Analysis III (BAUG) **Assignment 3**
Prof. Dr. Alessandro Sisto Due 11th October 2018

This exercise sheet has many Questions, it is ok to not solve all of them. The most important is Question 3, which has the "more standard exercises": it is important that you practice it.

Question 1

Let $a_0, \ldots, a_n$ be constants. Consider the function.

$$\phi(x) = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Show that $a_0 = \frac{1}{L}\int_0^L \phi(x)dx$.

Solution:
Since the integral is additive, we have

$$\int_0^L a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) dx = \int_0^L a_0 dx + \sum_{n=1}^{N} a_n \int_0^L \cos\left(\frac{n\pi x}{L}\right) dx.$$

It is easily seen that $a_0 = \frac{1}{L}\int_0^L a_0 dx$, thus we only need to show that $\int_0^L \cos\left(\frac{n\pi x}{L}\right) dx = 0$ for each $n > 0$.

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) dx = \left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L = 0 - 0 = 0.$$

Question 2

Consider the following IBVPs:

$$(1) \quad \begin{cases} \text{PDE}: & u_x(x,t) + e^{-t}u_{xt}(x,t) = 0 & \text{for } 0 < x < 3 \text{ and } t > 0 \\ \text{BC}: & u(0,t) = 0, \;\; u(3,t) = 0 & \text{for } t \geq 0 \end{cases}$$

$$(2) \quad \begin{cases} \text{PDE}: & u_x(x,t) + e^{-t}u_{xt}(x,t) = 0 & \text{for } 0 < x < 3 \text{ and } t > 0 \\ \text{BC}: & u(0,t) = t, \;\; u(3,t) = t^2 & \text{for } t \geq 0 \end{cases}$$

and suppose $v(x,t)$ satisfies (1) and $w(x,t)$ satisfies (2).

(i) Does $v(x,t) + w(x,t)$ satisfy (1)?

Solution:
No, because for example $v(0,t) + w(0,t) = 0 + t \neq 0$ for $t > 0$.

(ii) Does $v(x,t) + w(x,t)$ satisfy (2)?

Solution:
Yes. Since $v(x,t)$ and $w(x,t)$ satisfy the same homogeneous PDE as in (2), then so does $v(x,t) + w(x,t)$. Moreover, since $v(0,t) + w(0,t) = 0 + t = t$ and $v(3,t) + w(3,t) = 0 + t^2 = t^2$ for all $t \geq 0$, $v(x,t) + w(x,t)$ also satisfies the boundary conditions of (2).

(iii) Does $\alpha v(x,t) + \beta w(x,t)$ satisfy (2) for any constants α and β?

Solution:
No, because for example $\alpha v(0,t) + \beta w(0,t) = 0 + \beta t \neq t$ for $t > 0$ and $\beta \neq 1$.

(iv) Does $\alpha v(x,t) + w(x,t)$ satisfy (2) for any constant α?

Solution:
Yes. Since $v(x,t)$ and $w(x,t)$ satisfy the same homogeneous PDE as in (2), then so does $\alpha v(x,t) + w(x,t)$ for any constant α. Moreover, since $\alpha v(0,t) + w(0,t) = 0 + t = t$ and $\alpha v(3,t) + w(3,t) = 0 + t^2 = t^2$ for all $t \geq 0$, $\alpha v(x,t) + w(x,t)$ also satisfies the boundary conditions of (2) for any constant α.

Question 3

For each of the corresponding physical situations, write the corresponding IBVP and solve it for the given initial condition.

A) Consider a laterally insulated rod of length 3, parametrised using the coordinate x with $x \in (0,3)$, and with diffusion coefficient $\alpha^2 = 4$. Suppose heat flows in this rod starting with an initial temperature distribution given by $\phi(x)$ for $x \in (0,3)$ whilst both ends have their temperature fixed at 0 degrees at all times.

- Write the corresponding IBVP.

Solution:

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 4u_{xx}(x,t) && \text{for } 0 < x < 3 \text{ and } t > 0 \\ \text{BC}: &\quad u(0,t) = 0, u(3,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = \phi(x) && \text{for } 0 \leq x \leq 3 \end{aligned}$$

- Solve the IBVP in the case where $\phi(x) = 10\sin(2\pi x) - 4\sin(4\pi x)$.

Solution:
In this problem we have $\alpha^2 = 4$, $L = 3$ and $\phi(x) = 10\sin(2\pi x) - 4\sin(4\pi x)$. Thus, from the formulas seen in the lectures we conclude that the function

$$u(x,t) = 10e^{-16\pi^2 t}\sin(2\pi x) - 4e^{-64\pi^2 t}\sin(4\pi x)$$

is the unique solution to the IBVP.

B) Consider a laterally insulated rod of length 4, parametrised using the coordinate x with $x \in (0,4)$, and with diffusion coefficient $\alpha^2 = 4$. Suppose heat flows in this rod starting with an initial temperature distribution given by $\phi(x)$ for $x \in (0,4)$ whilst both ends are insulated, meaning that they do not allow any heat to pass.

- Write the corresponding IBVP.

Solution:

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 4u_{xx}(x,t) && \text{for } 0 < x < 4 \text{ and } t > 0 \\ \text{BC}: &\quad u_x(0,t) = 0, u_x(4,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = \phi(x) && \text{for } 0 \leq x \leq 4 \end{aligned}$$

- Solve the IBVP in the case where $\phi(x) = 5 - 2\cos\left(\frac{\pi x}{4}\right) + 3\cos(\pi x)$.

Solution:
In this problem we have $\alpha^2 = 4$, $L = 4$ and $\phi(x) = 5 - 2\cos\left(\frac{\pi x}{4}\right) + 3\cos(\pi x)$. Thus, from the formulas seen in the lectures we conclude that the function

$$u(x,t) = 5 - 2e^{-\left(\frac{\pi}{4}\right)^2 2^2 t}\cos\left(\frac{\pi x}{4}\right) + 3e^{-(\pi)^2 2^2 t}\cos(\pi x)$$

$$= 5 - 2e^{-\frac{\pi^2}{4}t} \cos\left(\frac{\pi x}{4}\right) + 3e^{-4\pi^2 t} \cos(\pi x)$$

is the unique solution to the IBVP.

C) Consider an insulated circular wire of length 10, parametrised using the coordinate x with $x \in (-5, 5)$, and with diffusion coefficient $\alpha^2 = 16$. Suppose heat flows in this wire starting from an initial temperature distribution given by $\phi(x)$ for $x \in (-5, 5)$.

- Write the corresponding IBVP.

Solution:

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 16u_{xx}(x,t) && \text{for } -5 < x < 5 \text{ and } t > 0 \\ \text{BC}: &\quad u(-5,t) = u(5,t),\ u_x(-5,t) = u_x(5,t) && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = \phi(x) && \text{for } -5 \leq x \leq 5 \end{aligned}$$

- Solve the IBVP in the case where $\phi(x) = -3 - 4\sin(3\pi x) + 30\cos(2\pi x)$.

Solution:
In this problem we have $\alpha^2 = 16$, $L = 5$ and $\phi(x) = -3 - 4\sin(3\pi x) + 30\cos(2\pi x)$. Thus, from the formulas seen in the lectures we conclude that the function

$$\begin{aligned} u(x,t) &= -3 - 4e^{-(3\pi)^2 4^2 t} \sin(3\pi x) + 30e^{-(2\pi)^2 4^2 t} \cos(2\pi x) \\ &= -3 - 4e^{-144\pi^2 t} \sin(3\pi x) + 30e^{-64\pi^2 t} \cos(2\pi x) \end{aligned}$$

is the unique solution to the IBVP.

Question 4

The goal of this exercise is to derive the solution of the following IBVP:

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = u_{xx}(x,t) && \text{for } 0 < x < L \text{ and } t > 0 \\ \text{BC}: &\quad u_x(0,t) = 0,\ \ u(L,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = \phi(x) && \text{for } 0 \leq x \leq L \end{aligned}$$

for a suitable initial distribution $\phi(x)$.

(i) Use the Ansatz $u(x,t) = f(x)g(t)$ and proceed as in the lecture (for a more precise reference, Lecture 2, part I) to obtain a non-zero solutions of the above PDE that depends on 3 parameters c_1, c_2, c_3.

Solution:
If $u(x,t) = f(x)g(t)$ then $u_t(x,t) = f(x)\frac{\mathrm{d}g}{\mathrm{d}t}(t) = f(x)g'(t)$ and $u_{xx}(x,t) = \frac{\mathrm{d}^2 f}{\mathrm{d}t^2}(x)g(t) = f''(x)g(t)$. Thus, since $u_t(x,t) = u_{xx}(x,t)$ it follows that $f(x)g'(t) = f''(x)g(t)$ and hence $\frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)}$.

Since the left-hand side only depends on t and the right-hand side only depends on x, it follows that both sides must equal a constant K. Hence, we conclude that $Kf(x) = f''(x)$ and $Kg(t) = g'(t)$ for some constant K. Now, we have to differentiate cases depending on K.

We know form previous courses that $g(t) = c_1 e^{Kt}$ is a solution for $Kg(t) = g'(t)$. Now consider $Kf(x) = f''(x)$. As in the lecture (Lecture notes 2, part 2, step I), we have different cases depending on the sign of K. If $K = 0$, then $f''(x) = 0$ and $g(t) = c_1$. Proceeding as in the lecture notes, we get that this would give the zero-solution.

If $K > 0$, then there is λ such that $K = \lambda^2$. We know from previous courses that $f(x) = c_2 e^{\lambda x} + c_3 e^{-\lambda x}$ is a solution for $\lambda^2 f(x) = f''(x)$, for some constants c_2 and c_3. Since, in this case, we can write $g(t) = c_1 e^{\lambda^2 t}$, we get that a solution in this case is

$$u(x,t) = c_1 e^{\lambda^2 t}\left(c_2 e^{\lambda x} + c_3 e^{-\lambda x}\right).$$

If $K < 0$, then again there is λ such that $K = -\lambda^2$. The general solution of this ODE is $f(x) = c_2 \sin(\lambda x) + c_3 \cos(\lambda x)$ for some constants c_1 and c_2. Again, we obtain that a solution of the PDE is

$$u(x,t) = c_1 e^{-\lambda^2 x}\left(c_2 \sin(\lambda x) + c_3 \cos(\lambda x)\right).$$

(ii) Plug in the expressions for f and g obtained into the boundary conditions and obtain conclusions on the parameters c_1, c_2, c_3. Remember that we are only interested in non-zero solutions.

Solution:
Since $u_x(x,t) = f'(x)g(t)$ and $u_x(0,t) = 0$ from the boundary conditions, we conclude that $f'(0)g(t) = 0$ for all $t \geq 0$. This implies that one of the following two cases holds:

- $g(t) = 0$ for every $t \geq 0$

- $f'(0) = 0$

If $g(t) = 0$ for every $t \geq 0$ then it follows that $u(x,t) = f(x)g(t) \equiv 0$. Since we are interested in the non-zero solutions, let's assume that this is not the case. We recall that at the moment we have two possible solutions, depending on the value of K.

If $K = \lambda^2$, for $\lambda > 0$, then $f(x) = c_2 e^{\lambda x} + c_3 e^{-\lambda x}$. Imposing the condition $f'(0) = 0$ we get

$$c_2 \lambda - c_3 \lambda = 0.$$

Since $\lambda > 0$, we conclude that $c_2 = c_3$.

From $u(L,t) = 0$ for each t, we get $f(L)g(t) = 0$ for each t. As before, we can assume that $f(L) = 0$, otherwise we will find the zero solution. We get $f(L) = c_2(e^{\lambda L} + e^{-\lambda L}) = 0$, thus $c_2 = 0$. In particular, this implies that every solution with $K > 0$ has to be the zero solution. Thus we can only consider the case $K < 0$.

Imposing the condition $f'(0) = 0$, we get

$$(c_2 \lambda \cos(0) - c_3 \sin(\lambda 0)) = 0.$$

Thus, since $\lambda > 0$, we have $c_2 = 0$.

Imposing the condition $f(L) = 0$, we get $c_3 \cos(\lambda L) = 0$. Since $\cos(x) = 0$ if and only if $x = \left(n + \frac{1}{2}\right)\pi$ for some integer n, we conclude that

$$\lambda L = \left(n + \frac{1}{2}\right)\pi \Leftrightarrow \lambda = \left(n + \frac{1}{2}\right)\frac{\pi}{L}$$

for some integer n.

Finally, note that if $c_1 = 0$ then $g(t) \equiv 0$ and so $u(x,t) \equiv 0$ as well. Thus, we have showed that either $u(x,t) \equiv 0$, or

$$u(x,t) = c_1 c_3 e{-}\lambda^2 t\left(\cos\left(\lambda x\right)\right),$$

where $\lambda = \left(n + \frac{1}{2}\right)\frac{\pi}{L}$, for some integer n.

(iii) Conclude that for each integer n the function

$$u_n(x,t) = e^{-\left(\left(n+\frac{1}{2}\right)\frac{\pi}{L}\right)^2 t} \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi x}{L}\right)$$

satisfies the PDE and the boundary conditions of the IBVP above.

Solution:
By taking $c_1 = 0$, $c_2 = c_3 = 1$ and $\lambda = \left(n + \frac{1}{2}\right)\frac{\pi}{L}$ in the expressions obtained in c) we get

- $f(x) = \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi x}{L}\right)$
- $g(t) = e^{-\left(\left(n+\frac{1}{2}\right)\frac{\pi}{L}\right)^2 t}$

and so in that case $u(x,t)$ is exactly the function $u_n(x,t)$ defined above. We conclude that $u_n(x,t)$ satisfies the PDE and the boundary conditions of the IBVP.

(iv) Using the superposition principle deduce that if $\phi(x) = \sum_{n=1}^{N} a_n \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi x}{L}\right)$, for some constants $a_1, \ldots, a_N$ and for some positive integer N, then

$$u(x,t) = \sum_{n=1}^{N} a_n e^{-\left(\left(n+\frac{1}{2}\right)\frac{\pi}{L}\right)^2 t} \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi x}{L}\right)$$

satisfies the IBVP above.

Solution:
We know that the functions $u_n(x,t)$ satisfy the PDE and the boundary conditions of the IBVP above. Note that we have:

$$u_n(x,0) = \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi x}{L}\right).$$

Thus, the function $u_n(x,t)$ satisfies the IBVP above if the initial condition is given by the function

$$\phi_n(x) = \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi x}{L}\right).$$

Since the PDE is linear and homogeneous and the boundary conditions are homogeneous we conclude from the superposition principle that if the initial condition $\phi(x)$ is of the form

$$\phi(x) = \sum_{n=1}^{N} a_n \phi_n(x)$$

then the function $u(x,t)$

$$u(x,t) = \sum_{n=1}^{N} a_n u_n(x,t)$$

satisfies the corresponding IBVP. Given the defitions of $u_n(x,t)$ and $\phi_n(x)$, this is exactly what we were asked to deduce, hence we are done.

(v) Can you conjecture which heat problem this IBVP models? Use the interpretation of the BCs used in the examples you saw in lectures.

Solution:
This IBVP can model the heat problem where we have a laterally insulated rod of length L with diffusion coefficient $\alpha^2 = 1$ where one end is insulated, the other is kept fixed at temperature 0 and heat flows starting from an initial temperature distribution given by $\phi(x)$.

Question 5

1. Indicate which of the following PDEs is linear:

   (i) $10xu_{xt} + e^{x+t}u_x u = 0$

   (ii) $u_{xx} + t^{100}u_{tt} - x^3 u_x = e^{xt}u_t$

   (iii) $u_{xxy} - 20u_{xyy} + 100u_{xx} + e^{x^2}u = x + 3y$

   (iv) $u_{xx} + 30u_{xy} + 2u_x u_y = x^{10} + u^2$

Solution:
The first PDE is not linear because of the non-linear term $u_x u$. The fourth PDE is not linear because of the non-linear terms $u_x u_y$ and u^2.

The second and third PDEs are linear because all the terms consist of u, its partial derivatives or 1 multiplied by functions independent of u or its derivatives.

2. Consider the following PDE:

$$u_{xx} + 2u_{xy} + 2u_{yy} + u_y = 0$$

   This PDE is Linear/Nonlinear? Homogenous/not homogenous? Parabolic/elliptic/hyperbolic? [Exam question, 2012-2013]

Solution:
The PDE given is linear because it can be written in the form:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A, B, C, D, E, F and G are functions independent of u and its partial derivatives. Moreover, since $G = 0$, the PDE is homogeneous. Finally, since $B^2 - 4AC = 2^2 - 4 \times 1 \times 2 = -4 < 0$, the PDE is elliptic.

3. Which of the following PDEs are homogeneous? (Check all that apply.)

   (i) $u_{xy} + e^{1+y^2} u_x + u = 2$

   (ii) $u_{xy} - u_{yz} + 100u_x - 2u_y = 30u$

   (iii) $u_{xxx} + e^{x^2} u_{xyx} - y^2 u_x = 0$

   (iv) $ye^x u_{xxy} + u_{xy} - 10u_y = e^{x+y^3}$

   (v) $u_{xy} + x^3 y u_{xx} + \log(1+x^2) - 10y u_{yy} + u = 0$

Solution:
The first PDE has the term 2 which makes it not homogeneous.

The fourth PDE has the term e^{x+y^3} which makes it not homogeneous.

The fifth PDE has the term $\log(1 + x^2)$ (which is not u or one of its partial derivatives multiplied by some function) which makes it not homogeneous.

The remaining two PDEs are linear combinations of u and its partial derivatives and hence are homogeneous.

Question 6

Note! This question is harder then the ones above, but it is more interesting. It is not compulsory to solve it, but we encourage you to try to do at least some parts of it.

Consider the following IVBP:

PDE $u_t - u_{xx} = 0$ for $(x, t) \in \Omega = [0, L] \times [0, \infty)$;

BC $u(0, t) = u(L, t) = 0$ for all $t > 0$;

IC $u(x, 0) = \phi(x)$.

The physical intuition tells us that, no matter which is the initial temperature $\phi(x)$ of the rod, after enough time the temperature will stabilize to 0 degrees. Indeed, for some explicit values of $\phi(x)$, we are able to compute an explicit solution and hence compute the limit $\lim_{t\to\infty} u(x, t)$ and observe that tends to zero, regardless of the point x. The goal of this exercise is to compute the, so called, solution at infinity for a general (!) function $\phi(x)$, and show that is, indeed, the constant function 0.

Part I Let $\psi, \psi' \colon [0, L] \to \mathbb{R}$ be two continuous functions such that, for each $x \in [0, L]$ one has $\psi(x) \leq \psi'(x)$. Let u be a solution of the IVBP with $\phi(x) = \psi(x)$ and v a solution $\phi(x) = \psi'(x)$.

Show that for each $(x, t) \in \Omega$, one has $u(x, t) \leq v(x, t)$.

Solution:

By the superposition principle, the function $v - u$ satisfy the IBVP with initial condition $\phi(x) = \psi'(x) - \psi(x)$. In particular, one has that $\min_{x \in [0,L]}\{\phi(x)\} \geq 0$ (in fact it is 0 since the temperature at the endpoints is the same). We want to show that for each $(x_0, t_0) \in \Omega$, one has $(v - u)(x_0, t_0) \geq 0$, that is $v(x_0, t_0) \geq u(x_0, t_0)$. Applying the maximum principle (Question 5 of the last exercise sheet) with domain $\overline{\Omega} = [0, L] \times [0, t_0 + 1]$, we get that

$$(v - u)(x_0, t_0) \geq \min_{x \in [0,L]}\{\phi(x)\} = 0.$$

Part II Let $\psi(x) \colon [0, L] \to \mathbb{R}$ be a function such that $\psi'(0) < \infty$, $\psi'(L) < \infty$. This is a technical condition that can be assumed to be true in any physical system.

Show that for u satisfying the above IVBP with $\phi(x) = \psi(x)$, for each $x_0 \in [0, L]$, we have

$$\lim_{t \to \infty} u(x_0, t) = 0.$$

Solution:

Consider the function

$$\psi'(x) = C \sin\left(\frac{\pi}{L}x\right).$$

Given any continuous function $\psi(x)$ such that $\psi(0) = \psi(L) = 0$, we can always find a C big enough such that $\psi'(x) \geq \psi(x)$ for all $x \in [0, L]$. Indeed, let $g_1(x)$ be the function $\frac{\phi(x)}{x}$. Intuitively, g_1 measures the slope of a segment in the plane connecting the origin to the point $(x, \phi(x))$. It is easily seen that g_1 is continuous and, by hypotheses we have that $\lim_{x \to 0} g_1(x) < \infty$. Moreover, $g_1(L) = 0$. Thus, it has a maximum C_1. This imply that for each $x \in [0, L]$, we have $\psi(x) \leq C_1 x$. Similarly, define g_2 as the function $g_2(x) = \frac{\psi(x)}{L-x}$. As above, it has a maximum C_2. Let $C = \max\{C_1, C_2\}$. Then the graph of the function ψ is below the lines emanating from $(0, 0)$ and $(0, L)$ with slopes C and $-C$ respectively.

This implies that for each $x \in [0, L]$, we have that $\psi(x) \leq C \sin\left(\frac{\pi}{L}x\right)$.

Let v be a solution of the IVBP with $\phi(x) = \psi'(x)$. By part I, we know that for each (x_0, t_0) we have that $u(x_0, t_0) \leq v(x_0, t_0)$. However, we can explicitly compute v and we get that

$$v(x,t) = Ce^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right).$$

Since, for each x_0 we have that $\lim_{t\to\infty} v(x_0, t) = 0$, we get the result.

Question 7

The goal of this exercise is to derive the solution of the following IBVP:

PDE $u_t - u_{xx} = 0$ for $(x,t) \in \Omega = [0, L] \times [0, \infty)$;

BC $u(0,t) = c_1$, $u(L,t) = c_2$ for all $t > 0$;

IC $u(x,0) = \phi(x)$.

(i) For a suitable ϕ, the above IBVP admits a solution u_0 that does not depend on t. Try to guess such an u_0 and verify that satisfy the PDE and BC. *Hint: A more formal strategy may be to use the Ansatz $u(x,t) = f(x)g(t)$ and consider the case $\frac{g'(t)}{g(t)} = K = 0$.*

Solution:
Let $u_0(x,t) = \frac{(L-x)}{L} c_1 + \frac{x}{L} c_2$. It is easily seen that $u_{0t} = u_{0xx} = 0$, and that the BC are satisfied. Choosing $\phi(x) = \frac{(L-x)}{L} c_1 + \frac{x}{L} c_2$, we get that u_0 is a solution of:

PDE $u_t - u_{xx} = 0$ for $(x,t) \in \Omega = [0, L] \times [0, \infty)$;

BC $u(0,t) = c_1$, $u(L,t) = c_2$ for all $t > 0$;

IC $u(x,0) = \frac{(L-x)}{L} c_1 + \frac{x}{L} c_2$.

(ii) Use the superposition principle to solve the above IBVP for all initial conditions ϕ of the form:

$$\phi(x) = u_0(x,0) + \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x}{L}\right).$$

Solution:
Let v be a solution of

PDE $v_t - v_{xx} = 0$ for $(x,t) \in \Omega = [0, L] \times [0, \infty)$;

BC $v(0,t) = v(L,t) = 0$ for all $t > 0$;

IC $v(x,0) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x}{L}\right)$.

The explicit formula of the lecture gives:

$$v(x,t) = \sum_{n=1}^{N} a_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

We claim that $u(x,t) = u_0(x,t) + v(x,t)$ is the required solution (of the first IBVP). Indeed, $u_t - u_{xx} = (v_t - v_{xx}) + (u_{0t} - u_{0xx}) = 0$, and $u(x,0) = u_0(x,0) + v(x,0)$. Moreover, $u(0,t) = u_0(0,t) + v(0,t) = c_1 + 0 = c_1$ and $u(L,t) = u_0(L,t) + v(L,t) = c_2 + 0 = c_2$.