Analysis III (BAUG) — **Assignment 6**

Prof. Dr. Alessandro Sisto — Due 3rd November 2017

Fourier expansions and IBVP

Question 1

Solve the following wave equations. (At least 2 exercises)

(i)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 2u_{xx}(x,t) && \text{for } 0 < x < 2\pi \text{ and } t > 0\\
\text{BC}: &\quad u(0,t) = 0, \;\; u(2\pi,t) = 0 && \text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = \sin(\tfrac{x}{2}) + \sin(\tfrac{5x}{2}) && \text{for } 0 \leq x \leq 2\pi\\
&\quad u_t(x,0) = 4\sin(3x) && \text{for } 0 \leq x \leq 2\pi
\end{aligned}$$

(ii)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 4u_{xx}(x,t) && \text{for } 0 < x < 10 \text{ and } t > 0\\
\text{BC}: &\quad u(0,t) = 0, \;\; u(10,t) = 0 && \text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = 2\sin(3\pi x) - 4\cos(\tfrac{(2x+5)\pi}{10}) && \text{for } 0 \leq x \leq 10\\
&\quad u_t(x,0) = 4\cos^2(\pi x - \tfrac{\pi}{4}) - 2 && \text{for } 0 \leq x \leq 10
\end{aligned}$$

(iii)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = 3u_{xx}(x,t) && \text{for } 0 < x < 6 \text{ and } t > 0\\
\text{BC}: &\quad u(0,t) = 0, \;\; u(6,t) = 0 && \text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = \sin(2\pi x) && \text{for } 0 \leq x \leq 6\\
&\quad u_t(x,0) = 3 && \text{for } 0 \leq x \leq 6
\end{aligned}$$

(iv)

$$\begin{aligned}
\text{PDE}: &\quad u_{tt}(x,t) = u_{xx}(x,t) && \text{for } 0 < x < \pi \text{ and } t > 0\\
\text{BC}: &\quad u(0,t) = 0, \;\; u(\pi,t) = 0 && \text{for } t \geq 0\\
\text{IC}: &\quad u(x,0) = x && \text{for } 0 \leq x \leq \pi\\
&\quad u_t(x,0) = x + 3 && \text{for } 0 \leq x \leq \pi
\end{aligned}$$

Question 2

Use d'Alembert's formula for the following problems about wave equations on infinite strings. (At least 2 exercises)

(i)

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = x && \text{for } -\infty < x < \infty \\ & u_t(x,0) = \cos(x) && \text{for } -\infty < x < \infty \end{aligned}$$

Compute $u(x,t)$ for all x and all $t \geq 0$.

(ii)

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = \begin{cases} 8x - 2x^2 & \text{for } 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases} \\ & u_t(x,0) = \begin{cases} 16 & \text{for } 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Compute $u(11,3)$ and $u(5,2)$.

[Exam question, 2013]

(iii)

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = 0 && \text{for } -\infty < x < \infty \\ & u_t(x,0) = \begin{cases} 1 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Compute $u(x,t)$ for all x and all $t \geq 0$.

[Exam question, 2008]

(iv)

$$\begin{aligned} \text{PDE}: \quad & u_{tt}(x,t) = u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\ \text{IC}: \quad & u(x,0) = \frac{1}{x^2+1} && \text{for } -\infty < x < \infty \\ & u_t(x,0) = \frac{1}{x^2} \end{aligned}$$

Compute $u(x,t)$ for all x and all $t \geq 0$.

Question 3

Let $f(x) = -2x$ for $x \in [0, 10]$. Choose the correct Fourier series (sines, cosines or normal one) and write f as a trigonometric series.

Question 4

Solve the following IBVP.

$$\begin{aligned} \text{PDE}: \quad & u_t(x,t) = u_{xx}(x,t) && \text{for } 0 < x < 10 \text{ and } t > 0 \\ \text{BC}: \quad & u(0,t) = 15, \quad u(10,t) = 35 && \text{for } t \geq 0 \\ \text{IC}: \quad & u(x,0) = 15 && \text{for } 0 \leq x < 10 \end{aligned}$$

Chain rule training

The goal of this part of exercises it to understand how to the change of variables can allow us to solve new IVPs, that is to solve Question 8. If you solve Question 8, you don't need to do any other question. However, Questions 5,6 and 7 are a training to Question 8. You should do as many point of them as you need to feel confident with the topics presented.

Question 5

Let $u(x, y)$ be a function satisfying $u_{xx} = u_{tt}$. For each of the following change of coordinates, choose the correct PDE satisfied by u with respect to the new coordinates.

(i) $\xi = 4x, \eta = 5t$.

- ☐ $u_{\xi\xi} = u_{\eta\eta}$;
- ☐ $4u_{\xi\xi} = 5u_{\eta\eta}$;
- ☐ $16u_{\xi\xi} - 40u_{\xi\eta} + 25u_{\eta\eta} = 0$;
- ☐ $16u_{\xi\xi} - 25u_{\eta\eta} = 0$.

(ii) $\xi = x + 3t, \eta = t$.

- ☐ $u_{\xi\xi} = u_{\eta\eta}$;
- ☐ $-8u_{\xi\xi} = 6u_{\xi\eta} + u_{\eta\eta}$;
- ☐ $9u_{\xi\xi} - u_{\eta\eta} = 0$;
- ☐ $u_{\xi\xi} = 9u_{\eta\eta} + 6u_{\eta\xi}$.

(iii) $\xi = 2x, \eta = xt$

- ☐ $\xi^2 u_{\xi\xi} = \eta^2 u_{\eta\eta}$;

☐ $\frac{-8}{3}\xi u_{\xi\xi} = 4\frac{\xi}{\eta}u_{\xi\eta} + u_{\eta\eta}$;

☐ $4u_{\xi\xi} + 4\eta^2 u_{\xi\eta} + \left(\left(\frac{1}{2\xi}\right)^2 - \eta^2\right) u_{\eta\eta} = 0$;

☐ $4(u_{\xi\xi} + \frac{2\eta}{\xi}u_{\xi\eta}) + \left(\left(\frac{2\eta}{\xi}\right)^2 - \left(\frac{1}{2}\xi\right)^2\right) u_{\eta\eta} = 0$.

Question 6

Consider a vibrating infinite string that vibrates with propagation speed c, where c represents the speed of light (for instance the wave caused by a light beam). Suppose that an observer is moving next to it with constant speed v (starting from $x = 0$ at time $t = 0$). If we consider Einstein's theory of relativity, the change of coordinates between the system of the string (coordinates x, t) and the observer (coordinates ξ, η) is $\xi = \gamma(x - vt)$ and $\eta = \gamma\left(t - \frac{vx}{c^2}\right)$, where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is a constant. What is the PDE satisfied by the string from the point of view of the observer?

Question 7

Consider the following IVP.

$$
\begin{aligned}
\text{PDE}: \quad & u_{tt}(x,t) = u_{xx} && \text{for } -\infty < x < \infty \text{ and } t > 0 \\
\text{IC}: \quad & u(x,0) = f(x) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\
& u_t(x,0) = g(x) && \text{for } -\infty < x < \infty \text{ and } t > 0
\end{aligned}
$$

For each coordinate change of exercise 5, write the corresponding IVP for $v(\xi, \eta) = u(\xi(x,t), \eta(x,t))$.

Question 8

(i) Consider a vibrating infinite string, and suppose that an observer is moving next to it with constant speed v (starting from $x = 0$ at time $t = 0$). What is the PDE satisfied by the string from the point of view of the observer? (Use Newtonian physic)

(ii) Solve the following IVP.

$$
\begin{aligned}
\text{PDE}: \quad & u_{tt}(x,t) - 2u_{tx} = 24u_{xx}(x,t) && \text{for } -\infty < x < \infty \text{ and } t > 0 \\
\text{IC}: \quad & u(x,0) = 0 && \text{for } -\infty < x < \infty \text{ and } t > 0 \\
& u_t(x,0) = \sin(x) && \text{for } -\infty < x < \infty \text{ and } t > 0
\end{aligned}
$$