \mathbb{R}$ be two continuous functions such that, for each $x \in [0, L]$ one has $\psi(x) \le \psi'(x)$. Let u be a solution of the IVBP with $\phi(x) = \psi(x)$ and v a solution $\phi(x) = \psi'(x)$.

Show that for each $(x, t) \in \Omega$, one has $u(x, t) \le v(x, t)$.

Part II Let $\psi(x) \colon [0, L] \to \mathbb{R}$ be a function such that $\psi'(0) < \infty$, $\psi'(L) < \infty$. This is a technical condition that can be assumed to be true in any physical system.

Show that for u satisfying the above IVBP with $\phi(x) = \psi(x)$, for each $x_0 \in [0, L]$, we have

$$\lim_{t \to \infty} u(x_0, t) = 0.$$

Question 7

The goal of this exercise is to derive the solution of the following IBVP:

PDE $u_t - u_{xx} = 0$ for $(x, t) \in \Omega = [0, L] \times [0, \infty)$;

BC $u(0, t) = c_1$, $u(L, t) = c_2$ for all $t > 0$;

IC $u(x, 0) = \phi(x)$.

(i) For a suitable ϕ, the above IBVP admits a solution u_0 that does not depend on t. Try to guess such an u_0 and verify that satisfy the PDE and BC. *Hint: A more formal strategy may be to use the Ansatz $u(x, t) = f(x)g(t)$ and consider the case $\frac{g'(t)}{g(t)} = K = 0$.*

(ii) Use the superposition principle to solve the above IBVP for all initial conditions ϕ of the form:

$$\phi(x) = u_0(x, 0) + \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x}{L}\right).$$