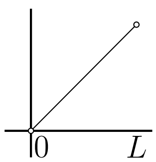
Analysis III (BAUG) **Assignment 4**

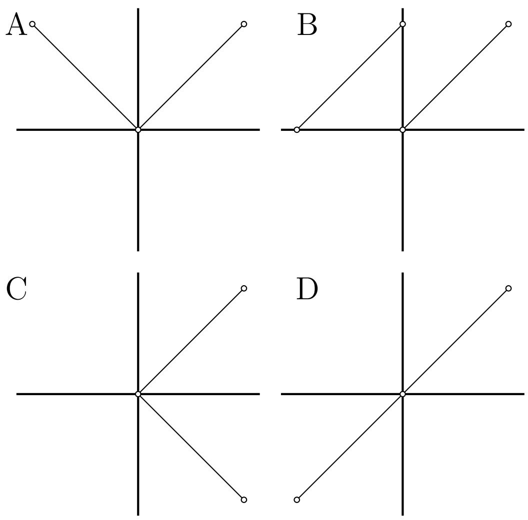
Prof. Dr. Alessandro Sisto Due 18th October 2018

Question 1

Consider the function $f(x) = x$ below, defined for $0 < x < L$,



and the following graphs:



(i) Which graph corresponds to the even extension of f?

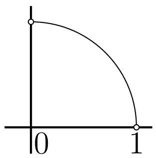
Solution: A

(ii) Which graph corresponds to the odd extension of f?

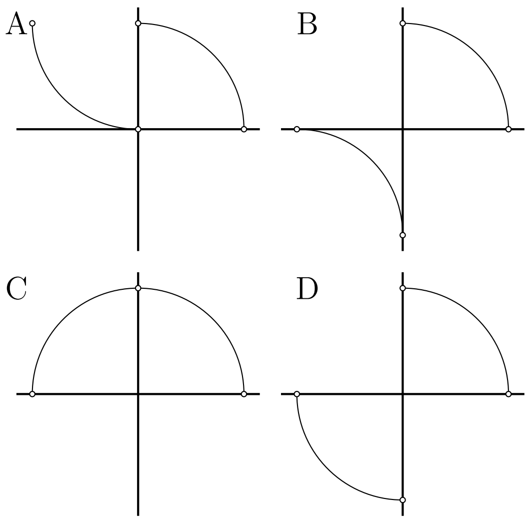
Solution: D

Question 2

Consider the function $f(x) = \sqrt{1-x^2}$ below, defined for $0 < x < 1$,



and the following graphs:



(i) Which graph corresponds to the even extension of f?

Solution: C

(ii) Which graph corresponds to the odd extension of f?

Solution: D

Question 3

(i) Are there functions that are even and odd? If yes, describe them.

Solution:
A function that is even and odd must be the constant zero function $f(x) = 0$. Indeed, for each x, we have that $f(x) = -f(-x) = -f(x)$, thus $f(x) = 0$.

(ii) Let f and g be odd functions. What can we say about $f \cdot g$?

Solution:
$(f \cdot g)(x) = f(x)g(x) = (-f(-x))(-g(-x)) = f(-x)g(-x) = (f \cdot g)(-x)$, so the function fg is even.

(iii) Let f and g be even functions. What can we say about $f \cdot g$?

Solution:
$(f \cdot g)(x) = f(x)g(x) = (f(-x))(g(-x)) = f(-x)g(-x) = (f \cdot g)(-x)$, so the function fg is even.

(iv) Let f be odd and g be even. What can we say about $f \cdot g$?

Solution:
$f \cdot g(x) = f(x) \cdot g(x) = -f(-x) \cdot g(-x) = -(f \cdot g)(-x)$, so the function fg is odd.

(v) Let f and g be odd functions. What can we say about $f + g$?

Solution:
$(f+g)(-x) = f(-x)+g(-x) = (-f(x))+(-g(x)) = -(f(x)+g(x)) = -(f+g)(x)$ so the function is odd.

(vi) Is it true that if f and g are odd functions then $(f+g)^2$ is odd?

Solution:
$f + g$ can be an arbitrary odd function, for instance $g(x) = 0$. By part (ii) we know $(f+g)(f+g) = f \cdot f$ is an even function. Thus we have that $(f+g)^2$ is not necessarily odd.

(vii) For which values of α and n the function $\sin^n(\alpha x)$ is odd? For which is even? For which is neither of the two?

Solution:
$\sin(x)$ is an odd function, so

$$\sin^n(-\alpha x) = (-\sin(\alpha x))^n = (-1)^n \sin^n(\alpha x).$$

If $\alpha = 0$, then the function is both even and odd. If $\alpha \neq 0$, then the function is odd whenever n is odd and even whenever n is even. In particular, it is always odd or even (or both).

Question 4

Compute the following integrals for fixed integers n and m. (Note that the result might depend on the value of m and n.)

(i) $\int_{-L}^{L} \sin(\frac{n\pi}{L}x)\cos(\frac{m\pi}{L}x)\,dx$

Solution:

$$\int_{-L}^{L} \sin(\tfrac{n\pi}{L}x)\cos(\tfrac{m\pi}{L}x)\,dx = \frac{1}{2}\int_{-L}^{L} \sin(\tfrac{(n+m)\pi}{L}x)\,dx + \frac{1}{2}\int_{-L}^{L} \sin(\tfrac{(n-m)\pi}{L}x)\,dx$$

$$= -\frac{L}{2(n+m)\pi}\left[\cos(\tfrac{(n+m)\pi}{L}x)\right]_{-L}^{L} - \frac{L}{2(n-m)\pi}\left[\cos(\tfrac{(n-m)\pi}{L}x)\right]_{-L}^{L} = 0 + 0 = 0$$

Except if $n+m$ is 0 then $\frac{L}{2(n+m)\pi}$ is undefined, but then $\sin(\frac{(n+m)\pi}{L}x) = 0$ so the corresponding integral is 0 anyways. Similarly for $n - m = 0$.

(ii) $\int_{-L}^{L} \sin(\frac{n\pi}{L}x)\sin(\frac{m\pi}{L}x)\,dx$

Solution:

$$\int_{-L}^{L} \sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{L}x)\,dx = \frac{1}{2}\int_{-L}^{L} \cos(\tfrac{(n-m)\pi}{L}x)\,dx - \frac{1}{2}\int_{-L}^{L} \cos(\tfrac{(n+m)\pi}{L}x)\,dx$$

$$= \frac{L}{2(n-m)\pi}\left[\sin(\tfrac{(n-m)\pi}{L}x)\right]_{-L}^{L} - \frac{L}{2(n+m)\pi}\left[\sin(\tfrac{(n+m)\pi}{L}x)\right]_{-L}^{L} = 0 + 0 = 0$$

Except if $n-m$ is 0 then $\cos(\frac{(n+m)\pi}{L}x) = 1$ so the first integral is L. Similarly if $n+m=0$ then the second integral is L. Therefore, the sum equals $-L$ if $n=-m\neq 0$, L if $n=m\neq 0$, and 0 otherwise.

Question 5

Write each of the following functions $\phi : [-L, L] \to \mathbf{R}$ as a sum of sines and cosines. In other words, find the coefficients of the Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L}x) + b_n \sin(\tfrac{n\pi}{L}x).$$

(i) $L = 2$ and

$$\phi(x) = \begin{cases} 1 & \text{for } x \in (-1, 2) \\ 0 & \text{for } x \in (-2, -1). \end{cases}$$

Solution:

$$a_0 = \frac{1}{4}\int_{-1}^{2} 1\,dx = 3/4$$

$$a_n = \frac{1}{2}\int_{-1}^{2}\cos(\tfrac{n\pi}{2}x)\,dx = \frac{1}{2}\left[\frac{2}{n\pi}\sin(\tfrac{n\pi}{2}x)\right]_{-1}^{2} = \frac{-1}{n\pi}\sin(\tfrac{-n\pi}{2}) = \frac{\sin(\frac{n\pi}{2})}{n\pi}$$

$$= \begin{cases} 0 & \text{for } n \equiv 0 \text{ or } 2 \pmod 4 \\ (n\pi)^{-1} & \text{for } n \equiv 1 \pmod 4 \\ -(n\pi)^{-1} & \text{for } n \equiv 3 \pmod 4 \end{cases}$$

$$b_n = \frac{1}{2}\int_{-1}^{2}\sin(\tfrac{n\pi}{2}x)\,dx = \frac{1}{2}\left[\frac{-2}{n\pi}\cos(\tfrac{n\pi}{2}x)\right]_{-1}^{2} = \frac{\cos(\frac{n\pi}{2}) - \cos(n\pi)}{n\pi}$$

$$= \begin{cases} (n\pi)^{-1} & \text{for } n \equiv 1 \text{ or } 3 \pmod 4 \\ 0 & \text{for } n \equiv 0 \pmod 4 \\ -2(n\pi)^{-1} & \text{for } n \equiv 2 \pmod 4 \end{cases}$$

(ii) $L = 1/2$ and

$$\phi(x) = \begin{cases} \sin(2\pi x) & \text{for } x \in (0, \frac{1}{2}) \\ 0 & \text{for } x \in (-\frac{1}{2}, 0). \end{cases}$$

[Exam question, 2001-02]

Solution:

$$a_0 = \int_0^{1/2} \sin(2\pi x)\,dx = \left[\frac{-\cos(2\pi x)}{2\pi}\right]_0^{1/2} = 1/\pi$$

$$\begin{aligned}
a_n &= 2\int_0^{1/2} \sin(2\pi x)\cos(2n\pi x)\,dx = \int_0^{1/2} \sin((1+n)2\pi x) + \sin((1-n)2\pi x)\,dx \\
&= \left[\frac{-1}{2(n+1)\pi}\cos((n+1)2\pi x) + \frac{1}{2(n-1)\pi}\cos((n-1)2\pi x)\right]_0^{1/2} \\
&= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{(n+1)\pi} - \frac{1}{(n-1)\pi} = \frac{2}{\pi(1-n^2)} & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

Note that for $n = 1$ the $\sin((1-n)2\pi x) = 0$ so the second term above does not exist. Similarly,

$$\begin{aligned}
b_n &= 2\int_0^{1/2} \sin(2\pi x)\sin(2n\pi x)\,dx = \int_0^{1/2} -\cos((n+1)2\pi x) + \cos((n-1)2\pi x)\,dx \\
&= \begin{cases} 1/2 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Question 6

Write each of the following functions $\phi : [0, L] \to \mathbf{R}$ as a sum of *sines* as follows:

- Extend $\phi(x)$ to $[-L, L]$ as an *odd* function.
- Then use the general methods to compute the coefficients of the Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L}x) + b_n \sin(\tfrac{n\pi}{L}x).$$

and note that $a_n = 0$ for every $n \geq 0$.

(i) $L = \pi$ and

$\phi(x) = 1$ for $x \in (0, \pi)$

[Exam question, 2010-11]

Solution:

The odd extension of $\phi(x)$ is $\phi(x) = \begin{cases} -1 & \text{for } x \in (-\pi, 0) \\ 1 & \text{for } x \in (0, \pi). \end{cases}$.

Since this is an odd function, $a_n = 0$ for every $n \geq 0$ (e.g. because $\phi(x)\cos(\frac{n\pi}{L}x)$ is odd, hence has integral 0 on $[-L, L]$).

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx = \frac{2}{\pi}\left[\frac{-1}{n}\cos(nx)\right]_0^{\pi}$$
$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd.} \end{cases}$$

(ii) $L = 2$ and
$\phi(x) = x^2$

Solution:

The odd extension of $\phi(x)$ is $\phi(x) = \begin{cases} -x^2 & \text{for } x \in (-2, 0) \\ x^2 & \text{for } x \in (0, 2). \end{cases}$.

Since this is an odd function, $a_n = 0$ for every $n \geq 0$.

$$b_n = \frac{1}{2}\int_{-2}^{2} \phi(x)\sin(\tfrac{n\pi}{2}x)\,dx = \int_0^2 x^2\sin(\tfrac{n\pi}{2}x)\,dx = \left[\frac{-2}{n\pi}x^2\cos(\tfrac{n\pi}{2}x)\right]_0^2 - \int_0^2 \frac{-2}{n\pi}2x\cos(\tfrac{n\pi}{2}x)$$
$$= \frac{-8\cos(n\pi)}{n\pi} + \left[\frac{8}{n^2\pi^2}x\sin(\tfrac{n\pi}{2}x)\right]_0^2 - \int_0^2 \frac{8}{n^2\pi^2}\sin(\tfrac{n\pi}{2}x)$$
$$= \frac{-8\cos(n\pi)}{n\pi} + 0 + \left[\frac{16}{n^3\pi^3}\cos(\tfrac{n\pi}{2}x)\right]_0^2$$
$$= -\frac{16}{n^3\pi^3} + \cos(n\pi)\cdot\left(\frac{16}{n^3\pi^3} - \frac{8}{n\pi}\right) = \begin{cases} \frac{-8}{n\pi} & \text{if n is even} \\ \frac{8}{n\pi} - \frac{32}{n^3\pi^3} & \text{if n is odd.} \end{cases}$$

Question 7

Write each of the following functions $\phi : [0, L] \to \mathbf{R}$ as a sum of *cosines* as follows:

- Extend $\phi(x)$ to $[-L, L]$ as an *even* function.

- Then use the general methods to compute the coefficients of the Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L}x) + b_n \sin(\tfrac{n\pi}{L}x).$$

and note that $b_n = 0$ for every $n \geq 1$.

(i) $L = \pi$ and

$$\phi(x) = \begin{cases} 0 & \text{for } x \in (0, \frac{\pi}{2}) \\ 2 & \text{for } x \in [\frac{\pi}{2}, \pi). \end{cases}$$

Solution:

The even extension of $\phi(x)$ is $\phi(x) = \begin{cases} 2 & \text{for } x \in (-\pi, -\frac{\pi}{2}) \text{ or } x \in (\frac{\pi}{2}, \pi) \\ 0 & \text{for } x \in (-\frac{\pi}{2}, \frac{\pi}{2}). \end{cases}$.

Since this is an even function, $b_n = 0$ for every $n \geq 0$ (e.g. because $\phi(x)\sin(\frac{n\pi}{L}x)$ is odd, hence has integral 0 on $[-L, L]$).

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi} 2\,dx = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos(nx)\,dx = \frac{2}{\pi}\int_{\pi/2}^{\pi} 2\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{2}{n}\sin(nx)\right]_{\pi/2}^{\pi}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n} & \text{if } n = 3 \pmod 4 \\ \frac{-4}{\pi n} & \text{if } n = 1 \pmod 4 \end{cases}$$

[Exam question, 2010-11]

(ii) $L = 2$ and

$\phi(x) = x^2$

Solution:

The even extension of $\phi(x)$ is $\phi(x) = x^2$.

Since this is an even function, $b_n = 0$ for every $n \geq 1$.

$$a_0 = \frac{1}{4}\int_{-2}^{2} x^2 = \frac{1}{2}\int_{0}^{2} x^2 = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} \phi(x)\cos(\tfrac{n\pi}{2}x)\,dx = \int_0^2 x^2\cos(\tfrac{n\pi}{2}x)\,dx = \left[\frac{2}{n\pi}x^2\sin(\tfrac{n\pi}{2}x)\right]_0^2 - \int_0^2 \frac{2}{n\pi}2x\sin(\tfrac{n\pi}{2}x)$$

$$= 0 - \left[\frac{-8}{n^2\pi^2}x\cos(\tfrac{n\pi}{2}x)\right]_0^2 + \int_0^2 \frac{-8}{n^2\pi^2}\cos(\tfrac{n\pi}{2}x)$$

$$= \frac{16\cos(n\pi)}{n^2\pi^2} - \left[\frac{16}{n^3\pi^3}\sin(\tfrac{n\pi}{2}x)\right]_0^2 = \frac{16\cos(n\pi)}{n^2\pi^2} = \begin{cases} \frac{16}{n^2\pi^2} & \text{if } n \text{ is even} \\ \frac{-16}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases}$$

Question 8

Note! This question is harder then the ones above, but it is more interesting. It is not compulsory to solve it, but we encourage you to try to do at least some parts of it.

In the previous exercises, we always considered a laterally insulated rod with some conditions at the endpoints. Now, we will consider a more general case. Namely, consider a rod of length 3π, and suppose that we light two candles near the points $\frac{\pi}{2}$ and $\frac{5\pi}{2}$. We can think that effect of the two candles is modeled by the function $1 + \sin(x)$, which is maximal at the points $\frac{\pi}{2}$ and $\frac{5\pi}{2}$. The corresponding IBVP has the following form:

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = u_{xx}(x,t) + 1 + \sin(x) && \text{for } 0 \le x \le 3\pi \text{ and } t \ge 0 \\ \text{BC}: &\quad u(0,t) = u(3\pi,t) = 0 && \text{for } t \ge 0 \\ \text{IC}: &\quad u(x,0) = \phi(x) && \text{for } 0 < x < 3\pi \end{aligned}$$

Find the steady-state solution, that is the limit

$$\lim_{t\to\infty} u(x,t),$$

and notice that does not depend on ϕ.

Solution:

We start finding a solution Q of the IBVP that does not depend on t. Thus we would have $Q_t = 0$ and $Q_{xx} = -1 + \sin(x)$. Integrating we get $Q(x) = \sin(x) - \frac{1}{2}x^2 - c_1 x - c_2$ for some c_1 and c_2.

We will now match c_1 and c_2 to the BC, that is need to choose c_1 and c_2 such that $Q(0) = 0$ and $Q(3\pi) = 0$. It is easily verified that $c_1 = -\frac{3}{2}\pi$ and $c_2 = 0$ are the desired ones.

Let v be a solution of the following IBVP:

$$\begin{aligned} \text{PDE}: &\quad v_t(x,t) = v_{xx}(x,t) &&\text{for } 0 \leq x \leq 3\pi \text{ and } t \geq 0 \\ \text{BC}: &\quad v(0,t) = v(3\pi,t) = 0 &&\text{for } t \geq 0 \\ \text{IC}: &\quad v(x,0) = \phi(x) - Q(x) &&\text{for } 0 < x < 3\pi \end{aligned}$$

Then $u = v + Q$ is a solution of the initial IBVP. Using Question 6 of the last exercise sheet we get that $\lim x \to \infty v(x,t) = 0 \forall x$. Thus, $\lim_{x\to\infty} u(x,t) = Q(x)$ for all x.

Question 9

1. Consider the following IBVP:

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 8u_{xx}(x,t) &&\text{for } -4 < x < 4 \text{ and } t > 0 \\ \text{BC}: &\quad u(-4,t) = u(4,t), \;\; u_x(-4,t) = u_x(4,t) &&\text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = 6 + \sin(\tfrac{5\pi}{4}x) - 3\cos(2x) &&\text{for } -4 < x < 4 \end{aligned}$$

Which of the following problems does the IBVP above model?

- The heat flow on a laterally insulated rod with the temperature fixed at both ends.
- The heat flow on a completely insulated rod.
- The heat flow on a circular wire.

Suppose $u(x,t)$ is a solution to the IBVP above and that

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-c_n t} \cos\left(\tfrac{n\pi}{4}x\right) + \sum_{n=1}^{\infty} b_n e^{-d_n t} \sin\left(\tfrac{n\pi}{4}x\right)$$

for some constants $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$. Which of the following are true?

(i) $a_0 = 3$

(ii) $a_2 = -3$

(iii) $a_8 = 3$

(iv) $d_5 = \frac{25\pi^2}{2}$

(v) $b_5 = 1$

Solution:
Note that $L = 4$ and $u(x,0)$ is a finite linear combination of terms of the form $\cos(\frac{n\pi}{L}x)$ and $\sin(\frac{n\pi}{L}x)$. Therefore the solution of the PDE is

$$u(x,t) = 6 + e^{-(\frac{5\pi}{4})^2 \cdot 8t} \sin(\tfrac{5\pi}{4}x) - 3e^{-(\frac{8\pi}{4})^2 \cdot 8t} \cos(2\pi x).$$

In particular, $a_0 = 6$, $a_8 = -3$, and $a_n = 0$ for all other n. Also, $b_5 = 1$ and $d_5 = \frac{25\pi^2}{2}$.