Analysis III (BAUG) **Assignment 5**

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Part I: Fourier series

The Fourier series is one of the central topics of the course. In this exercise sheet, you will find several exercises that uses Fourier Series. We don't expect you to solve all of them in one week, but solving at least two in the set $\{2, \ldots, 6\}$ and 2 in the set $\{7, 8, 9\}$ is expected. We encourage you, however, to solve all of them when preparing for the midterm/exam. Due to the high number of Questions, there is no hard exercise this week.

Question 1

Let $f(x)$ and $g(x)$ be two continuous functions defined on the interval $[-L, L]$. Suppose their Fourier series are $a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x)$ and $c_0 + \sum_{n=1}^{\infty} c_n \cos(\frac{n\pi}{L}x) + d_n \sin(\frac{n\pi}{L}x)$, respectively. Decide whether each of the following statements is true or false.

(i) The Fourier series of $f(x) + g(x)$ is $(a_0 + c_0) + \sum_{n=1}^{\infty}(a_n + c_n)\cos(\frac{n\pi}{L}x) + (b_n + d_n)\sin(\frac{n\pi}{L}x)$.

(ii) The Fourier series of $f(x)g(x)$ is $a_0c_0 + \sum_{n=1}^{\infty} a_nc_n \cos(\frac{n\pi}{L}x) + b_nd_n \sin(\frac{n\pi}{L}x)$.

(iii) Any continuous function defined on $[-L, L]$ has a Fourier series all of whose terms are sines.

(iv) Any odd continuous function defined on $[-L, L]$ has a Fourier series all of whose terms are sines.

(v) The Fourier series of $\frac{1}{2}(f(x) + f(-x))$ is $a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x)$.

(vi) The Fourier series of $\frac{1}{2}(f(x) - f(-x))$ is $a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x)$.

Question 2 [Midterm-like question]

Solve the following IBVP

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 3u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0 \\ \text{BC}: &\quad u(0,t) = 0, \ \ u(\pi,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = 1 && \text{for } 0 \leq x \leq \pi \end{aligned}$$

Which of the following are true?

☐ $a_3 = \frac{2}{3\pi}$

- ☐ $a_0 = 1$
- ☐ $b_4 = 0$
- ☐ $b_1 = \frac{4}{\pi}$
- ☐ $b_2 = 3\pi^2$

Question 3 [Midterm-like question]

Solve the following IBVP

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 3u_{xx}(x,t) &&\text{for } 0 < x < 2 \text{ and } t > 0 \\ \text{BC}: &\quad u(0,t) = 0, \;\; u(2,t) = 0 &&\text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = x^2 &&\text{for } 0 \leq x \leq 2 \end{aligned}$$

Which of the following are true?

- ☐ $a_0 = 0$
- ☐ $a_1 = \frac{-16}{\pi^2}$
- ☐ $b_1 = \frac{8(\pi^2-3)}{\pi^3}$
- ☐ $b_2 = 3\pi^2$
- ☐ $b_3 = \frac{-8}{3\pi}$

Question 4 [Midterm-like question]

Solve the following IBVP.

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 2u_{xx}(x,t) &&\text{for } 0 < x < \pi \text{ and } t > 0 \\ \text{BC}: &\quad u_x(0,t) = 0, \;\; u_x(\pi,t) = 0 &&\text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = \begin{cases} 0 & \text{for } x \in (0, \frac{\pi}{2}) \\ 2 & \text{for } x \in [\frac{\pi}{2}, \pi) \end{cases} \end{aligned}$$

Which of the following are true?

- ☐ $a_0 = 2$
- ☐ $b_1 = 0$
- ☐ $a_{16} = \frac{1}{4\pi}$
- ☐ $a_3 = \frac{4}{3\pi}$
- ☐ $b_2 = \frac{4}{\pi}$

Question 5 [Midterm-like question]

Solve the following IBVP.

$$\begin{aligned} \text{PDE :} \quad & u_t(x,t) = 2u_{xx}(x,t) && \text{for } 0 < x < 2 \text{ and } t > 0 \\ \text{BC :} \quad & u_x(0,t) = 0, \;\; u_x(2,t) = 0 && \text{for } t \geq 0 \\ \text{IC :} \quad & u(x,0) = x^2 && \text{for } 0 \leq x \leq 2 \end{aligned}$$

Which of the following are true?

- ☐ $a_0 = \frac{4}{3}$
- ☐ $a_5 = 0$
- ☐ $b_4 = 8\pi^2$
- ☐ $a_2 = \frac{4}{\pi}$
- ☐ $b_1 = \pi$

Question 6 [Midterm-like question]

Solve the following IBVP.

$$\begin{aligned} \text{PDE :} \quad & u_t(x,t) = 10u_{xx}(x,t) && \text{for } -2 < x < 2 \text{ and } t > 0 \\ \text{BC :} \quad & u(-2,t) = u(2,t), \;\; u_x(-2,t) = u_x(2,t) && \text{for } t \geq 0 \\ \text{IC :} \quad & u(x,0) = \begin{cases} 1 & \text{for } x \in (-1,2) \\ 0 & \text{for } x \in (-2,-1) \end{cases} \end{aligned}$$

Which of the following are true?

- ☐ $a_0 = \frac{3}{4}$
- ☐ $a_2 = 0$
- ☐ $b_1 = \frac{1}{\pi}$
- ☐ $a_3 = \frac{-1}{3\pi}$
- ☐ $b_2 = 0$

Question 7

Solve the following IBVP

$$\begin{aligned} \text{PDE :} \quad & u_t(x,t) = 10u_{xx}(x,t) && \text{for } -1/2 < x < 1/2 \text{ and } t > 0 \\ \text{BC :} \quad & u(-1/2,t) = u(1/2,t), \;\; u_x(-1/2,t) = u_x(1/2,t) && \text{for } t \geq 0 \\ \text{IC :} \quad & u(x,0) = \begin{cases} \sin(2\pi x) & \text{for } x \in (0, \frac{1}{2}) \\ 0 & \text{for } x \in (-\frac{1}{2}, 0) \end{cases} \end{aligned}$$

Question 8

Solve the following IBVP.

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 10u_{xx}(x,t) && \text{for } -2 < x < 2 \text{ and } t > 0 \\ \text{BC}: &\quad u(-2,t) = u(2,t),\ \ u_x(-2,t) = u_x(2,t) && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = x^2 && \text{for } -2 \leq x \leq 2 \end{aligned}$$

Question 9

Solve the following IBVPs

(i)

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0 \\ \text{BC}: &\quad u_x(0,t) = 0,\ \ u_x(\pi,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = 1 + \cos^2(x) && \text{for } 0 \leq x \leq \pi \end{aligned}$$

[Exam question, 2006]

(ii)

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 10u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0 \\ \text{BC}: &\quad u(0,t) = 0,\ \ u(\pi,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = x(\pi - x) && \text{for } 0 \leq x \leq \pi \end{aligned}$$

Part II: The wave equation

Question 10

Solve the following IBVP:

$$\begin{aligned} \text{PDE}: &\quad u_{tt}(x,t) = u_{xx}(x,t) && \text{for } 0 < x < L \text{ and } t > 0 \\ \text{BC}: &\quad u(0,t) = u(L,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = 0 && \text{for } 0 \leq x \leq L \\ &\quad u_t(x,0) = \sin\left(\frac{8\pi}{L}x\right)\cos\left(\frac{3\pi}{L}x\right) && \text{for } 0 \leq x \leq L. \end{aligned}$$

Question 11

Solve the following IBVP:

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 49u_{xx}(x,t) && \text{for } 0 < x < 1 \text{ and } t > 0 \\
\text{BC}: &\quad u(0,t) = u(1,t) = 0 && \text{for } t \geq 0 \\
\text{IC}: &\quad u(x,0) = 3\sin\left(\frac{2035\pi}{2}x\right)\sin\left(\frac{\pi(1999x+1)}{2}\right) && \text{for } 0 \leq x \leq 1 \\
&\quad u_t(x,0) = 0 && \text{for } 0 \leq x \leq 1.
\end{aligned}$$

Question 12

Solve the following IBVP:

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = u_{xx}(x,t) && \text{for } 0 < x < L \text{ and } t > 0 \\
\text{BC}: &\quad u(0,t) = u(L,t) = 0 && \text{for } t \geq 0 \\
\text{IC}: &\quad u(x,0) = \left(2\sin\left(\frac{(20x+L)\pi}{4L}\right)\right)^2 + 6\sin\left(\frac{\pi}{L}x\right) - 2 && \text{for } 0 \leq x \leq L \\
&\quad u_t(x,0) = 5\sin\left(\frac{10\pi}{L}x\right) + 11\sin\left(\frac{50\pi}{L}x\right) && \text{for } 0 \leq x \leq L.
\end{aligned}$$