Analysis III (BAUG) **Assignment 5**

Prof. Dr. Alessandro Sisto — Due 25th October 2018

Organizer: Davide Spriano

Part I: Fourier series

The Fourier series is one of the central topics of the course. In this exercise sheet, you will find several exercises that uses Fourier Series. We don't expect you to solve all of them in one week, but solving at least two in the set $\{2, \dots, 6\}$ and 2 in the set $\{7, 8, 9\}$ is expected. We encourage you, however, to solve all of them when preparing for the midterm/exam. Due to the high number of Questions, there is no hard exercise this week.

Question 1

Let $f(x)$ and $g(x)$ be two continuous functions defined on the interval $[-L, L]$. Suppose their Fourier series are $a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x)$ and $c_0 + \sum_{n=1}^{\infty} c_n \cos(\frac{n\pi}{L}x) + d_n \sin(\frac{n\pi}{L}x)$, respectively. Decide whether each of the following statements is true or false.

(i) The Fourier series of $f(x) + g(x)$ is $(a_0 + c_0) + \sum_{n=1}^{\infty}(a_n + c_n)\cos(\frac{n\pi}{L}x) + (b_n + d_n)\sin(\frac{n\pi}{L}x)$.

Solution:
This is true. We have that the Fourier series of $f(x) + g(x)$ is $a'_0 + \sum_{n=1}^{\infty} a'_n \cos(\frac{n\pi}{L}x) + b'_n \sin(\frac{n\pi}{L}x)$ for

$$a'_0 = \frac{1}{2L}\int_{-L}^{L} f(x) + g(x)\,dx = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx + \frac{1}{2L}\int_{-L}^{L} g(x)\,dx = a_0 + c_0$$

$$\begin{aligned} a'_n &= \frac{1}{L}\int_{-L}^{L}(f(x) + g(x))\cos(\frac{n\pi}{L}x), dx \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)\,dx + \frac{1}{L}\int_{-L}^{L} g(x)\cos(\frac{n\pi}{L}x)\,dx = a_n + c_n \end{aligned}$$

$$\begin{aligned} b'_n &= \frac{1}{L}\int_{-L}^{L}(f(x) + g(x))\sin(\frac{n\pi}{L}x), dx \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)\,dx + \frac{1}{L}\int_{-L}^{L} g(x)\sin(\frac{n\pi}{L}x)\,dx = b_n + d_n \end{aligned}$$

(ii) The Fourier series of $f(x)g(x)$ is $a_0c_0 + \sum_{n=1}^{\infty} a_nc_n \cos(\frac{n\pi}{L}x) + b_nd_n \sin(\frac{n\pi}{L}x)$.

Solution:
This is false. Eg $L = 1$, $f(x) = \sin(n\pi x)$, and $g(x) = \sin(n\pi x)$. Then $a_n = c_n = 0$ for all n, $b_1 = d_1 = 1$ and $b_n = d_n = 0$ for all $n \neq 1$.

However $f(x)g(x) = \sin^2(n\pi x) = (1-\cos(2n\pi x))/2$ which has $a_0 = 1$ and $a_2 = 1$.

(iii) Any continuous function defined on $[-L, L]$ has a Fourier series all of whose terms are sines.

Solution:
This is false. Eg $f(x) = 1$.

(iv) Any odd continuous function defined on $[-L, L]$ has a Fourier series all of whose terms are sines.

Solution:
This is true. This is because for all other terms we have:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)\,dx = 0$$

Both integrals are zero because they are integrals of odd functions from $-L$ to L.

(v) The Fourier series of $\frac{1}{2}(f(x) + f(-x))$ is $a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x)$.

Solution:
Let $g(x) = \frac{1}{2}(f(x) + f(-x))$ and suppose its Fourier series is

$$a_0' + \sum_{n=1}^{\infty} a_n' \cos(\tfrac{n\pi}{L}x) + \textstyle\sum_{n=1}^{\infty} b_n' \sin(\tfrac{n\pi}{L}x)$$

Note that

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} f(-x)\,dx$$

and so we know that:

$$a_0' = \frac{1}{2L}\int_{-L}^{L} g(x)\,dx = \frac{1}{2L}\int_{-L}^{L} \frac{1}{2}(f(x)+f(-x))\,dx = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = a_0.$$

Moreover, for any even function h defined on $[-L, L]$ we have

$$\int_{-L}^{L} f(x)h(x)\,dx = \int_{-L}^{L} f(-x)h(-x)\,dx = \int_{-L}^{L} f(-x)h(x)\,dx$$

and so taking $h(x) = \cos(\frac{n\pi}{L}x)$ we conclude that for $n \geq 1$:

$$\begin{aligned} a_n' &= \frac{1}{L}\int_{-L}^{L} g(x)\cos(\frac{n\pi}{L}x)\,dx \\ &= \frac{1}{L}\int_{-L}^{L} \frac{1}{2}(f(x)+f(-x))\cos(\frac{n\pi}{L}x)\,dx \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)\,dx = a_n. \end{aligned}$$

Finally, for any odd function h defined on $[-L, L]$ we have

$$\int_{-L}^{L} f(x)h(x)\,dx = \int_{-L}^{L} f(-x)h(-x)\,dx = -\int_{-L}^{L} f(-x)h(x)\,dx$$

and so taking $h(x) = \sin(\frac{n\pi}{L}x)$ we conclude that for $n \geq 1$:

$$\begin{aligned} b_n' &= \frac{1}{L}\int_{-L}^{L} g(x)\sin(\frac{n\pi}{L}x)\,dx \\ &= \frac{1}{L}\int_{-L}^{L} \frac{1}{2}(f(x)+f(-x))\sin(\frac{n\pi}{L}x)\,dx = 0. \end{aligned}$$

(vi) The Fourier series of $\frac{1}{2}(f(x) - f(-x))$ is $a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x)$.

Solution:
This is false. Eg take $L = 1$ and $f(x) = \sin(n\pi x)$. Then $\frac{1}{2}(f(x) - f(-x)) = \sin(n\pi x)$ which has $b_1 = 1$.

Question 2 [Midterm-like question]

Solve the following IBVP

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 3u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0 \\ \text{BC}: &\quad u(0,t) = 0, \;\; u(\pi,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = 1 && \text{for } 0 \leq x \leq \pi \end{aligned}$$

Which of the following are true?

- ☐ $a_3 = \frac{2}{3\pi}$
- ☐ $a_0 = 1$
- ☐ $b_4 = 0$
- ☐ $b_1 = \frac{4}{\pi}$
- ☐ $b_2 = 3\pi^2$

Solution:
To solve this IBVP we need first to compute the Fourier series of the odd extension of $u(x,0)$ to $[-\pi,\pi]$. This has been done in Assignment 4 and there we saw that it is:

$$u(x,0) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{\pi n} \sin(nx)$$

To get the solution of the IBVP we need to multiply the n-th term of the sum above by $e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} = e^{-\left(\frac{n\pi}{\pi}\right)^2 3t} = e^{-3n^2 t}$ and so we obtain the solution:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{\pi n} e^{-3n^2 t} \sin(nx)$$

Question 3 [Midterm-like question]

Solve the following IBVP

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 3u_{xx}(x,t) && \text{for } 0 < x < 2 \text{ and } t > 0 \\ \text{BC}: &\quad u(0,t) = 0, \;\; u(2,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = x^2 && \text{for } 0 \leq x \leq 2 \end{aligned}$$

Which of the following are true?

☐ $a_0 = 0$

☐ $a_1 = \frac{-16}{\pi^2}$

☐ $b_1 = \frac{8(\pi^2-3)}{\pi^3}$

☐ $b_2 = 3\pi^2$

☐ $b_3 = \frac{-8}{3\pi}$

Solution:
To solve this IBVP we need first to compute the Fourier series of the odd extension of $u(x,0)$ to $[-2,2]$. This has been done in Assignment 4 and there we saw that it is:

$$u(x,0) = \sum_{n=1}^{\infty} \left(\frac{8(-1)^{n+1}}{n\pi} - \frac{16(1-(-1)^n)}{n^3\pi^3} \right) \sin\left(\frac{n\pi}{2}x\right)$$

To get the solution of the IBVP we need to multiply the n-th term of the sum above by $e^{-\left(\frac{n\pi}{L}\right)^2\alpha^2 t} = e^{-\left(\frac{n\pi}{2}\right)^2 3t}$ and so we obtain the solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{8(-1)^{n+1}}{n\pi} - \frac{16(1-(-1)^n)}{n^3\pi^3} \right) e^{-\left(\frac{n\pi}{2}\right)^2 3t} \sin\left(\frac{n\pi}{2}x\right)$$

Question 4 [Midterm-like question]

Solve the following IBVP.

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 2u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0 \\ \text{BC}: &\quad u_x(0,t) = 0, \quad u_x(\pi,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = \begin{cases} 0 & \text{for } x \in (0, \frac{\pi}{2}) \\ 2 & \text{for } x \in [\frac{\pi}{2}, \pi) \end{cases} \end{aligned}$$

Which of the following are true?

☐ $a_0 = 2$

☐ $b_1 = 0$

☐ $a_{16} = \frac{1}{4\pi}$

☐ $a_3 = \frac{4}{3\pi}$

☐ $b_2 = \frac{4}{\pi}$

Solution:
To solve this IBVP we need first to compute the Fourier series of the even extension of $u(x,0)$ to $[-\pi,\pi]$. This has been done in Assignment 4 and there we saw that it is:

$$u(x,0) = 1 + \sum_{n=1}^{\infty} -\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(nx).$$

To get the solution of the IBVP we need to multiply the n-th term of the sum above by $e^{-\left(\frac{n\pi}{L}\right)^2\alpha^2 t} = e^{-2n^2t}$ and so we obtain the solution:

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{4\sin\left(\frac{n\pi}{2}\right)}{n\pi} e^{-2n^2t}\cos(nx).$$

Question 5 [Midterm-like question]

Solve the following IBVP.

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 2u_{xx}(x,t) && \text{for } 0 < x < 2 \text{ and } t > 0 \\ \text{BC}: &\quad u_x(0,t) = 0, \;\; u_x(2,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = x^2 && \text{for } 0 \leq x \leq 2 \end{aligned}$$

Which of the following are true?

- ☐ $a_0 = \frac{4}{3}$
- ☐ $a_5 = 0$
- ☐ $b_4 = 8\pi^2$
- ☐ $a_2 = \frac{4}{\pi}$
- ☐ $b_1 = \pi$

Solution:
To solve this IBVP we need first to compute the Fourier series of the even extension of $u(x,0)$ to $[-2,2]$. This has been done in Assignment 4 and there we saw that it is:

$$u(x,0) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right)$$

To get the solution of the IBVP we need to multiply the n-th term of the sum above by $e^{-\left(\frac{n\pi}{L}\right)^2\alpha^2 t} = e^{-\left(\frac{n\pi}{2}\right)^2 2t}$ and so we obtain the solution:

$$u(x,t) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} e^{-\left(\frac{n\pi}{2}\right)^2 2t} \cos\left(\frac{n\pi}{2}x\right)$$

Question 6 [Midterm-like question]

Solve the following IBVP.

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 10u_{xx}(x,t) && \text{for } -2 < x < 2 \text{ and } t > 0 \\ \text{BC}: &\quad u(-2,t) = u(2,t),\ \ u_x(-2,t) = u_x(2,t) && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = \begin{cases} 1 & \text{for } x \in (-1,2) \\ 0 & \text{for } x \in (-2,-1) \end{cases} \end{aligned}$$

Which of the following are true?

- ☐ $a_0 = \frac{3}{4}$
- ☐ $a_2 = 0$
- ☐ $b_1 = \frac{1}{\pi}$
- ☐ $a_3 = \frac{-1}{3\pi}$
- ☐ $b_2 = 0$

Solution:

To solve this IBVP we need first to compute the Fourier series of $u(x,0)$ on $[-2,2]$. This has been done in Assignment 4 and there we saw that it is:

$$u(x,0) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n\pi} \cos\left(\frac{n\pi}{2}x\right) + \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)}{n\pi} \sin\left(\frac{n\pi}{2}x\right)$$

To get the solution of the IBVP we need to multiply the n-th term of the sums above by $e^{-\left(\frac{n\pi}{L}\right)^2\alpha^2 t} = e^{-\left(\frac{n\pi}{2}\right)^2 10t}$ and so we obtain the solution:

$$u(x,t) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n\pi} e^{-\left(\frac{n\pi}{2}\right)^2 10t} \cos\left(\frac{n\pi}{2}x\right) + \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)}{n\pi} e^{-\left(\frac{n\pi}{2}\right)^2 10t} \sin\left(\frac{n\pi}{2}x\right)$$

Question 7

Solve the following IBVP

$$\begin{aligned}
\text{PDE}: \quad & u_t(x,t) = 10u_{xx}(x,t) && \text{for } -1/2 < x < 1/2 \text{ and } t > 0 \\
\text{BC}: \quad & u(-1/2,t) = u(1/2,t), \;\; u_x(-1/2,t) = u_x(1/2,t) && \text{for } t \geq 0 \\
\text{IC}: \quad & u(x,0) = \begin{cases} \sin(2\pi x) & \text{for } x \in (0, \frac{1}{2}) \\ 0 & \text{for } x \in (-\frac{1}{2}, 0) \end{cases}
\end{aligned}$$

Solution:
To solve this IBVP we need first to compute the Fourier series of $u(x,0)$ on $[-1/2, 1/2]$. This has been done in Assignment 4 and there we saw that it is:

$$u(x,0) = \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{1-(-1)^{n+1}}{\pi(1-n^2)} \cos(2n\pi x) + \frac{1}{2}\sin(2\pi x)$$

To get the solution of the IBVP we need to multiply the n-th term of the sum above by $e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} = e^{-(2n\pi)^2 10t}$ as well as the term $\sin(2\pi x)$ by $e^{-(2\pi)^2 10t}$ (corresponding to the first and only term of the "sine sum") and so we obtain the solution:

$$u(x,t) = \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{1-(-1)^{n+1}}{\pi(1-n^2)} e^{-(2n\pi)^2 10t} \cos(2n\pi x) + \frac{1}{2} e^{-(2\pi)^2 10t} \sin(2\pi x)$$

Question 8

Solve the following IBVP.

$$\begin{aligned}
\text{PDE}: \quad & u_t(x,t) = 10u_{xx}(x,t) && \text{for } -2 < x < 2 \text{ and } t > 0 \\
\text{BC}: \quad & u(-2,t) = u(2,t), \;\; u_x(-2,t) = u_x(2,t) && \text{for } t \geq 0 \\
\text{IC}: \quad & u(x,0) = x^2 && \text{for } -2 \leq x \leq 2
\end{aligned}$$

Solution:
To solve this IBVP we need first to compute the Fourier series of $u(x,0)$ on $[-2,2]$. This has been done in Assignment 4 and there we saw that it is:

$$u(x,0) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi}{2}x\right)$$

To get the solution of the IBVP we need to multiply the n-th term of the sums above by $e^{-\left(\frac{n\pi}{L}\right)^2\alpha^2 t} = e^{-\left(\frac{n\pi}{2}\right)^2 10t}$ and so we obtain the solution:

$$u(x,t) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} e^{-\left(\frac{n\pi}{2}\right)^2 10t} \cos\left(\frac{n\pi}{2}x\right)$$

Question 9

Solve the following IBVPs

(i)

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0 \\ \text{BC}: &\quad u_x(0,t) = 0, \;\; u_x(\pi,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = 1 + \cos^2(x) && \text{for } 0 \leq x \leq \pi \end{aligned}$$

[Exam question, 2006]

Solution:
We start first by computing the Fourier series of the even extension of $u(x,0)$ to the interval $[-\pi,\pi]$. Since $\cos(x)$ is an even function, the even extension of $u(x,0)$ to the interval $[-\pi,\pi]$ is $\phi(x) = 1+\cos^2(x)$ for $x \in [-,\pi,\pi]$. Moreover, note that:

$$\phi(x) = 1 + \cos^2(x) = 1 + \frac{1}{2}(1+\cos(2x)) = \frac{3}{2} + \frac{1}{2}\cos(2x)$$

Since this is already a linear combination of terms of the form $\cos(\frac{n\pi}{L}x) = \cos(nx)$, it is already in the form of a Fourier series. Thus, the solution of the above IBVP is given by:

$$u(x,t) = \frac{3}{2} + \frac{1}{2}e^{-\left(\frac{2\pi}{L}\right)^2 t}\cos(2x) = \frac{3}{2} + \frac{1}{2}e^{-4t}\cos(2x)$$

(ii)

$$\begin{aligned} \text{PDE}: &\quad u_t(x,t) = 10u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0 \\ \text{BC}: &\quad u(0,t) = 0, \;\; u(\pi,t) = 0 && \text{for } t \geq 0 \\ \text{IC}: &\quad u(x,0) = x(\pi - x) && \text{for } 0 \leq x \leq \pi \end{aligned}$$

Solution:
We start first by computing the Fourier series $a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x)$ of the odd extension of $u(x,0)$ to the interval $[-\pi, \pi]$. This is given by

$$\phi(x) = \begin{cases} x(\pi - x) & \text{for } x \in [0, \pi) \\ x(\pi + x) & \text{for } x \in (-\pi, 0) \end{cases}$$

Since ϕ is an odd function, we know that $a_n = 0$ for $n \geq 0$. Moreover, for $n \geq 1$ we have:

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L \phi(x) \sin\left(\frac{n\pi}{L}x\right) \, dx \\ &= \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin(nx) \, dx \\ &= \frac{2}{\pi} \left[-\frac{x(\pi - x)}{n} \cos(nx) \right]_0^\pi + \frac{2}{n\pi} \int_0^\pi (\pi - 2x) \cos(nx) \, dx \\ &= 0 + \frac{2}{n\pi} \left[\frac{\pi - 2x}{n} \sin(nx) \right]_0^\pi + \frac{4}{n^2\pi} \int_0^\pi \sin(nx) \, dx \\ &= 0 + \frac{4}{n^2\pi} \left[-\frac{\cos(nx)}{n} \right]_0^\pi = \frac{4}{n^3\pi}(1 - (-1)^n) \end{aligned}$$

Thus, the Fourier series of $\phi(x)$ is

$$\phi(x) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi}(1 - (-1)^n) \sin(nx)$$

To obtain the solution of the IBVP we just need to multiply each of terms in the sum above by $e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} = e^{-\left(\frac{n\pi}{\pi}\right)^2 10t} = e^{-10n^2 t}$. We conclude that the solution to the IBVP is:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi}(1 - (-1)^n) e^{-10n^2 t} \sin(nx)$$

Part II: The wave equation

Question 10

Solve the following IBVP:

$$
\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = u_{xx}(x,t) && \text{for } 0 < x < L \text{ and } t > 0 \\
\text{BC}: &\quad u(0,t) = u(L,t) = 0 && \text{for } t \geq 0 \\
\text{IC}: &\quad u(x,0) = 0 && \text{for } 0 \leq x \leq L \\
&\quad u_t(x,0) = \sin\left(\frac{8\pi}{L}x\right)\cos\left(\frac{3\pi}{L}x\right) && \text{for } 0 \leq x \leq L.
\end{aligned}
$$

Solution:
Using the trigonometric identity $\sin(\theta)\cos(\phi) = \frac{1}{2}\left((\sin(\theta+\phi) + \sin(\theta-\phi)\right)$, we obtain $u_t(x,0) = \frac{1}{2}\sin\left(\frac{11\pi}{L}x\right) + \frac{1}{2}\sin\left(\frac{5\pi}{L}x\right)$. Note that now both $u(x,0)$ and $u_t(x,0)$ are written as a Fourier series. Thus, we apply the formula from the lecture and obtain:

$$
u(x,t) = \frac{L}{10\pi}\sin\left(\frac{5\pi}{L}x\right)\sin\left(\frac{5\pi}{L}t\right) + \frac{L}{22\pi}\sin\left(\frac{11\pi}{L}x\right)\sin\left(\frac{11\pi}{L}t\right).
$$

Question 11

Solve the following IBVP:

$$
\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 49u_{xx}(x,t) && \text{for } 0 < x < 1 \text{ and } t > 0 \\
\text{BC}: &\quad u(0,t) = u(1,t) = 0 && \text{for } t \geq 0 \\
\text{IC}: &\quad u(x,0) = 3\sin\left(\frac{2035\pi}{2}x\right)\sin\left(\frac{\pi(1999x+1)}{2}\right) && \text{for } 0 \leq x \leq 1 \\
&\quad u_t(x,0) = 0 && \text{for } 0 \leq x \leq 1.
\end{aligned}
$$

Solution:
Using the trigonometric identity $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$, we obtain $u(x,0) = \frac{3}{2}\sin(2017\pi x) + \frac{3}{2}\sin(18\pi x)$. We apply the formula from the lecture and obtain:

$$
u(x,t) = \frac{3}{2}\sin(18\pi x)\cos(18\cdot 7\pi t) + \frac{3}{2}\sin(2017\pi x)\cos(2017\cdot 7\pi t).
$$

Question 12

Solve the following IBVP:

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = u_{xx}(x,t) && \text{for } 0 < x < L \text{ and } t > 0\\
\text{BC}: &\quad u(0,t) = u(L,t) = 0 && \text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = \left(2\sin\left(\frac{(20x+L)\pi}{4L}\right)\right)^2 + 6\sin\left(\frac{\pi}{L}x\right) - 2 && \text{for } 0 \leq x \leq L\\
&\quad u_t(x,0) = 5\sin\left(\frac{10\pi}{L}x\right) + 11\sin\left(\frac{50\pi}{L}x\right) && \text{for } 0 \leq x \leq L.
\end{aligned}$$

Solution:

We recall the trigonometric identities $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$ and $\cos(\theta) = -\sin((\theta - \frac{\pi}{2})$. Thus we have

$$\left(2\sin\left(\frac{(20x+L)\pi}{4L}\right)\right)^2 = 4\sin^2\left(\frac{(20x+L)\pi}{4L}\right) = 4\frac{1-\cos\left(\frac{(20x+L)\pi)}{2L}\right)}{2} =$$

$$2 + 2\sin\left(\frac{(20x+L)\pi}{2L} - \frac{\pi}{2}\right) = 2 + 2\sin\left(\frac{10\pi}{L}x\right).$$

Thus we have $b_1 = 6, b_{10} = 2, c_{10} = 5, c_{50} = 11$. Using the general formula we obtain:

$$\begin{aligned}
u(x,t) =& 6\sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi}{L}t\right) + \sin\left(\frac{10\pi}{L}x\right)\left[2\cos\left(\frac{10\pi}{L}t\right) + \frac{L}{2\pi}\sin\left(\frac{10\pi}{L}t\right)\right]\\
&+ \frac{11L}{50\pi}\sin\left(\frac{50\pi}{L}x\right)\sin\left(\frac{50\pi}{L}t\right).
\end{aligned}$$