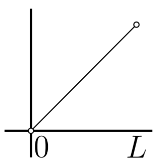
Analysis III (BAUG) — Assignment 4

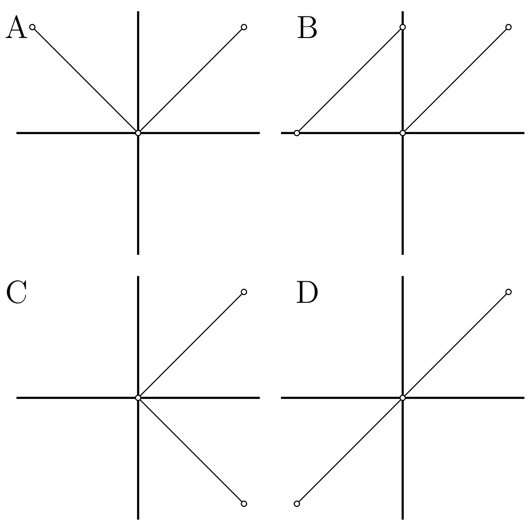
Prof. Dr. Alessandro Sisto — Due 18th October 2018

Question 1

Consider the function $f(x) = x$ below, defined for $0 < x < L$,



and the following graphs:

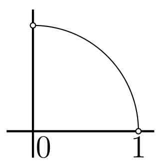


(i) Which graph corresponds to the even extension of f?

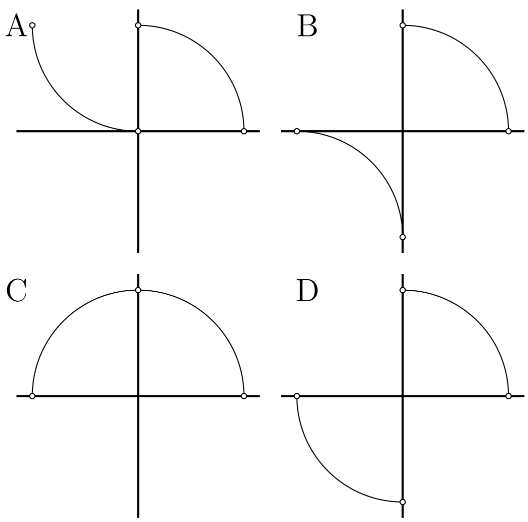
(ii) Which graph corresponds to the odd extension of f?

Question 2

Consider the function $f(x) = \sqrt{1 - x^2}$ below, defined for $0 < x < 1$,



and the following graphs:



(i) Which graph corresponds to the even extension of f?

(ii) Which graph corresponds to the odd extension of f?

Question 3

(i) Are there functions that are even and odd? If yes, describe them.

(ii) Let f and g be odd functions. What can we say about $f \cdot g$?

(iii) Let f and g be even functions. What can we say about $f \cdot g$?

(iv) Let f be odd and g be even. What can we say about $f \cdot g$?

(v) Let f and g be odd functions. What can we say about $f + g$?

(vi) Is it true that if f and g are odd functions then $(f + g)^2$ is odd?

(vii) For which values of α and n the function $\sin^n(\alpha x)$ is odd? For which is even? For which is neither of the two?

Question 4

Compute the following integrals for fixed integers n and m. (Note that the result might depend on the value of m and n.)

(i) $\displaystyle\int_{-L}^{L} \sin(\tfrac{n\pi}{L}x)\cos(\tfrac{m\pi}{L}x)\,dx$

(ii) $\displaystyle\int_{-L}^{L} \sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{L}x)\,dx$

Question 5

Write each of the following functions $\phi : [-L, L] \to \mathbf{R}$ as a sum of sines and cosines. In other words, find the coefficients of the Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L}x) + b_n \sin(\tfrac{n\pi}{L}x).$$

(i) $L = 2$ and

$$\phi(x) = \begin{cases} 1 & \text{for } x \in (-1, 2) \\ 0 & \text{for } x \in (-2, -1). \end{cases}$$

(ii) $L = 1/2$ and

$$\phi(x) = \begin{cases} \sin(2\pi x) & \text{for } x \in (0, \frac{1}{2}) \\ 0 & \text{for } x \in (-\frac{1}{2}, 0). \end{cases}$$

[Exam question, 2001-02]

Question 6

Write each of the following functions $\phi : [0, L] \to \mathbf{R}$ as a sum of *sines* as follows:

- Extend $\phi(x)$ to $[-L, L]$ as an *odd* function.
- Then use the general methods to compute the coefficients of the Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L}x) + b_n \sin(\tfrac{n\pi}{L}x).$$

  and note that $a_n = 0$ for every $n \geq 0$.

(i) $L = \pi$ and

$\phi(x) = 1$ for $x \in (0, \pi)$

[Exam question, 2010-11]

(ii) $L = 2$ and

$\phi(x) = x^2$

Question 7

Write each of the following functions $\phi : [0, L] \to \mathbf{R}$ as a sum of *cosines* as follows:

- Extend $\phi(x)$ to $[-L, L]$ as an *even* function.
- Then use the general methods to compute the coefficients of the Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L}x) + b_n \sin(\tfrac{n\pi}{L}x).$$

  and note that $b_n = 0$ for every $n \geq 1$.

(i) $L = \pi$ and

$$\phi(x) = \begin{cases} 0 & \text{for } x \in (0, \frac{\pi}{2}) \\ 2 & \text{for } x \in [\frac{\pi}{2}, \pi). \end{cases}$$

[Exam question, 2010-11]

(ii) $L = 2$ and

$\phi(x) = x^2$

Question 8

Note! This question is harder then the ones above, but it is more interesting. It is not compulsory to solve it, but we encourage you to try to do at least some parts of it.

In the previous exercises, we always considered a laterally insulated rod with some conditions at the endpoints. Now, we will consider a more general case. Namely, consider a rod of length 3π, and suppose that we light two candles near the points $\frac{\pi}{2}$ and $\frac{5\pi}{2}$. We can think that effect of the two candles is modeled by the function $1 + \sin(x)$, which is maximal at the points $\frac{\pi}{2}$ and $\frac{5\pi}{2}$. The corresponding IBVP has the following form:

$$\begin{array}{rll} \text{PDE}: & u_t(x,t) = u_{xx}(x,t) + 1 + \sin(x) & \text{for } 0 \leq x \leq 3\pi \text{ and } t \geq 0 \\ \text{BC}: & u(0,t) = u(3\pi, t) = 0 & \text{for } t \geq 0 \\ \text{IC}: & u(x,0) = \phi(x) & \text{for } 0 < x < 3\pi \end{array}$$

Find the steady-state solution, that is the limit

$$\lim_{t \to \infty} u(x,t),$$

and notice that does not depend on ϕ.

Question 9

1. Consider the following IBVP:

$$\begin{aligned} &\text{PDE}: && u_t(x,t) = 8u_{xx}(x,t) && \text{for } -4 < x < 4 \text{ and } t > 0 \\ &\text{BC}: && u(-4,t) = u(4,t), \quad u_x(-4,t) = u_x(4,t) && \text{for } t \geq 0 \\ &\text{IC}: && u(x,0) = 6 + \sin(\tfrac{5\pi}{4}x) - 3\cos(2x) && \text{for } -4 < x < 4 \end{aligned}$$

Which of the following problems does the IBVP above model?

- The heat flow on a laterally insulated rod with the temperature fixed at both ends.
- The heat flow on a completely insulated rod.
- The heat flow on a circular wire.

Suppose $u(x,t)$ is a solution to the IBVP above and that

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-c_n t} \cos\left(\tfrac{n\pi}{4}x\right) + \sum_{n=1}^{\infty} b_n e^{-d_n t} \sin\left(\tfrac{n\pi}{4}x\right)$$

for some constants $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$. Which of the following are true?

(i) $a_0 = 3$

(ii) $a_2 = -3$

(iii) $a_8 = 3$

(iv) $d_5 = \frac{25\pi^2}{2}$

(v) $b_5 = 1$